# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS



THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 54
1947

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND OBERLIN, OHIO

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 1


CONTENTS




JANUARY 1947

# The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)




EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the Secretary-Treasurer, W. B. Carver, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the first of the month in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---


Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# COORDINATING HIGH SCHOOL AND COLLEGE MATHEMATICS*

W. D. REEVE, Teachers College, Columbia University

**1. Introduction.** The problem of coordinating high school and college mathematics is one which both the high school and college teachers of that subject should cooperate in solving. Failure to coordinate these separate fields in the past has led to a great deal of confusion and genuine loss both for the pupils involved and also their teachers.

In the first place, the high schools have not and still do not succeed in teaching their pupils as much as they are capable of learning if only they would wake up to their responsibilities in the matter. Two obvious reasons may be given for our present failure to teach more mathematics to those who are capable of learning it. They are both related to the same general problem of organization of content material.

**2. Organization of subject matter.** Our traditional water-tight compartment method of teaching algebra, then plane geometry, then intermediate algebra, and so on, leads to a great deal of unnecessary repetition of subject matter throughout the secondary school period that results in the loss of a great deal of time and energy. We teach algebra in the ninth grade, then plane geometry in the tenth grade. By that time the pupils have forgotten the algebra they knew so that when intermediate algebra (which should be a half year course as such) is presented, usually in the eleventh grade, a large part of the time is spent, if not wasted, in re-teaching ninth grade algebra. As a result it often happens that the entire eleventh year is spent on algebra when with better organization at least the ordinary course in trigonometry might have been presented. Then, if the pupil goes on with mathematics, he may be taught college algebra where again a great deal of time is spent, if not lost, re-teaching elementary and intermediate algebra. And as is often the case with such practice, very little college algebra *per se* is presented. I recently looked through a college algebra text in which well over a hundred pages were devoted to elementary algebra.

What is even worse so far as the pupils and even their teachers are concerned is for the pupils to have to take algebra, trigonometry or other really high school mathematics courses in the colleges and the universities for one reason or another and thus often miss analytic geometry and the calculus.

One excellent way to avoid all of this loss is to organize a course in general mathematics extending all through the high school so that those who can take it will have had a chance to study all of the mathematics up to and including the fundamental ideas of the calculus before they leave the high school. The college teachers of mathematics could then go on from there and devote their time to reorganizing what should really be mathematics on the college level and in that way make it possible for those boys and girls whose life work de-

* Presented to the Metropolitan New York Section of the Mathematical Association of America at Cooper Union, New York City, on May 4, 1946.

pends upon a strong background of mathematics to secure the necessary training without so much loss in time and energy.

The idea that general mathematics is a course for dullards keeps bobbing up in educational circles. I would like to point out here that this idea is not only unjustifiable, it is absolutely untrue. This view is no doubt due to the fact that those who so consider general mathematics do not really know what it means and that they have had little if any real experience in organizing and administering such a course.

If algebra were made the core in the ninth grade, geometry (plane and solid) in the tenth, algebra and trigonometry and simple analytics in the eleventh, analytics and the calculus in the twelfth and the necessary arithmetic worked in the course all along the line, it is demonstrable that not only much time could be saved, but the pupils concerned would have a much better background for their future work in mathematics and related fields.

The fact is that in a real important sense it takes a higher type of pupil to succeed in the advanced points of such a general course than an inferior one.

**3. Individual differences.** Failure to take account of individual difference in ability among high school pupils to say nothing of differences in interest and experiences, has resulted in great educational loss. The most retarded pupil in the secondary school today is the gifted pupil, the one with a scholarly mind. The secondary school machine is all geared up to turn out a mediocre product. This is obviously also true in the colleges. New objectives and new standards are needed. As Olson [1] recently put it:

"An even more fundamental reason for insisting upon standards of some kind in every type of education is the value of self-discipline. Whatever may be said of the unwisdom of disciplinary measures thrust on a person by someone else, there can be no doubt that success in any field is achieved only by those who are able to determine their course and then hold themselves to it. Whether we interpret education as life itself or merely as preparation for life, it must, in some way, furnish this same basic life situation. Much harm is being done in our schools and colleges by lowering standards to such a point that the students find no stimulus in their work. We talk bravely about challenging out students, yet we set our objectives so low that there is no challenge for most of them. The solution to this difficulty would be to group students according to their interests and abilities, and then set up appropriate standards which would challenge each group. Modern psychology tells us that virtually none of us succeeds in developing all his potentialities to the highest possible degree. That is painfully evident in our schools today. Many students with genuine ability go through college with poor work habits, largely because they early found they could make satisfactory grades with very little effort. Because the work was planned for the poorer students, there was little incentive for those with ability to do more. Undeveloped potentialities are surely as great an evil as a frustrating experience of failure."

Again we must not forget that for some minds in these days of mass education the kind of mathematics that the gifted pupils need is not the same for the slower type of mind. We must remember that in a democracy we are under great responsibility not only to train leaders, but also to develop *intelligent* followers as far as possible in the schools. We should develop at least a two-track if not three-track system to meet individual needs [2].

**4. Reconciling points of view.** Another difficulty that we face in trying to organize an adequate program of mathematics in the secondary school is the matter of reconciling points of view particularly among the general educationists and teachers of mathematics in the schools and colleges.

President James Bryant Conant of Harvard University has recently given an excellent suggestion. In discussing the reasons why the lay critics of secondary education talk as they do, President Conant said:

"I am almost tempted to generalize that the more educated the person, the less his knowledge of secondary-school education. Certainly the lack of knowledge among the professors of arts and sciences in our colleges and universities is proverbial. And with lack of information goes lack of understanding and lack of sympathy. As a result, on more than one campus we have almost a state of civil war between those who profess a knowledge of education and those who profess a knowledge of subjects which constitute a modern educational curriculum.

"This academic war has been in a sense inevitable, as I propose to show by a brief résumé of history, but to my mind an armistice has been for some years overdue. And it is for such an armistice that I should like to put in a good word this afternoon (and I might remark parenthetically that it takes two to make an armistice quite as much as to make a quarrel). My belief in the need for the cessation of hostilities comes not only from my general tendency to favor pacific methods of handling academic controversy, but also because I am really worried about the present lay reaction to educational matters. I am distressed by both the vehemence and the ignorance with which views about education are expressed publicly and privately by many prominent people. Now we can hardly expect the public to have a very clear understanding about educational problems when education is a house warring against itself. Hence my plea for a 'cease firing' order" [3].

This is the attitude that some of us have been taking for a long time. When teachers of secondary mathematics, college mathematics, supervisors, administrators and general educationists sit down around the table to discuss what for all of them should be a common problem, then we can hope for some practical solution of what should constitute general education in the post-war years.

We have a few outstanding cases* of men and women in the colleges and universities who have taken great interest in the work of the secondary schools

* Witness the great interest of men like Professors Cairns, Hedrick, Kasner, Mitchell, Slaught and others.

and in organizations of teachers in secondary schools, but for the most part many of these people do not know what is going on in the schools and some of them seem to care less. And this in spite of the fact that some of the pupils that are being trained in the mathematics of the secondary schools are the same pupils who will sit at the feet of the college instructors in mathematics in the colleges and universities in the days ahead. There is a great need for a better working understanding and cooperation between teachers of mathematics in the secondary school and those of collegiate grade.

**5. What should be done.** Let us now consider what should be done all along the line to improve the mathematical background of our people so that in peace time as well as in war time they can function as intelligent citizens whether or not a definite crisis exists.

In a recent editorial in *The National Mathematics Magazine* Professor S. T. Sanders said:

"Many teachers are nervously concerned over what may be the post-war status of school mathematics. The enormous expansion of the technical applications of the science under pressure of war has brought about a world-wide strengthening of mathematics in the school curriculum. Can this current academic primacy of mathematics be made permanent? Such is the question raised by those keenly mindful of the scant attention paid to this subject by the less recent curriculum makers.

"A careful study of the matter should not discount the fact that in respect to mathematics, the war has served only to bring about greatly multiplied *uses* of mathematics, a large proportion of which were already in existence. For, even in pre-war times, there had been for many years a steady growing public emphasis upon *applied* mathematics, rather than upon the logical or cultural aspects of the science.

"In the light of this definite trend, a trend not rooted in any war, it could well be that the post-war school effort should first be directed to discovering the mathematical aids or needs of all the major peacetime industrial enterprises. Cooperative programs initiated between industry and the schools would then have sounder foundations. Who shall say that the cultures of mathematics would be impaired by being stemmed in its utilities?" [4]

That great interest is being manifested in what place mathematics is to have in the schools after the war is evidenced by the many discussions that one hears, the various articles now appearing in current magazines and editorial comments like that above. Moreover, the National Council of Teachers of Mathematics has a *Commission on Post-War Plans* in mathematics, the first report of which appeared in *The Mathematics Teacher* for May, 1944. A second and more inclusive report of progress of this commission appeared in the May, 1945, issue of that journal.

The problem about which we are concerned here is that we do not have agreement in all quarters as to what should be done. We cannot take the space to

survey all of the recent opinions and articles, but a few typical ones will show how the wind is blowing.

After discussing the high place which mathematics once held in the schools and the poor results in mathematical education shown by recent Army reports, Professor H. L. Dorwart recently said:

"At this point, it may be asked why so many of our young people ceased to study mathematics some years ago. Many mathematics teachers say that it is all the fault of the educationists with their half-baked theories of the non-transfer of training and of the removal of everything difficult from education in order to prevent harmful personality development. The retort of the educationists is that mathematics teachers are just a lot of sadistic drill-masters who do more harm than good anyway. I will pass over this charge and countercharge in favor of what may, it is hoped, be a somewhat more helpful point of view.

"But, first, let us face the fact that there have been, and still are, both poor and poorly prepared teachers of mathematics, and that they have repelled many good students. Many high-school principals must be forced to give up the idea that anyone who possesses the credits in methods-of-education courses specified by law is thereby qualified to teach algebra or geometry. Also, college presidents and deans should investigate carefully the personalities of the instructors assigned to elementary courses. Even if the instructor is a recognized authority in his field, the conceited, arrogant, show-off type (fortunately few in number) should be used elsewhere than in elementary courses.

"I now propose the thesis that, once the requirements were withdrawn, students ceased to study mathematics principally because they did not recognize the fundamental rôle that it plays in modern civilization, or that by omitting the study of mathematics they were thereby imposing large restrictions on their future choice of profession or employment. In short, they did not recognize and usually have not been told that, in addition to serving, mathematics is a queen in her own right, a queen who will richly reward her followers, but only if they follow her diligently from their youth through a long period of time, even when the going is tough and when the path ahead is not always crystal-clear" [5].

In commenting on Dorwart's article, the late Professor Bagley said:

"The leading article in last week's number, *Mathematics—Queen and Handmaiden*, represents a type of discussion that is likely to become more and more important as the educational trends of the past few decades come increasingly to be scrutinized and questioned in the light of the educational weakness that the experiences of the war have revealed. Among these trends, probably none have been more debilitating than have the theories and arguments used to discount and discredit the subjects that are, to use a favorite phrase of the present writer, "exact and exacting, systematic and sequential"—more specifically, on the secondary level, mathematics and Latin.

"It is clear enough now that an educational development, fundamental in character and of far-reaching scope, lay back of the readiness with which

American education accepted theories and postulates that served to rationalize this discreditment. The upward expansion of mass education, which has resulted in what is virtually a "universal" high school, and which is now extending beyond the secondary level, would have been impossible without a relaxation of standards that are perhaps beyond the capacity and certainly alien to the tastes of a significant proportion of each generation of pupils and students.

"All this is water over the dam. The re-instating of Latin and of mathematics beyond the simple arithmetical process as secondary school requirements would doubtless be as unwise as it would be impossible. But it would be equally unwise, and it is not at all necessary, to continue allegiance to an educational theory that in effect encourages the following of the lines of least resistance by those who are competent to do the more difficult types of learning.

"Expert educational guidance, coupled with such masterful teaching as Dr. Dorwart suggests, can do much to solve the problem. But there should be as well, we think, a systematic, even a militant, effort to enlighten both youth and the public generally as to the basic significance of the subjects that constituted the core of liberal secondary education over the long period during which the secondary school was a highly selective institution. It was no more an accident that this core comprised Latin and mathematics than it was an accident that the core of elementary education from the outset comprised the reading and writing for the mother tongue and the primary arts of number. (After all, civilization itself began with the invention of writing and development of computation and measurement.) Liberal education on the secondary and higher levels has been based upon a refinement and expansion of the symbolism of conceptual thought. This has meant mathematics, and, in English-speaking countries, Latin, which, in a very real sense (as can be shown by abundant evidence), is the 'mother tongue of our mother tongue' in so far as the symbolism of advanced conceptual thought is concerned" [6].

Thus far the comments and articles referred to have been favorable to mathematics. However, we must present a typical point of view which raises questions as to whether mathematics is as important in the secondary school as some people think. Professor F. N. Freeman, Dean of the School of Education at the University of California, recently said:

"One of the outcomes of the war, in the opinion of many officers of the Army and Navy and of many observant laymen, is the revelation of gross inadequacy in the teaching of mathematics. The experience of the armed forces and of industry is supposed to have shown that vastly larger numbers of students should study mathematics and that they should study more mathematics—up to and including trigonometry. We may look, therefore, for a campaign after the war, to require the study of mathematics through trigonometry by a large share of high-school students. If such a campaign is launched, it will grow out of an idea that is about as sound as the belief that the psychological tests given in World War I showed 40% of the male population to be morons.

"The reasons that no such conclusion follows from the experience of the

Army and Navy, not to speak of industry, are, first, that the demands of the services in war time are no reflection of their demands in times of peace, and, second, that the number of men who need straight mathematics to perform duties required of them is much fewer than has been implied by the published statements. To say that the schools should, in peace time, give all the preparatory training that may be needed in time of war is like saying that industry must be kept in continual readiness to produce 10,000 planes a month. Again, the number of men who acquire techniques in the Army or Navy which are based on mathematics is vastly greater than the number who are required to understand the principles of mathematics and their application. The same is true of industry. The implied statement that all the men in the armed forces and industry who perform technical operations require a knowledge of higher mathematics for such performance is the wildest exaggeration. The war does not teach that mathematics through trigonometry is a practical necessity for a large proportion of men, let alone women. The question of the kind and amount of mathematics which is good for the average person and should be an element in general education still remains open" [7].

In the same article Dean Freeman presents a series of propositions which he thinks may be taken as a platform for the reorganization of mathematics in general education as follows:

"1. Only that mathematics is important for general education which the individual will use.
2. The individual will use only those mathematical ideas and operations which he has learned by use or in use.
3. The individual will actually use only those processes that he has mastered and made thoroughly familiar to himself.
4. Understanding is desirable, but it comes best through familiar use first, and formal explanation afterward.
5. Mathematics is, or contains, a form of language which formulates and defines the quantitative aspect of experience and, therefore, stimulates and largely creates quantitative ideas and forms of thought.
6. Mathematics may properly be thought of as a language—that is, as a particular set or particular sets of symbols which represent special aspects of reality.
7. To have meaning in the thinking of the child or of the ordinary person in general, the use of mathematical symbols and operations should be dedeveloped in intimate and continual association with the real world of experience.
8. The mathematics for the millions is that which gives clearer and more effective ways of thinking about the real world of experience because it has been developed out of this world of experience" [7].

I think Dean Freeman's points represent the thoughts of a great many powerful people in the education field.

It is gratifying to know that the Mathematical Association of America and

The National Council of Teachers of Mathematics are now cooperating in a plan which is intended to coordinate the studies of the two organizations in Mathematical Education. The Association has appointed a Committee under the chairmanship of Professor C. V. Newsom and the Council one under Professor F. L. Wren. Professor Newsom's committee has already issued a memorandum recommending that each Section of the Association give active attention to a number of points;* perhaps part of the time at each annual meeting could be devoted to a discussion of them. The Committee also recommended that each Section organize a committee to push the cause of mathematical education and to maintain contact with the central group.

**6. Mathematics in an Air-Conditioned Civilization.** In this complex civilization school administrators and teachers alike are faced with the problem of providing a better type of education for the seven million or so pupils who now make up our secondary school population. This means, not only drastic changes in the education of those pupils of the academic type to meet changing demands, but also an altogether different type of curriculum for those pupils who, under the traditional system, "leave school without skills or orientation of any kind with respect to social institutions" [**8**].

We have all heard the familiar remark, "We live in a changing world." The fact is that at the present time we not only live in a changing world, but the rate of change is so much more rapid than previously that few people are willing even to predict with any degree of precision just what will come to pass in the next fifteen or twenty years.

This generation and the one to follow are going to be "air conditioned," so to speak, and the sooner everybody adjusts his life to this situation the better for everybody concerned. The fact is that we must become air conditioned in order to survive.

The significance of all this for science in general and mathematics in particular is obvious to all who think seriously and fairly about the situation. These subjects, or at least their fundamental and useful aspects, should be *must* studies for most of our school population in the days to come.

Those who have been thinking about the matter already realize that in any well-organized course in air navigation many of the principles of mathematics, physics, and the other sciences previously taught in the secondary school can be applied in ways that will make these subjects interesting, meaningful, and more useful than ever before.

If we are to furnish an adequate basis for the education of pupils in the fundamental principles of aeronautics, then we shall have to get rid of the "deadwood" (the debris) that clutters up the courses in mathematics, physics, and other subjects in our secondary schools. Thus, in order to have time for some of the real life applications of mathematics to air navigation and other similarly im-

* Interested persons may obtain copies of this memorandum by writing to Professor C. V. Newsom, Oberlin College, Oberlin, Ohio.

portant life interests, we shall have to eliminate all such obsolete processes as certain types of factoring, complicated processes with polynomials, and the like, from the lower years of the secondary school and teach then in the later years only if and when they are needed. Moreover, this can be done without any loss to any important interest. But progress along this line is too slow. We need action and we need it quickly.

What groups of American citizens then are likely to need further training in mathematics in the days ahead and what type of organizations of subject matter and methods of instruction are likely to prove most efficient?

The purpose of a course in mathematics in the secondary school is to meet the needs of four groups of pupils:

1. Those who intend to go on to colleges and technical schools.
2. Those who are going to specialize in commercial work or vocational pursuits that require mathematics, especially algebra.
3. Those who intend to major in science.
4. Those who elect mathematics because they like it.

There is no question about the need for mathematical training for the pupils in the first group who are going to be engineers, statisticians, actuarial workers, certified public accountants and the like. The main problem here is one of guidance. If we can be certain that pupils are on the right track, then we can and should clear the way so that such pupils can be given not only four years of mathematics but mathematics of a very high order. To permit such pupils to take less mathematics and waste time in their courses by going at the rate of the less superior pupils will be tragic for them and the country as well. The same line of thought follows for those pupils who plan to go on to college where they will specialize in mathematics and then teach the subject, carry on research, or be applied mathematicians of one type or another.

A second group of skilled workers in business and industry will need to know certain mathematical facts and be able to use certain mathematical skills in the new era that will open up now that the war is over. Their work is extremely important, but they will not have to be as highly trained as those in the first group mentioned above. The possible future needs of such a group need to be studied carefully and a suitable course of study worked out to meet such needs.

A third group consists of those who intend to specialize in science. We ought to be able to assume that for them mathematical training of a high order would be basic if it were not for the fact that it does not work out that way. A careful study of the needs of this group should be made and the necessary content material organized for teaching purposes.

The last and by far the largest group of our citizens will be those who take mathematics because they are made to realize that without a reasonable understanding of the fundamental ideas they will not be competent to carry on as they should. The skills and habits of quantitative thinking must be acquired by such pupils if they are to meet the reasonable demands placed upon them.

In the new era just ahead many of our pupils will need to know how to solve

the problems of air and marine navigation. This will necessitate an understanding of the facts and processes of algebra, geometry and at least numerical trigonometry to an extent that will make the offering of at least a two-year course in mathematics starting with the ninth grade.

In the fields of business and industry where the war created new problems, different emphases may have to be made particularly where some of these problems of war are also the problems of the peace days ahead.

Let us also hope that more and better prepared teachers will be available and at salaries that will enable them to have a decent living. No matter what happens, however, we should not expect miracles and it is clear to those who understand the present situation that in many places better teachers, higher salaries, and more money for better books, equipment and the like may not be forthcoming.

In any case teachers who remain in the classroom because they want to do so in spite of handicaps must somehow learn to teach in better and more interesting ways many of the things that are now in the course of study. This need not be discouraging, for this can be done, if and only if, teachers are inspired to carry on the work as outlined by the commissions which have tried so hard to help.

## REFERENCES

1. Olson, Claire C., And Gladly Would He Learn, School and Society, April 27, 1946, pp. 289–291.

2. High-School Methods with Superior Students and High-School Methods with Slow Learners, National Education Association Research Bulletin No. 4, Vol. 12, Sept., 1941 and No. 3, Vol. 21, Oct., 1943.

3. Conant, James Bryant, A Truce Among Educators, Teachers College Record, December, 1944, pp. 157–163.

4. Sanders, S. T., Post-War Planning in Mathematics, The National Mathematics Magazine, October, 1943, p. 2.

5. Dorwart, H. L., Mathematics—Queen and Handmaiden, School and Society, October 14, 1944, pp. 241–243.

6. Bagley, W. C., The Present and Future of Latin and Mathematics in Secondary Education, School and Society, October 21, 1944, pp. 259–260.

7. Freeman, F. N., Teaching Mathematics for the Million, California Journal of Secondary Education, May, 1944, pp. 246–254. See also Casewell, H. L., Progressive Education Principles Used in the War Effort, Teachers College Record, March, 1944, pp. 386–397.

8. Judd, C. H., The Real Youth Problem, School and Society, January 10, 1942, p. 32.

# THE EULER COLUMN

W. B. CATON, University of Maine

**1. Introduction.** We shall consider the classical problem of the buckling of a uniform column under an axial load. For a detailed elementary discussion of this question, we refer the reader to Kármán and Biot [1, p. 295]. The reader may consult Love [2, Chap. 19] for a more advanced discussion of the problem under consideration.

We shall denote by $a$ the length of the chord subtended by the central axis of the column, and the maximum deflexion of the central axis from its chord by $d$. The Taylor series expansion of the load $P$ in terms of $d^2/a^2$ is obtained. Also, a similar expansion for $P$ in terms of $d^2/l^2$ is found. Finally, estimates of the error committed in stopping with the $p$'th term of either series are found.

The actual formulas appearing in this note appertain to the first buckling mode only, but it is clear from the analysis that these results can be extended to the $k$'th mode at once. One has only to replace $l$ and $a$ wherever they appear in the formulas by $l/k$ and $a/k$ respectively.

**2. Preliminary formulas and definitions.** We consider a slender uniform column of length $l$, for which Young's modulus is $E$. Let $I$ be the moment of inertia of the cross section about an axis through its centroid at right angles to the plane of bending. The lower end of the column is placed at the origin, and the central axis of the column is made to coincide with the $x$-axis taken vertically. The $y$-axis is taken horizontally, and directed to the right.

A load $P$, directed down along the $x$-axis, is then applied at the upper end of the column. The column buckles if $P$ is sufficiently large, and the upper end descends along the $x$-axis to a point $(a, 0)$ at which equilibrium is attained. It is supposed that the lower end remains at $(0, 0)$, and that no restraints are placed on the slope of the central axis at either end.

The above considerations together with the differential equation of the elastic curve,

$$\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-3/2} = -\frac{Py}{EI}, \tag{1}$$

imply

$$l = 2\sqrt{\frac{EI}{P}}\,K(\pi/2, k), \tag{2}$$

$$a = 2\sqrt{\frac{EI}{P}}\,\{2E(\pi/2, k) - K(\pi/2, k)\}, \tag{3}$$

$$k^2 = Pd^2/4EI.$$

In the display above, $K(\pi/2, k)$ and $E(\pi/2, k)$ denote the complete elliptic integrals of the first and second kinds respectively.

Formulas (2) and (3) appear in Love **[2,** p. 403**]**, but it is believed that some observations on their proofs would be of interest to readers of this note. Proofs of (2) and (3) will be sketched in section 3.

Elementary considerations show that $k^2 \leqq 1/2$ if the angle between the tangent at the origin and the chord is less than or equal to 90°.

It is perhaps worth while to notice in passing that (2) and (3) imply

$$\frac{l-a}{d} = \frac{2}{k}\left\{K(\pi/2,\, k) - E(\pi/2,\, k)\right\},$$

and this in turn gives

$$l - a = \frac{\pi d^2}{4}\sqrt{\frac{P}{EI}}\left\{1 + O(k^2)\right\}.$$

We set $w = k^2$ and put

$$K_1(w) = K(\pi/2,\, k), \qquad E_1(w) = E(\pi/2,\, k).$$

Also, we introduce

$$K^*(w) = \{K_1(w)\}^2, \qquad E^*(w) = \{2E_1(w) - K_1(w)\}^2.$$

Next, we define $r_1$ and $r_2$ as follows:

$$r_1 = \min_{0<R<1}\left\{\frac{1}{R}\max_{|\zeta|=R} |K^*(\zeta)|\right\}, \tag{4}$$

$$r_2 = \min_{0<R<1}\left\{\frac{1}{R}\max_{|\zeta|=R} |E^*(\zeta)|\right\}. \tag{5}$$

In the definitions of $r_1$ and $r_2$, $\zeta$ denotes a complex variable and $|\zeta| = R$ is the circle in the $\zeta$ plane with center at the origin and radius $R$. We may suppose that $r_1$ and $r_2$ are taken on for $R = R_1$ and $R_2$ respectively. A simple argument, based on the principle of the maximum, shows that $R_1$ and $R_2$ are unique.

**3. On formulas** (2) **and** (3).

We set

$$\frac{dy}{dx} = p, \qquad \frac{d^2y}{dx^2} = p\frac{dp}{dy},$$

in (1), and after integration, obtain

$$(1 + p^2)^{-1/2} = \frac{Py^2}{2EI} + c_1, \tag{6}$$

where $c_1$ is the constant of integration.

We now notice that since the beam is uniform, and the buckling mode is of

the first type, $x=a/2$ implies $p=0$ and $y=d$, where $d$ is the maximum deflexion. Hence,

$$c_1 = 1 - Pd^2/2EI = 1 - 2k^2. \tag{7}$$

Next, (6), (7), and the usual formula for arc-length from the calculus, gives

$$l/2 = \int_0^d \left\{1 - \left(c_1 + \frac{Py^2}{2EI}\right)^2\right\}^{-1/2} dy. \tag{8}$$

Introduce $k' > 0$,

$$k^2 + k'^2 = 1.$$

Formula (8) may now be put in the form

$$l/2 = \sqrt{\frac{EI}{P}} \int_0^{1/k'} \{(1 + k^2u^2)(1 - k'^2u^2)\}^{-1/2} dy.$$

Now from (3, p. 494),

$$l = 2\sqrt{\frac{EI}{P}}\, sd^{-1}(1/k'),$$

which implies (2). In the notation of Glaisher (explained in *op. cit.*), $sd^{-1}(u)$ is the inverse of the elliptic function

$$sd(u) \equiv \frac{sn(u)}{dn(u)}\cdot$$

The proof of (3) is much the same. From (6) we have

$$\left(c_1 + \frac{Py^2}{2EI}\right)\left\{1 - \left(c_1 + \frac{Py^2}{2EI}\right)^2\right\}^{-1/2} dy = dx,$$

with the value of $c_1$ obtained from (7). Integrating the right hand member between 0 and $a/2$, and the left hand member between the corresponding limits 0 and $d$, we obtain with the help of (8),

$$\frac{c_1 l}{2} + \frac{P}{2EI}\int_0^d \left\{1 - \left(c_1 + \frac{Py^2}{2EI}\right)^2\right\}^{-1/2} y^2 dy = a/2. \tag{9}$$

Formula (9) is next put up in the form

$$\frac{c_1 l}{2} + 2k^2k'^2\sqrt{\frac{EI}{P}}\int_0^d \{(1 + k^2u^2)(1 - k'^2u^2)\}^{-1/2} u^2 du = a/2. \tag{10}$$

By means of the transformation

$$u = sd(v),$$

the integral in (10) can be reduced to

$$\int_{l_1}^{l_2} sd^2(v)dv, \tag{11}$$

with

$$l_1 = sd^{-1}(0) = 0, l_2 = sd^{-1}(1/k') = K(\pi/2, k) \equiv K.$$

Now, integral (11) is well known, and we have

$$k^2k'^2 \int_0^K sd^2(v)dv = E(\pi/2, k) - k^2K(\pi/2, k). \tag{12}$$

Formula (12) follows from a result in Whittaker and Watson [3, p. 516].

Finally, substitute the value of the integral obtained in (12) back in (10) to get (3).

**4. Main results and their proofs.**

THEOREM 1. *If* $w = Pd^2/4EI$, *then*

$$w = \sum_1^\infty \frac{t^n}{n!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\{K^*(\zeta)\}^n\right]_{\zeta=0},$$

*valid for* $t = d^2/l^2 < 1/r_1$.

The remainder after $p$ terms of the series in Theorem 1 is not greater than

$$-R_1(tr_1)^p \log(1 - tr_1)$$

if $tr_1 < 1$.

THEOREM 2. *If* $w = Pd^2/4EI$, *then*

$$w = \sum_1^\infty \frac{r^n}{n!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\{E^*(\zeta)\}^n\right]_{\zeta=0},$$

*valid for* $r = d^2/a^2 < 1/r_2$.

An estimate of the remainder after $p$ terms of the type obtained for the series in Theorem 1 holds here also.

COROLLARY 1.

$$P = \frac{\pi^2EI}{l^2}\left\{1 + \frac{\pi^2d^2}{2^3l^2} + \frac{19\pi^4d^4}{2^9l^4} + \cdots\right\},$$

*valid for* $d^2/l^2 < 1/r_1$.

COROLLARY 2.

$$P = \frac{\pi^2EI}{a^2}\left\{1 - \frac{3\pi^2d^2}{2^3a^2} + \cdots\right\},$$

*valid for* $d^2/a^2 < 1/r_2$.

COROLLARY 3.

$$\left| P - \frac{\pi^2 EI}{l^2} \right| < - \frac{4EIR_1 r_1}{l^2} \log\left(1 - \frac{d^2 r_1}{l^2}\right),$$

*if* $d^2/l^2 < 1/r_2$.

We observe that from $d/l$ or $d/a$ one could compute $P$ from Corollary 1 or 2. Notice too that in Corollary 3 we have an estimate of the error committed when one uses the common engineering formula,

$$P \cong \frac{\pi^2 EI}{l^2}\cdot$$

The proof of Theorem 1 will now be sketched, that of Theorem 2 is much the same, so will be omitted.

Equation (2) implies that $\zeta = w$ is a root of

$$\zeta = tK^*(\zeta).$$

The Lagrange expansion theorem, (see Whittaker and Watson [3, p. 132]), is now applied. We take the $f(\zeta)$ of the theorem to be $\zeta$ and $a = 0$. Also, as the integration contour we choose

$$|\zeta| = R_1.$$

Then, on this contour

$$\left| \frac{K^*(\zeta)}{\zeta} \right| \leqq r_1,$$

and this implies

$$t | K^*(\zeta) | < | \zeta |,$$

for any $\zeta$ on the contour $|\zeta| = R_1$, and any $t < 1/r_1$.

Thus, the conditions of the Lagrange theorem are satisfied and so we get the expansion of Theorem 1 at once.

For the estimate of the remainder, we notice that the Cauchy integral formula applied to

$$\frac{1}{(n-1)!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\{K^*(\zeta)\}^n\right]_{\zeta=0},$$

shows that the remainder of the series in Theorem 1 after $p$ terms is not greater than

$$R_1 \sum_{p+1}^{\infty} \frac{(tr_1)^n}{n} \tag{13}$$

if $tr_1 < 1$. From (13), the estimate given follows at once.

For the proof of Theorem 2, we start with (3), and it implies $\zeta = w$ is a root of

$$\zeta = rE^*(\zeta).$$

The rest of the proof is the same as above.

BIBLIOGRAPHY

1. T. V. Kármán and M. A. Biot, Mathematical Methods in Engineering, New York, 1940.
2. A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th edition, Cambridge, 1927.
3. E. T. Whittaker and G. N. Watson, Modern Analysis, 4th edition, Cambridge, 1927.

# CONJUGO-CONJUGATE COUPLES IN INVOLUTION

ERNEST MITCHELL, Millsaps College

**1. Introduction.** That a certain type of involution of points on a line, namely, the hyperbolic, has imaginary elements is a fact well known and long known. [1] Geometers in general have been inclined to "bypass" this interesting assemblage going on to the higher and wider aspects of the subject of involution. Since, however, this assemblage of imaginary elements is infinite, it seems that it deserves further consideration and explication, and such is the object of this paper.

It not infrequently happens when a problem of geometry has its origin on the algebraical side of analysis, or when the algebra is called into service, complete generality being sought, that the imaginary obtrudes itself demanding consideration. This, too, notwithstanding the fact that the constants and parameters employed are all real. Such is the case of the involution of points on a line generated, say, by a pencil of quadratics, the roots of the pencil being, of course, the coordinates of the conjugate point-pairs of the involution, [2] or, and probably, more directly the result of normalizing [3] or polarizing [4] a single quadratic, the roots of the quadratic being the double points of the involution. We shall incline rather to this latter approach.

It turns out, as we shall see in this paper, that in the involution of points on a line, real constants being employed, there exists a single infinitude of point-pairs which on the one hand are conjugate harmonically with respect to the double points, and on the other hand are conjugate imaginaries. The object of this paper, therefore, is primarily: To "locate" this infinite assemblage of imaginaries, and to give them a "real" representation; secondarily: To give a complete characterization of the type of involution (hyperbolic) in which these elements appear; and finally: To give a complete set of constructions for finding the double-points of the involution determined by any two given point-pairs.

**2. Definition.** Because of the two-fold bond (yoke) pairing the elements of this infinite assemblage—conjugate harmonically and conjugate imaginary—we shall refer to them as the conjugo-conjugate couples in involution. In speaking

of point-pairs in general in the remainder of this paper, when the pairing is effected through an involution, we shall employ the term couple since it has a stronger connotation than that of point-pair. The involution is, of course, taken on a line, and by line we shall mean a straight line having a real trace but endowed with imaginary as well as real points. The constants (coefficients and parameters) employed shall be real numbers. The line on which the involution lies we shall call the base line, and the circle centered on the center of the involution and passing through the double-points we shall call the base circle. The double points we shall refer to as doubles.

**3. Generation.** The symmetrical lineo-linear relation:

$$ax_1x_2 + b(x_1 + x_2) + c = 0 \tag{1}$$

defines an involution of points on a line if we agree that the numbers $x_1$ and $x_2$ shall be the coordinates of points, $X_1$ and $X_2$ say, referred to some convenient origin: $X_1(x_1)$, $X_2(x_2)$. Given any point $X(x)$ on the base line, its mate is given by (1) and together they constitute an involutory couple. There are two points each of which is its own mate. Together they constitute the doubles of the involution. These doubles are determined by $x_1 = x_2$ and are furnished by

$$ax^2 + 2bx + c = 0. \tag{2}$$

If we represent the doubles by $D_1(d_1)$ and $D_2(d_2)$ then:

$$d_1 \,|\, d_2 = (-b + \,|\, - \sqrt{\Delta})/a \quad \text{where} \quad \Delta = b^2 - ac. \tag{3}$$

The type of double and in turn the type of involution is determined by $\Delta$. For $\Delta > 0$ the doubles are real and distinct and the involution is hyperbolic; for $\Delta < 0$ the doubles are conjugate imaginary and the involution is elliptic; for $\Delta = 0$, the doubles double up, and the involution is parabolic.

Each real point of the base line has a real harmonic conjugate with respect to the doubles. Together they constitute a real couple, and we may pair them ad libitum by assuming real values for $x_1$ and calculating the corresponding value of $x_2$ employing (1). This furnishes a single infinitude of real couples which are harmonic conjugates with respect to the double points.

We discover an enlarged situation, however, when we pose the question: Is every point on the base line the mid-point of a couple? The answer is: Yes, but the couples are not always real. Let us call the mid-point of a couple $M(\mu)$ and let $\delta$ be the digression of the individuals of the couple from $M$; that is, the individual points of the couple have for coordinates $\mu+\delta$ and $\mu-\delta$: $X_1(\mu+\delta)$, $X_2(\mu-\delta)$. Then we shall have from (1)

$$a(\mu^2 - \delta^2) + 2b\mu + c = 0, \quad \text{or}$$

$$\delta^2 = \left(\mu + \frac{b}{a}\right)^2 - \frac{\Delta}{a^2}, \quad \text{or} \tag{4}$$

$$\delta^2 = (\mu - d_1)(\mu - d_2). \tag{5}$$

From (4) we see that $\delta^2$ is positive whenever $\Delta \leqq 0$. These are the elliptic and parabolic types. Hence, neither type has conjugo-conjugate couples. When $\Delta > 0$ there is the possibility of $\delta^2$ negative for certain values of $\mu$. If we consider $d_1 < d_2$ we see from (5) that when $\mu < d_1$ and when $\mu > d_2$, $\delta^2$ is positive. That is, when $M$ is without the segment $D_1D_2$ or outside the base circle, it is the mid-point of a real couple. When, however, $d_1 > \mu > d_2$ the first factor of the second member of (5) is positive and the second is negative; hence $\delta^2$ is negative and $\delta$ is pure imaginary, and therefore the corresponding couples are conjugate imaginary. This situation occurs only in the hyperbolic type of involution [5].

The geometry may be called into case here to explicate the situation still further. The equation of the base circle is

$$a(x^2 + y^2) + 2bx + c = 0$$

and the equation of the congruence of circles orthogonal thereto is

$$a(x^2 + y^2) - 2a\mu x - 2a\nu y - 2b\mu - c = 0.$$

The parameters are $\mu$ and $\nu$, $\mu$ being the coordinate of the mid-point $M$ of a conjugate couple; for, letting $y = 0$ and solving for $x$ we have $x_1, x_2 = \mu \pm \delta$, as we should. The congruence may be written in the form

$$(x - \mu)^2 + (y - \nu)^2 = \nu^2 + \delta^2;$$

hence $(\mu, \nu)$ is the center of the circle of the congruence and $\sqrt{\nu^2 + \delta^2}$ is its radius. Now, as we have seen, there are values of $\mu$ which make $\delta^2$ negative, namely $d_1 > \mu > d_2$. For all such values of $\mu$ the radius of the congruence is less than $\nu$ the ordinate of the center of the circle.

Let us construct the lines $x = d_1$ and $x = d_2$ (Fig. 2). Then every circle having its center without the strip determined by these lines will have its radius greater than $\nu$ and will cut the base line in a real conjugate couple as (1); every circle having its center on one of these lines will have a radius equal to $\nu$ and will cut the base line in one of the doubles as (2); and, finally, every circle having its center within the strip will have a radius less than $\nu$ and will cut the base line in a conjugo-conjugate couple as (3). Thus, the mid-points of our conjugo-conjugate couples all lie within the strip determined by two lines through the double points perpendicular to the base line.

**4. Location.** Stated geometrically, then, we have the following theorem:

*Every point on the base line of an involution of points is the mid-point of a conjugate couple; if the point is without the base circle, the couple is real, and there is a single infinitude of them; if the point is on the base circle, the individuals of the couple coalesce, and we have the doubles; if the point is within the base circle, the couple is (conjugate) imaginary, and there is a single infinitude of them.*

These latter are our conjugo-conjugate couples in a quadratic involution of points on a line, and thus are they "located."

**5. Representation.** We have succeeded in "locating" our conjugo-conjugate couples in the involution of points on a line. We propose now to give them a (real) representation. The method of representing imaginary configurations by real ones employed by Laguerre lends itself both readily and adequately. We make a brief statement of the method omitting the proofs [6].

The complex projective plane is four-dimensional ($\infty^4$) in complex points, the term complex comprehending both the real and the imaginary. In this quadruple infinitude there is a sub-assemblage consisting of a double infinitude ($\infty^2$) of real points constituting the real (projective) plane. The imaginary points of the quadruple infinitude are paired conjugately. The multiplicity of points in the complex plane is co-dimensional with the multiplicity of point-pairs in the real plane; so that if we had a scheme setting up a one-to-one relationship (transformation) pairing each (finite) point of the complex plane with each (finite) point-pair of the real plane and vice versa, then we should have a (real) representation of the imaginary points in the complex plane. This is exactly what the Laguerre representation does, according to the following free translation from the French:

(a) "Let $\alpha$ be an imaginary point of a plane; through this point pass an isotropic of the first system having one real point $A$ and an isotropic of the second system having one real point $A'$. It is clear that these two points are completely determined by the point $\alpha$, and that reciprocally $\alpha$ is determined without ambiguity by the points $A$ and $A'$.

"I shall say that $AA'$ is the representative segment of the point $\alpha$, $A$ being the initial point and $A'$ the terminal point of this segment."

(b) "If $AA'$ represents an imaginary point $\alpha$, then $A'A$ represents its conjugate $\bar{\alpha}$."

(c) "The real line which joins two conjugate imaginary points $\alpha$, $\bar{\alpha}$ is the perpendicular bisector of their representative segment $AA'$."

(d) "Having given a real circle, in order that a segment $AA'$ may represent a point $\alpha$ on this circle, it is necessary and sufficient that the points $A$ and $A'$ be mutually inverse in respect to this circle." [7]

Thus, in the light of the first statement of Laguerre, if $A$ and $\bar{A}$ are two conjugate imaginary points and if $P$ and $P'$ are the initial and terminal points of the real ordered point-pair which constitutes the real representation of $A:A(P\rightarrow P')$, then reversing the order we have the real representation of $\bar{A}$, the conjugate of $A:\bar{A}(P'\rightarrow P)$. We shall employ this notation in the designation of an imaginary point and its (real) representation.

From the third statement we see that the (real) representation of the complex line with a real trace is the ensemble of ordered point-pairs which are symmetrical, the one the reflection of the other, with respect to the real trace. And again from the fourth statement we have it that the (real) representation of the complex circle having a real trace is the assemblage of ordered point-pairs which are mutually inverse with respect to the real trace. Both the complex line and the complex circle are two-dimensional ($\infty^2$) in complex points. Then every point

of the finite projective real plane is both the initial point and the terminal point in the representation of complex points on the line and the circle. Hence to find the representation of their intersection when their traces do not meet is to find that ordered point-pair which they have in common.

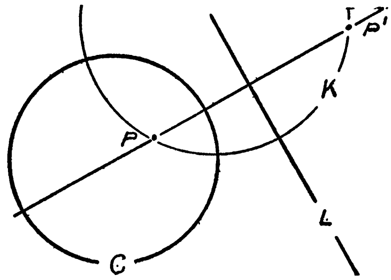



FIG. 1

Thus, in Figure 1, to find the (real) representation of the intersection of the line $L$ and the circle $(C)$ we seek that ordered point-pair which is common to both. By reason of the symmetry involved we see that the required point-pair will lie on a line through the center of the circle perpendicular to the line. The pair is discovered when we cut this line with a circle centered on $L$ and orthogonal to $(C)$ as for example $(K)$. Hence if $A$ and $\overline{A}$ are the intersections of $L$ and $(C)$, then $A(P\rightarrow P')$ and $\overline{A}(P'\rightarrow P)$.

Let us return now to our problem. The parameter $\nu$ is really redundant, the problem does not really require a congruence of circles, but only a pencil. We may choose this pencil out of our congruence in a variety of ways: circles having a common point, having a common radius, centered on a line oblique to the base line. We shall choose that pencil of circles for which $\nu$ is constant, say $\nu_0$. Any finite value chosen for $\nu_0$ is adequate but if we choose $\nu_0<\Delta$ some of the circles of the pencil will be imaginary, that is, they will have imaginary radii and hence will not have real traces. This case is taken care of by the Laguerre representation, of course, but introduces an unnecessary element into the problem which we shall side step by taking $\nu_0>\Delta$.

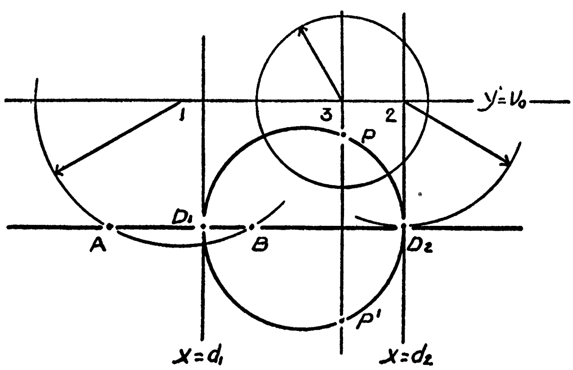



FIG. 2

Referring now to Figure 2, the line of centers of the pencil of circles which cuts the involution under consideration from the base line is the line $y=\nu_0$, $\nu_0>\Delta$. The centers of the circles then have for coordinates $(\mu, \nu_0)$. When the center lies without the strip $x=d_1$, $x=d_2$, the circle will cut out a real conjugate couple, as circle (1). When the center is on one of the lines $x=d_1$ or $x=d_2$, the circle is tangent to the base line, and hence cuts it in a double as circle (2). When the center of the circle lies within the strip, the circle cuts the base line in one of our conjugo-conjugate couples, as circle (3). Our problem is to find the real representation of this conjugo-conjugate couple. Referring to Figure 1 we see at once that it is the point-pair $P\rightleftarrows P'$. That is, if the conjugo-conjugate couple is $A$, $\bar{A}$, then $A(P\rightarrow P')$ and $\bar{A}(P'\rightarrow P)$. There is a single infinitude of such points and their (real) representation is the assemblage of ordered point-pairs on the base circle which are symmetric (the one the reflection of the other) with respect to the base line.

**6. Characterization.** We may now give a complete characterization of the hyperbolic type of involution on a line having a real trace, real quantities only being employed in its generation.

*The double points are real. The conjugate point-pairs consist on the one hand of a single infinitude of real pairs whose mid-points lie without the segment determined by the double points; and, on the other hand, of a single infinitude of imaginary pairs whose mid-points lie within the segment determined by the double points.*

If we agree to speak of the real representation of the imaginary pairs as of the pairs themselves, and that is the function of a representative; and if we further agree to speak of reflection in a line as an inversion, and that is really what it is; then we may say:

*The hyperbolic involution of points on a line with a real trace consists of a single infinitude of real point-pairs and a single infinitude of imaginary point-pairs. The real point-pairs are those points on the base line which are mutually inverse with respect to the base circle, while the imaginary point-pairs are those points on the base circle which are mutually inverse with respect to the base line.*

**7. Illustration.** The Apollonian Circles of a triangle determine an elliptic involution of circles orthogonal to the circumcircle. [8] This pencil of circles cuts from any diameter of the circumcircle a hyperbolic involution of points. In particular it cuts such an involution from the Euler line $E$. The line of centers of the Apollonian Circles is the Lemoine Line $L$. All the circles of the pencil whose centers lie within the segment determined by the tangents to the circumcircle perpendicular to the Euler Line will cut from the Euler Line the conjugo-conjugate couples with which this paper is concerned. One of the Apollonian Circles in Figure 3 is such a circle. Incidentally, the common points of the Apollonian Circles are the Isodynamic Points, $I$, $I'$ of the triangle whose properties are noted by Johnson, and Altschiller-Court. [8] In addition, we call attention to the fact that when ordered they constitute the real representation of the intersection of the Lemoine Line and the circumcircle. Again they are the

real representation of the double points of the elliptic involution determined by the Apollonian Circles on the Lemoine Line.*

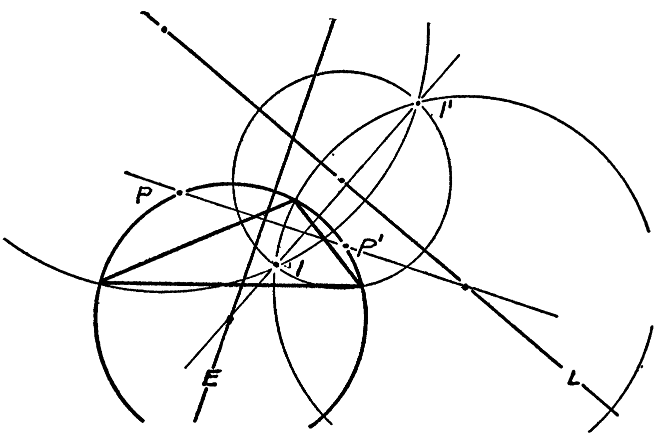



FIG. 3

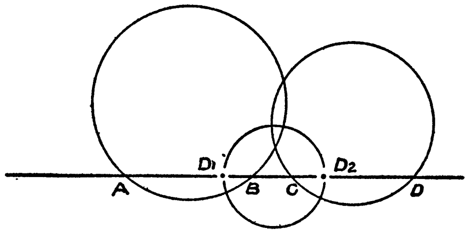



FIG. 4

**8. Construction.** We conclude with a problem of construction: Given two point-pairs on a line to construct the involution, *i.e.* find the double points, determined thereby. The point-pairs may not interlace and they may not share a mutual individual point since these determine the elliptic and parabolic types respectively. Thus we recognize three cases:

1. Both point-pairs real (and noninterlaced)
2. One pair real, one conjugate imaginary
3. Both pairs conjugate imaginary.

The construction is the same for all three cases: (a) Construct a circle on each point-pair, (b) construct the radical axis (so called) of the circles in (a); this line determines the center of involution, (c) construct the circle centered on the center of involution and orthogonal to the circles in (a). This circle cuts the base line in the required double points.

---

* We have employed this configuration merely as an illustration, but it seems to deserve consideration per se: for it is a mutual relationship between the Lemoine Line and the Euler Line and this mutuality extends to the given triangle and a certain triangle of which the Lemoine Line is the Euler Line and the Euler Line of the original triangle the Lemoine Line.

Case 1. Figure 4. Given the point-pairs $A$, $B$; $C$, $D$.

Case 2. Figure 5. Given the point-pairs $A$, $B$; $C$, $\bar{C}$; $C(P \to P')$, $\bar{C}(P' \to P)$.

Case 3. Figure 6. Given the point-pairs $A$, $\bar{A}$; $B$, $\bar{B}$; $A(Q \to Q')$, $\bar{A}(Q' \to Q)$; $B(P \to P')$, $\bar{B}(P' \to P)$.

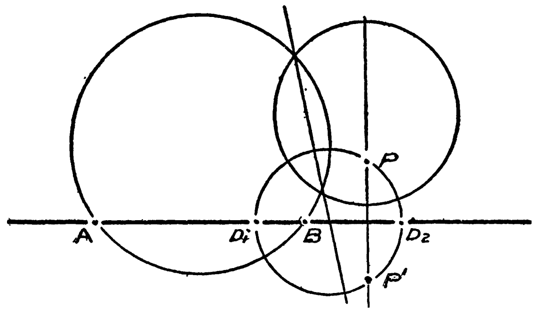



FIG. 5

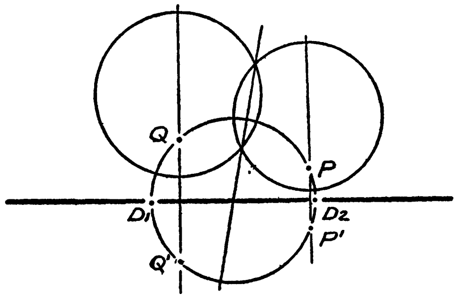



FIG. 6

If one pair is a double point, no modification of the method is necessary to secure the other double point.

The subject of involution on a complex line and its accompanying problems of construction has been generalized by the author and may merit publication in the future [9].

## REFERENCES

1. Scott, Modern Analytical Geometry, Secs. 50, 175.
2. Scott, Modern Analytical Geometry, Secs. 174, 175, p. 165.
3. Morley and Morley, Inversive Geometry, §27.
4. Winger, Projective Geometry, Sec. 89, p. 167.
5. Scott, Modern Analytical Geometry, Sec. 175, p. 167.
6. Laguerre, Oeuvres, Tome II, p. 89.
7. Laguerre, Oeuvres, Tome II (a) p. 352; (b) and (c) p. 94; (d) p. 103. Reference should also be made to Study, Vorlesungen über Ausgewählte Gegenstände der Geometrie, erstes Heft, 1911, pp. 8–10, 25–32; and to Coolidge, The Geometry of the Complex Domain, 1924, pp. 85–91.
8. Johnson, Modern Geometry, Sec. 490 et seq. Altschiller-Court, College Geometry, Sec. 466 et seq.
9. This MONTHLY, Vol. 44 1937, No. 6: pp. 344, 345; and Vol. 45, No. 9: pp. 569, 570.

# THE JEEP PROBLEM

N. J. FINE, Washington, D. C.

**1. Introduction.** The problem in logistics with which this paper deals was proposed to the author by Gail Young and Ivan Niven, both of Purdue University, in the latter part of 1945. The original source is unknown to the author. At that time Niven had obtained a partial solution based on certain assumptions. After the first submission of this paper it was learned that L. Alaoglu had also obtained a complete solution. He mentioned that similar problems had arisen in air transport operations in the China theater. It has also been suggested that there may be applications to Arctic expeditions and interplanetary travel. This paper, however, will confine itself to the lowly jeep.

Suppose that a jeep can carry a maximum load of $n$ gallons of gas and can travel $c$ miles per gallon. The jeep is required to cross a desert $x$ miles wide. Our problem is to prescribe a method for making the journey most economically and to find the least sufficient amount of gas. It is not obvious that such a method exists, and it would be more exact to speak of the greatest lower bound, until the existence of the minimum is established.

We shall assume that $n$ and $c$ are both unity. This involves no loss of generality; it is equivalent to taking as our unit of distance $nc$, the number of miles that the jeep can travel on a full load.

If $x \leqq 1$, the problem is trivial. If $x$ exceeds 1, however, gas dumps will have to be established at various points along the way. It will be convenient to take the path of the jeep along the positive $x$-axis, starting at $x$ and ending at the origin. The gas dumps will then form a subdivision $\sigma$ of the interval $(0, x)$:

$$\sigma: \quad 0 < x_1 < x_2 < \cdots < x_r < x,$$

in which the $x_i$ denote the positions of the dumps (assumed to be finite in number). If $z$ is any non-negative number less than $x$, the subdivision $\sigma$ induces a subdivision of $(0, z)$ by deletion of all the stations to the right of $z$. There will be no ambiguity if we refer to this induced subdivision by the same symbol, $\sigma$. Other subdivisions will be denoted by $\sigma'$, $\sigma^*$, and so forth. If all the stations (points of division) of $\sigma$ are contained among the stations of $\sigma'$, we shall say that $\sigma'$ is a *refinement* of $\sigma$, written $\sigma' < \sigma$.

We may now rephrase our problem. Once a subdivision is fixed, the amount of gas required is still a function of the method of establishing and employing its stations. We shall denote by $f(x, \sigma)$ the greatest lower bound of this amount for all possible methods, and by $f(x)$ the greatest lower bound of $f(x, \sigma)$ for all possible subdivisions $\sigma$. Our task is to discover the form of $f(x)$.

In §2 we introduce the *standard* method of establishing and using the stations of a given subdivision $\sigma$, and we prove that this method is at least as economical as any other. This enables us to determine $f(x, \sigma)$ in §3. A rather surprising application of the standard method leads to the result (§4) that if $\sigma' < \sigma$, then $f(x, \sigma') \leqq f(x, \sigma)$. In §5 we determine criteria for non-improvement

by refinement. These criteria lead us to the construction of an optimum $\sigma^*$ and to the explicit representation of $f(x, \sigma^*) = f(x)$ (§6). In §7 we derive a simple and accurate asymptotic formula for $f(x)$. The last section is devoted to a few remarks, including a comparison of the exact solution with the result obtained by considering the stations equally spaced (one of Niven's assumptions).

**2. The standard method.** One very natural method of employing the stations of a given $\sigma$ is to build up the stockpile of gasoline at $x_r$ by making all the trips between $x$ and $x_r$ before going to to $x_{r-1}$, and to continue in this way throughout the journey. In other words, once we go beyond any station $x_i$ we never return to the preceding one, $x_{i+1}$.

Suppose that, by some other method, $m$ complete round trips are made starting at $x$, followed by a last, one-way trip from $x$ to $x_r$. The $i$th one of the round trips consists of $A_i$, the one-way trip from $x$ to $x_r$; $B_i$, the round trip starting and ending at $x_r$; and $C_i$, the return trip from $x_r$ to $x$. Let $g_i$ be the amount of gas in the jeep at the start of $A_i$. Since $2(x-x_r)$ is the amount used in performing trips $A_i$ and $C_i$, the amount $g_i - 2(x-x_r)$ plus the residue of the preceding trips is sufficient to perform $B_i$. If we replace the sequence $A_1, B_1, C_1, A_2, B_2, C_2, \cdots, A_m, B_m, C_m, A_{m+1}$, by $A_1, C_1, A_2, C_2, \cdots, A_m, C_m, A_{m+1}$, and deposit at $x_r$ the amount $g_i - 2(x-x_r)$ after each $A_i (i=1, 2, \cdots, m)$, and $g_{m+1} - (x-x_r)$ after $A_{m+1}$, we shall then be in a position to perform all the $B_i$ in exactly the same order as before. When this has been done, the final configuration will not have been altered and no more gas will have been used. The same reasoning applies to all the trips starting at $x_r$, and so, by induction, the standard method is established as being at least as economical as any other. Henceforth we shall assume its use.

**3. Determination of $f(x, \sigma)$.** Now we suppose that there is given a subdivision

$$\sigma: \quad 0 = x_0 < x_1 < \cdots < x_r < x_{r+1} = x,$$

and that $f(x_{t-1}, \sigma)$ has already been determined. Clearly $f(x_0, \sigma) = 0$, so we have the initial step in the inductive definition of $f(x_t, \sigma)$. Let $k_t$ be the number of trips to be made from $x_t$ to $x_{t-1}$. Obviously $k_t \geqq 1$. No gas is to be left behind at $x_t$, since that would imply waste. Hence the difference between $f(x_t, \sigma)$ and $f(x_{t-1}, \sigma)$ must be accounted for by the amount used in the $2k_t - 1$ trips between the two stations. Writing $\Delta_t = x_t - x_{t-1}$, we have

$$f(x_t, \sigma) - f(x_{t-1}, \sigma) = (2k_t - 1)\Delta_t \qquad (t = 1, \cdots, r+1). \tag{1}$$

We must now determine $k_t$. The maximum amount of gas that can be transported on each of the first $k_t - 1$ trips is $1 - 2\Delta_t$; on the last, $1 - \Delta_t$, since there is no return. The total must not be less than the amount required to proceed from $x_{t-1}$. Hence

$$k_t(1 - 2\Delta_t) + \Delta_t \geqq f(x_{t-1}, \sigma). \tag{2}$$

From (1) it is clear that the number of trips must be as small as possible so that

$$(3)\qquad (k_t - 1)(1 - 2\Delta_t) + \Delta_t < f(x_{t-1}, \sigma),$$

provided that $k_t > 1$. It is easy to see that $k_t = 1$ for $x_t \leqq 1$, and that $f(x_t, \sigma) = x_t$ in this case. If $x_t > 1$, we have $k_t > 1$. In this case, (2) and (3) determine the integer $k_t$ uniquely, and $f(x_t, \sigma)$ is then obtained from (1). We remark that if the equality

$$m(1 - 2\Delta_t) + \Delta_t = f(x_{t-1}, \sigma)$$

holds for some integer $m$, then $k_t = m$.

We shall now derive a useful relationship between $k_t$ and $f(x_t, \sigma)$. If we eliminate $f(x_{t-1}, \sigma)$ between (1) and (2), we have, for all $t \geqq 1$,

$$(4)\qquad k_t \geqq f(x_t, \sigma).$$

Similarly, (1) and (3) yield

$$(5)\qquad k_t - 1 < f(x_t, \sigma),$$

provided that $k_t > 1$. But we see directly that (5) is also valid for $k_t = 1$, so (4) and (5) hold for all $t \geqq 1$. If we define $\{a\}$ as the least integer not less than $a$, then for all $t \geqq 1$ we may write

$$(6)\qquad k_t = \{f(x_t, \sigma)\}.$$

Since $f(z, \sigma)$ is an increasing function of $z$,

$$(7)\qquad k_t \leqq k_{t+1}, \qquad t \geqq 1.$$

Summing (1), we obtain

$$(8)\qquad f(x, \sigma) = \sum_{t=1}^{r+1} (2k_t - 1)\Delta_t.$$

**4. Refinements of subdivisions.** Let $\sigma'$ be a refinement of $\sigma$. The quantity $f(x, \sigma)$ may be thought of as the result obtained by applying a *non-standard* method to $\sigma'$, namely, passing over those stations of $\sigma'$ which do not belong to $\sigma$. It follows immediately from §2 that $f(x, \sigma')$ is not greater than $f(x, \sigma)$, that is,

$$(9)\qquad f(x, \sigma') \leqq f(x, \sigma) \qquad \textit{if } \sigma' < \sigma.$$

If $(x_{t-1}, x_t)$ is an interval of $\sigma$, with the associated parameter $k_t$, and if $(x_{t-1} = y_0, y_1), (y_1, y_2), \cdots, (y_{p-1}, y_p = x_t)$ are intervals of $\sigma'$, with parameters $k', k'', \cdots, k^{(p)}$, then

$$(10)\qquad k^{(p)} = \{f(x_t, \sigma')\} \leqq \{f(x_t, \sigma)\} = k_t.$$

Using (7), we obtain

$$(11)\qquad k' \leqq k'' \leqq \cdots \leqq k^{(p)} \leqq k_t.$$

From (8),

$$f(x_t, \sigma') - f(x_{t-1}, \sigma') = \sum_{i=1}^{p} (2k^{(i)} - 1)(y_i - y_{i-1}). \tag{12}$$

Equation (1) may be written in the form

$$f(x_t, \sigma) - f(x_{t-1}, \sigma) = \sum_{i=1}^{p} (2k_t - 1)(y_i - y_{i-1}). \tag{13}$$

Subtracting the members of (12) from those of (13), we have

$$f(x_t, \sigma) - f(x_t, \sigma') = f(x_{t-1}, \sigma) - f(x_{t-1}, \sigma') + 2\sum_{i=1}^{p} (k_t - k^{(i)})(y_i - y_{i-1}). \tag{14}$$

We observe that all the differences in (14) are non-negative. From this we deduce that actual improvement by refinement takes place if and only if we can find an interval $(x_{t-1}, x_t)$ and an integer $i$ such that $k_t > k^{(i)}$. By (11), this is equivalent to $k_t > k'$.

**5. Properties of $\sigma^*$.** Our problem will be solved if we can find a subdivision $\sigma^*$ for which

$$f(x, \sigma^*) \leqq f(x, \sigma) \quad \textit{for every } \sigma. \tag{A}$$

We can bring to bear the results of §4 by proving that any $\sigma^*$ which satisfies (A) also satisfies (B) that follows, and conversely.

$$f(x, \sigma^*) = f(x, \sigma') \quad \textit{for every} \quad \sigma' < \sigma^*. \tag{B}$$

Clearly, (A) and (9) imply (B). Conversely, suppose that (B) is satisfied, and let $\sigma$ be any subdivision whatsoever. We choose for $\sigma'$ the common refinement of $\sigma$ and $\sigma^*$. From (B), $f(x, \sigma^*) = f(x, \sigma')$; another application of (9) shows that $f(x, \sigma') \leqq f(x, \sigma)$. Combining these we obtain (A).

Using the criterion established at the end of §4, we find that (B) is equivalent to

(C) *For every $t = 1, 2, \cdots, r+1$, and for every $y$ such that $x_{t-1} < y \leqq x_t$,*

$$k' \equiv \{f(y, \sigma^*)\} = k_t \equiv \{f(x_t, \sigma^*)\}.$$

We shall now show that (C) is equivalent to (D):

(D) *For every $m = 1, 2, \cdots, [f(x, \sigma^*)]$, there exists an integer $s$ such that $f(x_s, \sigma^*) = m$.*

Suppose first that (C) fails for some $t$ and $y$. Since $f(z, \sigma^*)$ is strictly increasing,

$$f(x_{t-1}, \sigma^*) < f(y, \sigma^*). \tag{15}$$

By the definitions of $k'$ and the function $\{\ \ \}$,

$$f(y, \sigma^*) \leqq k'. \tag{16}$$

Since (C) fails, $k' < k_t$, that is,

$$k' \leqq k_t - 1. \tag{17}$$

Finally, since $k_t = \{f(x_t, \sigma^*)\}$ cannot exceed $f(x_t, \sigma^*)$ by as much as unity,

$$k_t - 1 < f(x_t, \sigma^*). \tag{18}$$

Combining (15), (16), (17), and (18), we see that the integer $k'$ lies strictly between $f(x_{t-1}, \sigma^*)$ and $f(x_t, \sigma^*)$. By monotonicity, $k'$ cannot be equal to $f(x_s, \sigma^*)$ for any $s$, so (D) fails.

Conversely, if $m$ is an integer satisfying

$$f(x_{t-1}, \sigma^*) < m < f(x_t, \sigma^*), \tag{19}$$

we can find a number $y$ such that

$$f(y, \sigma^*) = m, \tag{20}$$

and (C) is violated. To prove this, set

$$y = x_{t-1} + \frac{m - f(x_{t-1}, \sigma^*)}{2m - 1}, \tag{21}$$

and let $\Delta = y - x_{t-1}$. Clearly $\Delta$ is positive, and

$$m(1 - 2\Delta) + \Delta = f(x_{t-1}, \sigma^*). \tag{22}$$

Referring to the remark in §3, we see that $m$ is the parameter associated with the interval $(x_{t-1}, y)$, and

$$f(y, \sigma^*) = f(x_{t-1}, \sigma^*) + (2m - 1)(y - x_{t-1}) = m, \tag{23}$$

which proves (20). This completes the proof that (D) is equivalent to (C), (B), and (A).

**6. Construction of $\sigma^*$.** It is now almost trivial to construct a $\sigma^*$ satisfying (D) and therefore (A). Merely choose the stations $x_t^*$ so that $f(x_t^*, \sigma^*) = t$. Clearly this can be done for $t = 1$. Suppose that $x_1^*, \cdots, x_{t-1}^*$ have been found. We must determine $x_t^*$ by

$$k_t = \{f(x_t^*, \sigma^*)\} = t, \tag{24}$$

$$f(x_t^*, \sigma^*) - f(x_{t-1}^*, \sigma^*) = (2k_t - 1)(x_t^* - x_{t-1}^*). \tag{25}$$

The left member of (25) equals unity; hence

$$x_t^* - x_{t-1}^* = (2t - 1)^{-1}. \tag{26}$$

Therefore

$$x_t^* = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2t - 1} \cdot \tag{27}$$

It is easy to verify that (27) leads to the required equations $k_t=f(x_t^*, \sigma^*)=t$. The subivision $\sigma^*$ so determined evidently satisfies (D). If $r$ is the greatest integer for which $x_r^*$ does not exceed $x$, we may write

$$f(x) = f(x, \sigma^*) = r + (2r+1)(x - x_r^*), \tag{28}$$

and

$$0 \leqq x - x_r^* < (2r+1)^{-1}. \tag{29}$$

It has now been shown that $f(x)$, which represents the number of gallons needed to take the jeep $x$ miles, is a function which is piecewise linear over intervals of length 1, 1/3, 1/5, 1/7, and so on, the slope of the graph over the $n$th interval being the $n$th odd number so that the function takes consecutive integral values at the corner points.

**7. An asymptotic formula for $f(x)$.** From equation (1) it is possible to get a rough idea about the order of magnitude of $f(x)$. We have approximately

$$\Delta f = (2k - 1)\Delta x,$$
$$k = f(x).$$

Neglecting the $-1$ compared with $k$, we find

$$\frac{\Delta f}{f} = 2\Delta x;$$
$$\log f = 2x + C_1,$$
$$f = C_2 e^{2x}.$$

We shall not attempt to make these heuristic methods precise, but shall proceed directly to a derivation based on the exact solution obtained in the preceding section. Let us define

$$S(r) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}\cdot \tag{30}$$

It is well known that for large $r$,

$$S(r) = \log r + C + \frac{1}{2r} + O\left(\frac{1}{r^2}\right), \tag{31}$$

where $C$ is Euler's constant, .577 $\cdots$, and the $O(1/r^2)$ denotes an error term whose absolute value does not exceed a certain constant, independent of $r$, multiplied by $1/r^2$. This constant may vary from one equation to the next. From (27), (30), and (31), we have

$$x_r^* = S(2r) - \frac{1}{2}S(r) = \left(\log(2r) + C + \frac{1}{4r} + O\left(\frac{1}{r^2}\right)\right)$$
$$- \left(\frac{1}{2}\log r + \frac{1}{2}C + \frac{1}{4r} + O\left(\frac{1}{r^2}\right)\right)\cdot \tag{32}$$

Therefore,

$$x_r^* = \log (2\sqrt{r}) + \frac{1}{2} C + O\left(\frac{1}{r^2}\right).$$

Taking exponentials,

$$\exp\left(x_r^* - \frac{C}{2}\right) = 2\sqrt{r} \exp\left(O\left(\frac{1}{r^2}\right)\right) = 2\sqrt{r}\left(1 + O\left(\frac{1}{r^2}\right)\right). \tag{33}$$

Squaring, we obtain

$$\frac{1}{4} \exp (2x_r^* - C) = r + O\left(\frac{1}{r}\right). \tag{34}$$

Thus, writing $g(u) = \frac{1}{4} \exp (2u - C)$,

$$f(x_r^*) = r = g(x_r^*) + O\left(\frac{1}{r}\right) = g(x_r^*) + O(e^{-2x_r^*}). \tag{35}$$

Now for any $u$ satisfying

$$x_r^* < u \leqq x < x_{r+1}^*, \tag{36}$$

we have, by equation (28),

$$f'(u) = 2r + 1. \tag{37}$$

Also,

$$g'(u) = 2g(u) > 2g(x_r^*) = 2r + O\left(\frac{1}{r}\right), \tag{38}$$

$$g'(u) < 2g(x_{r+1}^*) = 2r + 2 + O\left(\frac{1}{r}\right), \tag{39}$$

$$f'(u) - g'(u) = O(1). \tag{40}$$

Integrating (40) between $x_r^*$ and $x$, we find

$$\begin{aligned} f(x) - g(x) &= (f(x_r^*) - g(x_r^*)) + \int_{x_r^*}^{x} (f'(u) - g'(u))du \\ &= O\left(\frac{1}{r}\right) + O(x - x_r^*) = O\left(\frac{1}{r}\right) \\ &= O(e^{-2x}). \end{aligned} \tag{41}$$

Therefore, for all $x$,

$$f(x) = \tfrac{1}{4} \exp (2x - C) + O(e^{-2x}). \tag{42}$$

It will be observed that the error term is not only of lower order than the principal term, but it actually tends to zero exponentially. Also, the approximate

equation derived by such high-handed methods at the beginning of this section turns out to be much more accurate than might have been expected.

**8. Remarks.** Suppose we decide to use equal subdivisions of the interval $(0, x)$, $N$ in number. We can prove that if $N$ is chosen to vary with $x$ so that $x^2/N$ tends to zero, then the increase in the amount of gas required (over the exact solution) is less than 50% for large $x$. To counterbalance this increase, however, the number of stations required will be very much smaller. For example, if $N$ varies as the cube of $x$, it will also vary as the cube of log $r$. Hence, if we take into account the cost (in time, energy, material, and so on) of setting up the stations, it might very well happen that the equal subdivisions would be more economical. Of course, we should then have an entirely new problem.

For a fixed $x$ and sufficiently large $N$, we can come as close as we please to the minimum $f(x)$ by means of $N$ equal subdivisions. An amusing fact here is that the minimum can be *attained* if and only if $x$ is rational.

We close with several remarks about the character of the solution $\sigma^*$ we have obtained here. It is obvious that this solution is not unique, since every refinement of $\sigma^*$ is also a solution. It can be shown that the converse is also true; that is, every solution is a refinement of $\sigma^*$. Furthermore, if we consider the class of subdivisions $\sigma$ for which the function $f(x, \sigma)$ is *continuous*, we can prove that this class is identical with the class of all solutions. It would be of interest to see whether this criterion can be obtained directly, and whether the minimum $f(x)$ can be derived from it.

# MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Send material for this department directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

## A LINEAR DIOPHANTINE EQUATION

H. S. GRANT, Rutgers University

In solving the linear Diophantine equation

$$\sum_{i=1}^{n} a_i x_i = K, \qquad (a_1, a_2, \cdots, a_n) = 1,$$

where $(a_1, a_2 \cdots, a_n)$ denotes the greatest common divisor of $a_1, a_2, \cdots, a_n$, attention should be paid to the number of variables that may be considered arbitrary. Thus, if

$$(1) \qquad (a_{i_1}, a_{i_2}, \cdots, a_{i_s}) = 1,$$

it is evident that $n-s$ of the variables may be assigned arbitrary values. The

maximum number of variables to which may be assigned arbitrary values is therefore $n-s$, where $s(\geqq 1)$ is the smallest number for which (1) holds, $i_1$, $i_2, \cdots, i_s$ being any of the $C(n, s)$ choices of $s$ subscripts from $1, 2, \cdots, n$.

Suppose now that we have

$$a_1x_1 + a_2x_2 + \cdots + a_sx_s = K_1, \tag{2}$$

where $s>2$, and that

$$(a_1, a_2, \cdots, a_s) = 1,$$

while the latter condition does not hold for any proper sub-set of the $a_i$'s. Subtracting $a_ix_i$ from both members of (2), $1 \leqq i \leqq s$, we have

$$K_1 - a_ix_i = dt,$$

where

$$d = (a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_s) > 1,$$

with obvious modification for $i=1$ or $s$; further, $(a_i, d)=1$. Solving for $x_i$ and $t$, we have

$$\frac{a_1}{d}x_1 + \frac{a_2}{d}x_2 + \cdots + \frac{a_{i-1}}{d}x_{i-1} + \frac{a_{i+1}}{d}x_{i+1} + \cdots + \frac{a_s}{d}x_s = t,$$

$$\left(\frac{a_1}{d}, \frac{a_2}{d}, \cdots, \frac{a_{i-1}}{d}, \frac{a_{i+1}}{d}, \cdots, \frac{a_s}{d}\right) = 1;$$

and the number of variables has been reduced by one. Continuing this process, we thus arrive at the general solution through successive solutions of the case involving two variables only. This may afford a rapid method of solution, especially if, at some step of the process, a coefficient 1 should appear or two coefficients should be relatively prime.

As an example, we consider the equation

$$99x + 79y + 55z + 33w = K,$$

discussed by D. H. Lehmer in this MONTHLY, vol. 48, 1941, p. 245. Since $(99, 79) = 1$, we may consider $z$ and $w$ arbitrary and write

$$99x + 79y = K - 55z - 33w. \tag{3}$$

Solving (3) by any of the well known methods, for example the Euclidean algorithmic process, we obtain

$$\begin{aligned} x &= \phantom{-}4K - 220z - 132w + 79t, \\ y &= -5K + 275z + 165w - 99t, \end{aligned}$$

with $z$, $w$, and $t$ arbitrary. The reader may readily verify that this solution is equivalent to Lehmer's solution (21), p. 245, *loc. cit.*, namely

$$\begin{aligned} x &= \phantom{-}6K + 79L - 198M - 330N, \\ y &= -5K - 66L + 165M + 275N \\ z &= N, \\ w &= -6K - 79L + 199M + 330N, \end{aligned}$$

where $L$, $M$, and $N$ are arbitrary; the equivalence is established by applying the linear transformation

$$\begin{aligned} L &= 10K - 550z - 330w + 199t, \\ M &= 4K - 220z - 131w + 79t, \\ N &= z. \end{aligned}$$

## AN ARTILLERY PROBLEM

R. J. WALKER, Cornell University

A gun, shooting a heavy, low-velocity projectile, is trying to hit a target which need not lie on the same level as the gun. At what elevation should the gun fire, assuming a vacuum trajectory?

In an attempt to give a simple graphical method of determining the possible angles of elevation, the following construction was found.

Let the gun be at the origin of a rectangular coördinate system with horizontal $x$-axis and vertical $y$-axis, and let the target be at $(x_0, y_0)$ in the $xy$-plane. Let

$$\xi_0 = \frac{gx_0}{V^2}, \qquad \eta_0 = \frac{gy_0}{V^2},$$

where $V$ is the muzzle velocity of the projectile and $g$ is the acceleration due to gravity. On the figure we carry out the following construction:

1. Through $T$: $(\xi_0, \eta_0)$ draw a line parallel to the $y$-axis, meeting the hyperbola $xy=1$ in $A$.
2. Through $A$ draw a line $AC$ parallel to the $x$-axis.
3. Through the origin $O$ draw $OC$ perpendicular to $OT$.
4. With $C$, the intersection of $AC$ and $OC$, as center, draw a circle through $O$, cutting the line $x=1$ in $R_1$ and $R_2$.
5. Draw $OR_1$ and $OR_2$, making angles of $\alpha_1$ and $\alpha_2$ with the positive $x$-axis.

Now $\alpha_1$ and $\alpha_2$ are the required angles of elevation.

The proof of this construction is simple. In terms of the normalized distances $\xi = gx/V^2$ and $\eta = gy/V^2$ and the normalized time $\tau = gt/V$, the equations of motion of the projectile are

$$\xi = \tau \cos \alpha, \qquad \eta = \tau \sin \alpha - \tfrac{1}{2}\tau^2.$$

Eliminating $\tau$ gives

$$\eta = m\xi - \tfrac{1}{2}(1 + m^2)\xi^2,$$

where $m=\tan\alpha$. We wish $m$ to satisfy

$$\eta_0 = m\xi_0 - \tfrac{1}{2}(1+m^2)\xi_0^2. \tag{1}$$

Now the circle

$$\eta_0 x = y\xi_0 - \tfrac{1}{2}(x^2+y^2)\xi_0^2 \tag{2}$$

intersects the line $x=1$ in points whose ordinates are roots of (1), and so the lines joining these points to $O$ have for slopes the required values of $m$. It is readily verified that $C$ is the center of (2).

Additional information can be obtained from the figure that follows:

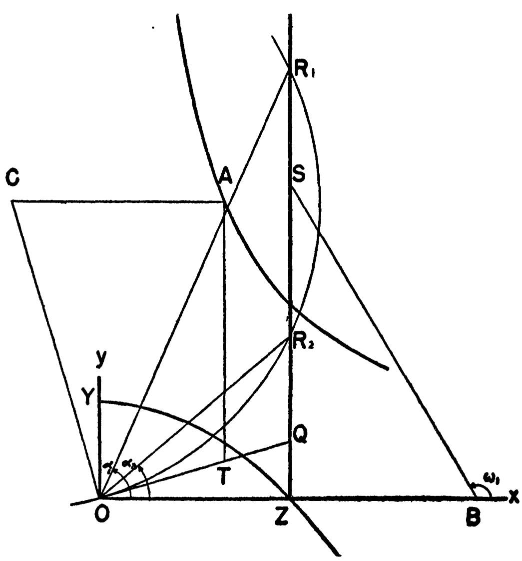
1. The normalized time of flight is $\tau_0=\xi_0\sec\alpha$. This is just the distance from $O$ to the point in which $OR$ intersects $TA$.
2. The parabola $YZ$, with focus at $O$, determines the maximum range of the projectile. $T$ must be inside this parabola.
3. The angle of fall $\omega$ of the projectile at $T$ is given by

$$\tan\omega = \left(\frac{d\eta}{d\xi}\right)_T = m - (1+m^2)\xi_0.$$

Substituting for $1+m^2$ from (1) gives

$$\tan\omega = \frac{2\eta_0}{\xi_0} - m.$$

Now $\omega$ is readily constructed by extending $OT$ to meet the line $x=1$ in $Q$ and marking off $RS=2QZ$ downward from $R$. Then $\omega$ is the angle between the positive $x$-axis and $BS$, where $B$ is the point (2, 0).



# CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.*

## A RIGOROUS TREATMENT OF THE FIRST MAXIMUM PROBLEM IN THE CALCULUS

J. L. WALSH, Harvard University

Even some of the best calculus texts fail to set forth adequately the logic underlying the usual maximum and minimum problems, studied subsequently to curve-plotting. The present note outlines the treatment the writer has used for a number of years, in a course primarily for freshman and sophomores.

PROBLEM. *A piece of sheet tin three feet square is to be made into a rectangular box open at the top, by cutting out equal squares from the corners and bending up the sides of the resulting piece parallel with the edges. Among all such boxes, to find the box of greatest volume.*

We consider an arbitrary box that can be made in this way. Let $x$ denote the length in feet of the side of the square corners cut out, so that $x$ is also the height of the box. The volume $V$ of the box is then (in cubic feet)

$$V = V(x) = x(3 - 2x)^2. \tag{1}$$

We study $V$ as a function of $x$ in the *closed* interval $0 \leqq x \leqq 3/2$. When $x$ is very small and positive, so also is the volume $V$, as appears both by (1) and by considering the shallow box formed; we have $V(0) = 0$. When $x$ is smaller than but near $3/2$, the volume $V$ is also very small and positive as is seen both by (1) and by considering the small area of the base of the corresponding box, we have $V(3/2) = 0$. Between these two values $x = 0$ and $x = 3/2$, the continuous function $V$ is always positive and hence possesses a maximum; the maximum $V$ is characterized by the vanishing of $D_x V$ for the corresponding value of $x$. We find

$$D_x V = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 = (3 - 2x)(3 - 6x).$$

Between the two values 0 and $3/2$ the only value of $x$ for which $D_x V$ vanishes is $x = 1/2$, which therefore yields the box of maximum volume.

The logic underlying this formal work can be analyzed as follows:

1. The function $V(x)$ is continuous in the closed interval $0 \leqq x \leqq 3/2$, hence possesses a maximum* there. [Any function continuous in a closed finite interval possesses a maximum there.]

2. The function $V$ vanishes for $x = 0$ and $x = 3/2$ and is positive between

* To be more explicit, $V(x)$ possesses a (weak) absolute maximum, which is necessarily also a (weak) relative maximum.

those values, so the maximum occurs at some point or points in the ***interior*** $0<x<3/2$ of the original interval.

3. At this maximum of $V(x)$ we must have $D_x V=0$. [If $D_x V>0$, the function $V$ increases as $x$ increases; if $D_x V<0$, the function $V$ decreases as $x$ increases and increases as $x$ decreases; in either case $V$ has no maximum.]

4. Among the values $0<x<3/2$ we have $D_x V=0$ if and only if $x=1/2$.

5. Hence $x=1/2$ yields the maximum value of $V$.

It will be noticed that $x=1/2$ cannot yield a minimum of $V$, nor a point of inflection. Moreover, the function $V(x)$ possesses a *unique* maximum in the interval $0\leqq x\leqq 3/2$; there is no choice here among several relative maxima.

The logic as given is complete (although the theorem quoted in 1 can hardly be proved at this stage), and will satisfy the most critical student even at any stage of his later career. This logical analysis, with suitable modifications, applies to all maximum-minimum problems in one variable that the student is likely to meet.

## TEACHING TRIGONOMETRY

E. P. VANCE, Oberlin College

A frequent source of confusion in mathematics courses in college is the meaning of $\sin x$, $\cos x$, and so forth, when $x$ is a general real variable. In dealing with periodic behavior or phenomena resulting from combinations of periodic behavior, such as in the study of electricity, one frequently encounters expressions such as $\sin \omega t$. Because of the manner in which the circular functions were originally defined, often the student will ask: "Where is the angle?" Beginners are in the habit of thinking of these functions solely as functions of angles and many are incapable of understanding any generalization. The author feels that steps should be taken to clarify this difficulty in the students' minds.

On an experimental basis we are attacking this problem in some classes at Oberlin by defining the functions at the outset as functions of a real variable $\theta$. The method consists of drawing a unit circle with its center at the origin of a rectangular axis system. Then corresponding to the real variable $\theta$, an arc of length $\theta$ is laid off from $(1, 0)$ along the circle; if $\theta$ is positive the arc is laid off counterclockwise, but if $\theta$ is negative the direction of the arc is clockwise. Letting $P$ be the endpoint of this arc, the $x$-coördinate of $P$ is defined to be $\cos\theta$, and the $y$-coördinate of $P$ is defined to be $\sin\theta$. The other trigonometric functions may then be defined in terms of the ratios and reciprocals of the sine and cosine functions. It will be immediately apparent that all the properties of the circular functions easily follow: $\cos 0=1$, $\sin 0=0$, $\sin(-\theta)=-\sin\theta$, $\sin(\pi/2-\theta)=\cos\theta$, $\sin^2\theta+\cos^2\theta=1$, and so forth. Problems in analytic trigonometry, such as identities and equations, are always phrased in terms of the real variable $\theta$. The graphs and the use of graphs play a dominate role, thus emphasizing the important periodic properties of these functions. In constructing the graph of $y=\sin x$, care is taken to use the same type of units for both $x$ and $y$, although not necessarily the same scale. It is then pointed out that the radian measure of

an angle is equal to the value of $\theta$ of the intercepted arc, and thus the functions of an angle are derived as special cases. The angle approach is used only for the solutions of triangles.

An alternative approach is to use standard textbooks up to the point where graphs are introduced. Then the generalization may be made as described above. Care, however, should be taken to bridge the gap between the notions of sin (2 radians) and sin 2; for in the former, angle is still predominant while in the latter the full transfer to real variables has been carried out.

The sophisticate may object that in this approach, trigonometry is based upon the notion of arc length, generally agreed to be a more advanced mathematical topic. In fact, G. H. Hardy refuses to discuss arc length even in a book as advanced as his *Pure Mathematics* because of its complexity. The reply is that to freshmen arc length is a perfectly clear concept, and its use is very helpful in the understanding of this matter. A rigorous approach in terms of integrals leading to inverse functions may well find its place in the calculus.

## UNDETERMINED COEFFICIENTS IN INTEGRATION

J. B. Reynolds, Lehigh University

Teachers of calculus usually find difficulty in getting students to be able to integrate readily expressions in which the integrand is a product of an exponential term and a binomial factor each term of which contains a sine or a cosine of an angle. The integration is usually accomplished by integrating by parts twice. Sometimes a table of integrals is resorted to and the student never really understands how the integral was found. The integration of the forms cited can be performed readily by the use of undetermined coefficients. Suppose we want to integrate the following product

$$e^{3x}(2 \sin 5x - 4 \cos 5x)dx.$$

A little thought shows that the only terms the integral can contain are given in

$$\int e^{3x}(2 \sin 5x - 4 \cos 5x)dx = e^{3x}(A \sin 5x + B \cos 5x) + C$$

in which $A$ and $B$ are coefficients to be determined. Upon taking the derivative of both sides with respect to $x$ we have

$$e^{3x}(2 \sin 5x - 4 \cos 5x) = e^{3x}([3A - 5B] \sin 5x + [3B + 5A] \cos 5x).$$

Upon cancelling $e^{3x}$ from both sides and equating coefficients of sin $5x$ and cos $5x$ we have

$$3A - 5B = 2 \quad \text{and} \quad 3B + 5A = -4$$

whence $A=-7/17$ and $B=-11/17$ and, therefore,

$$\int e^{3x}(2 \sin 5x - 4 \cos 5x)dx = - e^{3x}(7 \sin 5x + 11 \cos 5x)/17 + C.$$

It is equally easy to apply this method to the integral of the product of a sine and a cosine of two angles. Thus, if we asume

$$\int \sin 3x \cos x \, dx = A \sin 3x \sin x + B \cos 3x \cos x + C$$

we find after differentiating and equating coefficients of like terms $A-3B=1$ and $3A-B=0$, giving $A=-1/8$ and $B=-3/8$.

Of course the method can be used to derive general formulas for the integration of these and other forms.

---

# ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

## PROBLEMS FOR SOLUTION

E 751. *Proposed by Alan Wayne, Flushing, N. Y.*

Find the digits represented by the letters in the following addition, if no two different letters represent the same digit:

$$\begin{array}{r} FORTY \\ TEN \\ TEN \\ \hline SIXTY \end{array}$$

E 752. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that in a right triangle the twelve points of contact of the inscribed and escribed circles form two groups of six points situated on two circles which cut each other orthogonally at the points of intersection of the circumcircle with the line joining the midpoints of the legs of the triangle.

E 753. *Proposed by L. M. Kelly, University of Missouri*

How can one convince a class in elementary analytics that if the inside of a

race track is a non-circular ellipse, and the track is of constant width, then the outside is not an ellipse?

E 754. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A finite sequence of positive integers will be said to be *skew ordered* if either each integer in an even position of the sequence is greater than or each such integer is less than its immediate neighbors. If the eight integers $1, \cdots, 8$ are placed in random order in a sequence, what is the probability that the sequence will be skew ordered?

E 755. *Proposed by Alfred Brauer, University of North Carolina*

Let $a_1, a_2, a_3, a_4$ be relatively prime integers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 = 0. \tag{1}$$

Let $\alpha_\nu(\nu=1, 2, 3, 4)$ be the smallest non-negative residue of $a_\nu$ (mod 6). Then

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \pmod 6. \tag{2}$$

In his paper, *Residual Types of Partitions of* 0 *into Four Cubes* (The Mathematics Student, vol. 13, 1945, pp. 47–48), A. K. Srinivasan tries to find solutions of (1) for each set of numbers $\alpha_\nu$ satisfying (2). For instance, for $\alpha_1=\alpha_2=0$, $\alpha_3=1$, $\alpha_4=5$, he gives the solution $a_1=12$, $a_2=-54$, $a_3=19$, $a_4=53$. In the following cases he did not succeed in finding examples:

$$\begin{cases} \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 4; \\ \alpha_1 = \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 3; \\ \alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 3, \quad \alpha_4 = 5; \end{cases} \tag{3}$$

and in the cases obtained from (3) if each $\alpha_\nu$ is replaced by $6-\alpha_\nu$. He conjectures that these cases are impossible. Prove that this conjecture is true.

## SOLUTIONS

### A Four Digit Square

E 722 [1946, 270]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a square of four digits such that if one interchanges the two middle digits the new number is a square for radix 9.

*Solution by Walter Penney, Washington, D. C.* Let the four digits, in order, be $a$, $b$, $c$, $d$. The only final digits for $(\text{squares})_{10}$ are 0, 1, 4, 5, 6, and 9, and for $(\text{squares})_9$ are 0, 1, 4, and 7. Therefore $d$ must be 0, 1, or 4.

If $d=0$, $c$ must be 0, since all $(\text{squares})_{10}$ ending in 0 end in 00. All $(\text{squares})_9$ ending in 0 end in 10, 40, or 70. Since no $(\text{square})_{10}$ can end in 700, $b$ must be 1 or 4. The only possibility is $(8100)_{10}$. But $(8010)_9$ is not a square. Therefore $d\neq 0$.

Similarly, considering the penultimate digit when $d=1$, we find that no $(\text{square})_{10}$ is also a $(\text{square})_9$ when $b$ and $c$ are interchanged. Therefore $d\neq 1$.

If $d=4$, we find

$$(3364)_{10} = 58^2 \quad \text{and} \quad (3634)_9 = 57^2,$$
$$(4624)_{10} = 68^2 \quad \text{and} \quad (4264)_9 = 62^2.$$

These are therefore the only two solutions.

Also solved by W. E. Byrne, Monte Dernham, and E. P. Starke. Margaret Olmsted found the solution 3364; Paul Bateman, C. Perry, and the proposer found the solution 4624. Starke pointed out that tables of squares in other systems than the decimal are to be found in the proposer's pamphlet, *Les Recréations Mathêmatiques* (parmi les nombres curieux), Supplément a Mathesis, 1943. With these in hand the answers can be read off directly.

**Trigonometric Inequalities**

E 723 [1946, 271]. *Proposed by A. W. Goodman, Columbia University*

Let $A$ and $B$ be any real quantities, and let $k$ be a positive integer. Show that

$$\left|\frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A}\right| \leqq k^2 - 1, \tag{1}$$

$$\left|\frac{\cos kB - \cos kA}{\cos B - \cos A}\right| \leqq k^2, \tag{2}$$

equality holding only in the limit as $B$ and $A$ approach zero.

I. *Solution by A. S. Peters, New York University.* The fundamental inequality upon which the proposed inequalities depend is

$$|\sin rx| \leqq r|\sin x|, \tag{3}$$

where $r$ is a positive integer. This inequality is easily proved, for

$$\begin{aligned} |\sin rx| &= |\sin[(r-1)x + x]| \\ &= |\sin(r-1)x \cos x + \cos(r-1)x \sin x| \\ &\leqq |\sin(r-1)x| + |\sin x|, \end{aligned}$$

from which (3) follows by iteration. Now, from (3),

$$\left|\frac{\sin rx \sin sy}{\sin x \sin y}\right| \leqq rs,$$

or, more symmetrically

$$\frac{1}{2}\left|\frac{\sin rx \sin sy + \sin sx \sin ry}{\sin x \sin y}\right| \leqq rs, \tag{4}$$

where $r$ and $s$ are positive integers and the equality sign holds only if $r=s=1$; or $x \to 0$ and $y \to 0$. With the substitutions $r=m+n$, $s=m-n$, $x=(A+B)/2$,

$y=(A-B)/2$, and the use of some trigonometric identities, (4) becomes

$$\left|\frac{\cos mB \cos nA - \cos mA \cos nB}{\cos B - \cos A}\right| \leqq m^2 - n^2, \tag{5}$$

where $m>n$ and $m=(r+s)/2$, $n=(r-s)/2$. The inequalities (1) and (2) are particular cases of (5). If $m=k$, $n=1$, then (5) reduces to (1). If $m=k$, $n=0$, then (5) reduces to (2). Other inequalities of the same nature as (5) can be deduced by analyses similar to the above.

II. *Solution by R. C. Buck, Harvard University.* Let $T_n(t)$ be the Tchebycheff polynomial of degree $n$; this has the value $\cos n\theta$ if $t=\cos\theta$. Let $P$ and $Q$ be the points on the curve $y=T_k(x)$ with $x$-coordinates $\cos B$ and $\cos A$, and let $L$ be the line through them. Then, the left member of (1) is the absolute value of the $y$-intercept of $L$, and the left member of (2) is the absolute value of the slope of $L$. Since $L$ is a secant of the curve, its slope cannot exceed the greatest value of $|T_k'(t)|$, which is known to be $k^2$, thus proving (2). Since the curve lies entirely within the 2-unit square, the greatest value of the $y$-intercept of $L$ is obtained by finding the greatest value of the $y$-intercept of a tangent line for $|t|\leqq 1$. But, for tangents, the intercept is given by $|T_k(t)-tT_k'(t)|$, whose greatest value is $k^2-1$, thus proving (1).

The facts stated above for the Tchebycheff polynomials are most easily seen from the relations

$$\begin{aligned} T_n'(t) &= n \sin n\theta/\sin\theta, \\ tT_n'(t) - T_n(t) &= n \sin (n-1)\theta/\sin\theta + (n-1)\cos n\theta. \end{aligned}$$

Thus

$$|T_n'(t)| \leqq n\,|\sin n\theta/\sin\theta| \leqq n^2,$$

while

$$|T_n(t) - tT_n'(t)| \leqq n(n-1) + n - 1 = n^2 - 1.$$

Equality in (1) and (2) holds as $A$ and $B$ approach 0 or $\pi$.

Also solved by Paul Brock and Alan Wayne.

### Admissible Numbers

E 724 [1946, 271]. *Proposed by N. J. Fine, University of Pennsylvania, and Ivan Niven, Purdue University*

Define an $n$-admissible number $k$ as one such that an $n$-dimensional cube may be subdivided into $k$ cubes. Prove that for each $n$ there exists an integer $A_n$ such that all integers exceeding $A_n$ are $n$-admissible.

*Solution by Fritz Herzog, Michigan State College.* Let $k$ be an $n$-admissible number, and let an $n$-dimensional cube be subdivided into $k$ cubes. If one of these $k$ cubes is further subdivided into $a^n$ equal cubes, where $a$ is any integer

greater than unity, then we obtain a subdivision of the original cube into $k+(a^n-1)$ cubes. Thus, if $k$ is $n$-admissible, so is $k+(a^n-1)$. Since unity is $n$-admissible, we conclude that all numbers of the form

$$1 + p(a^n - 1) + q(b^n - 1), \qquad a \geqq 2, b \geqq 2, p \geqq 0, q \geqq 0, \tag{1}$$

are $n$-admissible. Choose $a$ and $b$ so that $(a^n-1,\ b^n-1)=1$, which is always possible, for instance by taking $a=2$, $b=2^n-1$. Then, by a well known fact in number theory (see, *e.g.*, Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 4, Problem 26), every integer $k>(a^n-1)(b^n-1)$ can be written in the form (1) in at least one way, and is, therefore, $n$-admissible.

Also solved by Paul Bateman, J. B. Kelly, Leo Moser, William Scott, and the proposers.

*Editorial Note.* The problem of finding the largest number not $n$-admissible appears not to be easy. For $n=2$ the answer is 5. Scott showed this by proving that the only numbers not 2-admissible are 2, 3, and 5. Following Herzog's suggestion of taking $a=2$, $b=2^n-1$, we find that all numbers greater than 2394 are 3-admissible. This was greatly improved upon by Scott, who showed that 1, 20, 38, 39, 49, 51, 61 are all 3-admissible. These numbers, along with those obtained by adding multiples of 7 to these, then constitute a set of 3-admissible numbers, and it is easily shown that every number greater than 54 is in this set. The question as to whether 54 is 3-admissible or not is still open.

Suppose we define a number $k$ to be *strictly* $n$-admissible if an $n$-dimensional cube may be subdivided into *k different* cubes. For some time the conjecture was that there are no numbers strictly 2-admissible. This conjecture was proved false in the paper *The Dissection of Rectangles into Squares*, by C. A. B. Smith, A. H. Stone, W. T. Tutte, Duke Mathematics Journal, Dec. 1940. This same paper proved that there are no numbers which are strictly 3-admissible.

Bateman remarked that the proposed problem appeared in Ripley's "Believe It or Not" column.

### Correlation Coefficient Between Two Events

E 725 [1946, 271]. *Proposed by Henry Scheffé, University of California at Los Angeles*

If two events have probabilities $p_1$ and $p_2 (0<p_i<1)$, and correlation coefficient $\rho$, show that the range of possible $\rho$ is the interval $-f(a_1a_2)\leqq\rho\leqq f(a_1/a_2)$, where $a_i=[p_i/(1-p_i)]^{1/2}$ and $f(x)=\min\ (x,\ 1/x)$. The correlation coefficient between two events $E_1$ and $E_2$ may be defined as that between $x_1$ and $x_2$, where $x_i=1$ if $E_i$ happens and $x_i=0$ if not-$E_i$ happens.

*Solution by Ellen Buck, Wellesley College.* By definition

$$\rho = \frac{M(x_1x_2) - M(x_1)M(x_2)}{[M(x_1^2) - (M(x_1))^2]^{1/2}[M(x_2^2) - (M(x_2))^2]^{1/2}}$$

where $M(x_i)$ denotes the mean value of $x_i$. Now, if $p$ is the probability that both $E_1$ and $E_2$ happen, we have

$$M(x_i) = p_i, \qquad M(x_i^2) = p_i, \qquad M(x_1x_2) = p,$$
$$\max(p_1 + p_2 - 1, 0) \leqq p \leqq \min(p_1, p_2).$$

Therefore

$$\frac{\max(p_1 + p_2 - 1, 0) - p_1p_2}{p_1(1 - p_1)p_2(1 - p_2)} \leqq \rho \leqq \frac{\min(p_1, p_2) - p_1p_2}{p_1(1 - p_1)p_2(1 - p_2)}.$$

The numerators may be simplified to

$$\max[-(1 - p_1)(1 - p_2), -p_1p_2], \qquad \min[p_1(1 - p_2), p_2(1 - p_1)]$$

respectively. Then, in terms of the $a_i$, our inequality becomes

$$-\min(a_1a_2, 1/a_1a_2) \leqq \rho \leqq \min(a_1/a_2, a_2/a_1),$$

which is the desired relation.

Also solved by Paul Bateman.

### Professor Umbugio's Prediction

E 716 [1946, 219], *Proposed by H. E. G. P.*

On April 1, 1946, the Erehwon Daily Howler carried the following item: "The famous astrologer and numerologist of Guayazuela, the Professor Euclide Paracelso Bombast Umbugio, predicts the end of the world for the year 2141. His prediction is based on profound mathematical and historical investigations. Professor Umbugio computed the value of the formula

$$1492^n - 1770^n - 1863^n + 2141^n$$

for $n=0, 1, 2, 3$, and so on, up to 1945, and found that all the numbers which he so obtained in many months of laborious computation are divisible by 1946. Now, the numbers 1492, 1770, and 1863 represent memorable dates: the Discovery of the New World, the Boston Massacre, and the Gettysburg Address. What important date may 2141 be? That of the end of the world, obviously."

Deflate the Professor! That is, show with little computation that the formula proposed is divisible by 1946 for $n=0, 1, 2, 3, \cdots$.

*Solution by E. P. Starke, Rutgers University.* That $x-y$ is a divisor of $x^n-y^n$ for $n=0, 1, 2, \cdots$, is the only principle required. Let the Professor's number be $F(n)$. Then, since $2141-1863=1770-1492=278$, we certainly have $F(n)$ is always divisible by 278. Similarly, since $2141-1770=1863-1492=371$, $F(n)$ is always divisible by 371. But 278 and 371 are relatively prime, whence $F(n)$ is always divisible by $(278)(371)=(53)(1946)$, and hence, of course, by 1946 itself.

Also solved by D. W. Alling, Murray Barbour, Joshua Barlaz, Paul Bateman, Barney Bissinger, W. G. Brady, Paul Brock, D. H. Browne, R. C. Buck, Roy Dubisch, George Grossman, E. S. Keeping, W. J. LeVeque, C. L. Peny, Jr., P. A. Pizá, J. G. Wendel, Maud Willey, R. H. Wilson, Jr., and the proposer.

Buck pointed out that more generally

$$A^n + (A + r + s)^n - (A + r)^n - (A + s)^n$$

is always divisible by the least common multiple of $r$ and $s$.

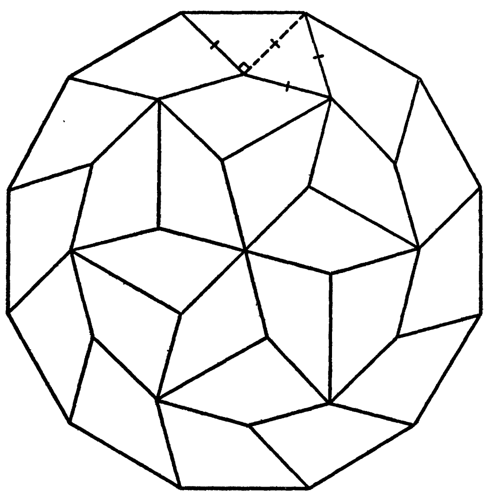
Pizá remarked that the extra factor 53 may have significance, because that is the number of cards in a pack when the joker is included, and surely the proposer of the problem must have been the joker.

### A Dodecagon Dissection Puzzle

E 721 [1946, 270]. *Proposed by Joseph Rosenbaum, The Milford School, Conn.*

Given two equal regular dodecagons. Show how to dissect one of them into twelve congruent pieces which can be fitted to the other to form another larger regular dodecagon.

*Solution by J. M. Kingston, University of Washington.* The dissection is apparent from the following figure. Note that six of the quadrilaterals must be turned over after dissection.



Also solved, in the same way, by E. S. Smith and the proposer.

*Editorial Note.* For other dissection puzzles see Dudney, *Amusements in Mathematics*, and Ball-Coxeter, *Mathematical Recreations and Essays.* The latter gives further references.

### Matching Chips

E 717 [1946, 219]. *Proposed by Orrin Frink, Jr., State College, Pa.*

In the game called "matching chips" each player selects, independently of the other, either a red or a white chip from his store of chips. If both chips are found to be of the same color, then player $A$ keeps both chips; otherwise player $B$ keeps both chips. Red chips are worth $r$ cents each, and white chips are worth

$w$ cents each. Develop the theory of this game.

*Solution by the Proposer.* If $A$ selects red chips with probability (or frequency) $p$, and $B$ selects red chips with probability $q$, then $A$'s mathematical expectation $M$ is: $M=rpq+w(1-p)(1-q)-rp(1-q)-wq(1-p)$. Hence $M=(2q-1)(rp+wp-w)$.

It follows that $A$'s "best strategy" is to select red chips with probability $w/(r+w)$, and $B$'s "best strategy" is to select red and white chips with equal probability $\frac{1}{2}$. If *either* player adopts his best strategy, the mathematical expectation of *both* players is zero. Hence the game is "fair."

If $B$ departs from his best strategy, then $A$ can take advantage of this error by imitating it; while if $A$ departs from his best strategy, $B$ can penalize him by reversing the error.

A general theory of games of this sort is given in *Theory of Games and Economic Behavior* by von Neumann and Morgenstern, Princeton, N. J., 1944.

**Derangements**

E 719 [1946, 220]. *Proposed by George Grossman, New York City*

A man addresses $n$ envelopes and writes $n$ checks in payment of $n$ bills.

(a) If the $n$ bills are placed at random in the $n$ envelopes, what is the probability that each bill is placed in a wrong envelope?

(b) If the $n$ bills and the $n$ checks are placed at random in the $n$ envelopes, one of each in each envelope, what is the probability that in no instance are the enclosures completely correct?

(c) In (b), what is the probability that each bill and each check will be in a wrong envelope?

*Solution by Paul Bateman, Philadelphia, Pa.* We use the well known combinatorial theorem: If there are $N$ objects, of which $N(A_1)$ have the property $A_1$, $N(A_2)$ have the property $A_2, \cdots, N(A_1A_2)$ have both $A_1$ and $A_2, \cdots, N(A_1A_2A_3)$ have $A_1$, $A_2$, and $A_3, \cdots$, and so on, then the number of objects which have none of $A_1, A_2, A_3, \cdots$ is

$$N - N(A_1) - N(A_2) - \cdots + N(A_1A_2) + \cdots - N(A_1A_2A_3) - \cdots .$$

Part (a) is the classical "problème des rencontres." The total number of ways of putting $n$ bills in $n$ envelopes is $n!$. If these $n!$ ways are regarded as the objects of the preceding theorem and a way is considered to have property $A_1$ if under it the first envelope has the right bill in it, to have property $A_2$ if the second envelope has the right bill in it, *etc.*, then we find for the number of ways such that no envelope has the right bill:

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\binom{n}{n}$$
$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).$$

Hence the desired probability is

$$\sum_{i=2}^{n}(-1)^i/i!.$$

A similar application of the combinatorial theorem to the $(n!)^2$ ways of putting $n$ bills and $n$ checks in $n$ envelopes gives the following as the number of ways such that in no envelope are both enclosures correct:

$$(n!)^2-\binom{n}{1}\{(n-1)!\}^2+\binom{n}{2}\{(n-2)!\}^2-\cdots+(-1)^n\binom{n}{n}$$
$$=(n!)^2\left\{1-\frac{1}{1!n}+\frac{1}{2!n(n-1)}-\cdots+\frac{(-1)^n}{n!n(n-1)\cdots 1}\right\}.$$

Hence the desired probability for part (b) is

$$\sum_{i=0}^{n}(-1)^i(n-i)!/i!n!.$$

The probability asked for in part (c) is merely the square of that obtained for part (a).

A somewhat more difficult problem is that of finding the probability that each bill and each check will be in a wrong envelope and no bill will be in the same envelope with the check which is supposed to go with it.

Also solved by D. W. Alling, Murray Barbour, D. H. Browne, Harley Flanders, and E. P. Starke.

Browne pointed out that the distribution in part (b) is ${}_2H_n$ of John Riordan's solution to $E$ 589 [1944, 287]. He also showed that with increasing $n$ the three probabilities of parts (a), (b), and (c) respectively approach $e^{-1}$, 1, $e^{-2}$.

*Editorial Note.* For a proof (and several interesting applications) of the combinatorial theorem referred to see, *e.g.*, Chapter V of *Elementary Number Theory* by Uspensky and Heaslet. Also, in connection with this problem, see problem 4146 and editorial note [1946, 107].

### The Extended Coin Problem

E 712 [1946, 156]. *Proposed by Donald Eves, Paterson, N. J.*

A man has twelve coins, all of which appear exactly alike, but one of which is counterfeit and does not weigh the same as a genuine coin. He has at his disposal a delicate set of balances, but no weights. How can he detect the false coin, and whether light or heavy, in not more than three weighings? (*Cf.* E 651 [1945, 397].)

I. *Solution by E. D. Schell, Arlington, Va.* Label the coins 1, 2, $\cdots$, 12. When an equal number of coins are placed on each side of the scale, the left side may be heavier, the coins may balance, or the right side may be heavier. Denote these three situations by $L$, $B$, and $R$ respectively. Further, let

$(a, b, c, \cdots)-(x, y, z, \cdots)$ mean the coins $a, b, c, \cdots$ are weighed on the left against the coins $x, y, z, \cdots$ on the right. $H(x)$ will mean that coin $x$ is counterfeit and heavier than the others, while $S(x)$ will mean coin $x$ is counterfeit and short in weight. A method of weighing which leads to a solution is as follows:

$$
(1, 2, 3, 4)\text{–}(5, 6, 7, 8)
\begin{cases}
L\ (1, 2, 6)\text{–}(3, 4, 5) & \begin{cases}
L\ (1)\text{–}(2) & \begin{cases} L\ H(1) \\ B\ S(5) \\ R\ H(2) \end{cases} \\
B\ (7)\text{–}(8) & \begin{cases} L\ S(8) \\ R\ S(7) \end{cases} \\
R\ (3)\text{–}(4) & \begin{cases} L\ H(3) \\ B\ S(6) \\ R\ H(4) \end{cases}
\end{cases} \\
B\ (1, 2, 3)\text{–}(10, 11, 12) & \begin{cases}
L\ (10)\text{–}(11) & \begin{cases} L\ S(11) \\ B\ S(12) \\ R\ S(10) \end{cases} \\
B\ (1)\text{–}(9) & \begin{cases} L\ S(9) \\ R\ H(9) \end{cases} \\
R\ (10)\text{–}(11) & \begin{cases} L\ H(10) \\ B\ H(12) \\ R\ H(11) \end{cases}
\end{cases} \\
R\ (1, 2, 6)\text{–}(3, 4, 5) & \begin{cases}
L\ (1)\text{–}(2) & \begin{cases} L\ S(2) \\ B\ H(5) \\ R\ S(1) \end{cases} \\
B\ (7)\text{–}(8) & \begin{cases} L\ H(7) \\ R\ H(8) \end{cases} \\
R\ (3)\text{–}(4) & \begin{cases} L\ S(4) \\ B\ H(6) \\ R\ S(3) \end{cases}
\end{cases}
\end{cases}
$$

It is interesting to note that it is sufficient to be given twelve coins in order to determine the existence of the counterfeit among them and its identity. That is, the existence of the counterfeit coin is not used in solving the problem and is not essential to the hypothesis.

In general we may determine the existence of a counterfeit and its identity among $(3^n-1)/2$ coins in $n$ weighings if we are given an additional $3^{n-1}$ coins known not to be counterfeit. From this result it is easy to show that the exist-

ence and identity of a counterfeit coin may be determined in $n$ weighings if $(3^n-3)/2$ coins are given without any additional coins.

II. *Solution by Joseph Rosenbaum, The Milford School, Conn.* Label the coins, 1, 2, $\cdots$, 12, and make the following three weighings:

| | | | |
|---|---|---|---|
| (1) | 1, 2, 3, 4 | against | 5, 6, 7, 8, |
| (2) | 1, 2, 3, 5 | against | 4, 9, 10, 11, |
| (3) | 1, 6, 9, 12 | against | 2, 5, 7, 10. |

It will now be shown how the counterfeit coin can be detected, and whether light or heavy, from the results of these three weighings. Designate the relationships "is lighter than," "is the same as," "is heavier than" by $L$, $S$, $H$, and denote by $x$, $y$, $z$, respectively, the observed relationships of the left to the right in (1), (2), (3). It is now easy to verify that out of the $27=3^3$ permutations with repetitions of $L$, $S$, $H$ the three $(x, y, z)=(S, S, S)$, $(L, H, H)$, and $(H, L, L)$ cannot happen. For the remaining 24 permutations the following key may be verified:

light coin

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *LLL* | *LLH* | *LLS* | *LHS* | *HLH* | *HSL* | *HSH* | *HSS* | *SHL* | *SHH* | *SHS* | *SSL* |

heavy coin

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *HHH* | *HHL* | *HHS* | *HLS* | *LHL* | *LSH* | *LSL* | *LSS* | *SLH* | *SLL* | *SLS* | *SSH* |

Also solved by D. W. Alling, LeRoy Babcock, Murray Barbour, Paul Brock, D. H. Browne, Mannis Charosh, Monte Dernham, Clara Fellers, Harley Flanders, G. E. Forsythe, Frank Herlihy, Vern Hoggatt, J. A. Jenkins, A. E. Karp, Nobert Kaufman, Ralph Keffer, N. D. Lane, H. R. Leifer, C. O. Oakley, Victor Perlo, C. L. Perry, C. F. Pinzka, Joseph Rosenbaum (also like I), Cedric Smith, Waldo Steiner, G. Sved, P. D. Thomas, J. A. Waidelich, Jr., F. M. Wood, and the proposer.

Forsythe and Pinzka gave solutions like II; all other solutions were similar to I. Babcock, Feller, Forsythe, Jenkins, Kaufman, Keffer, Perlo, Smith and Sved gave generalizations. These generalizations will be considered later when the solutions to problem 4203 [1946, 278] are published.

C. D. Olds drew attention to the generalizations of "bad" coin problems published in the *Mathematical Gazette*, 1945, p. 227. Forsythe mentioned the interesting problem: Given 31 coins, of which exactly one is false, being of weight unequal to that of the other 30, and also given a *spring* balance. In five weighings determine (1) which coin is false, (2) the weight of the false coin, (3) the weight of the good coins.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4229. *Proposed by Paul Erdös, University of Michigan*

Let $f(z)=z^n+\cdots+a_n$, $g(z)=z^m+\cdots+b_m$ be two polynomials. Denote by $A$ the region where $|f(z)|\leqq 1$ and by $B$ the region where $|g(z)|\leqq 1$. Prove that $A$ cannot properly contain $B$.

4230. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration in which the base $B$ is divisible by two or more distinct primes but not by 3 or any prime of the form $6k+1$, the numbers which have the property that they are reproduced in the right-hand digits of their squares are the same as those which are reproduced at the right of their fourth powers. (For example: $B=10$, $76^2=5776$, $76^4=33362176$.)

4231. *Proposed by Paul Nemenyi, Washington State College*

Show that any parabola $y=ax^n(a\neq 0,\ n>0)$ has the following property: If through the vertex any ray is drawn, the ratio of the area of the segment to that of the largest inscribed triangle is independent of the direction of the ray. Are there other curves with the same property?

4232. *Proposed by H. D. Ruderman, New York City*

$A:(0,\ 0)$ and $B:(0,\ u)$ are joined by the straight line segment $AB$ and a curve $\pi$ to enclose a region of area $s$. Let $k$ represent the length of $\pi$ from $A$ to $B$. Find the equation of $\pi$ such that the ratio $s/k^2$ is a maximum.

4233. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Parallel lines, of arbitrary direction, through the vertices $A'$, $B'$, $C'$, $D'$ of a tetrahedron $A'B'C'D'$ intersect in $A_1$, $B_1$, $C_1$, $D_1$ the faces $BCD$, $CDA$, $DAB$, $ABC$ of a homothetic tetrahedron $ABCD$. If $k$ is the homothetic ratio, $V$ the volume of $ABCD$, and $V_1$ that of $A_1B_1C_1D_1$, then

$$V_1 = -\ k^2(2k+1)V.$$

4206 [1946, 341]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider spheres with centers at the vertices of a tetrahedron $ABCD$ and radii equal respectively to $k$ times the sum of the squares of the three opposite

edges. Show that the sum of the squares of the distances from the four spheres to the center of the sphere orthogonal to the four spheres is equal to

$$[2(4k+1)]^2R^2 - 2k(2k+1)\Sigma,$$

where $R$ is the radius of the circumsphere and $\Sigma$ means the sum of the squares of the six edges. Consider particular cases.

4208 [1946, 342]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron. If two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere, belonging to the linear net determined by the circumscribed and conjugate spheres, and whose center is the complementary point of the orthocenter with respect to the tetrahedron.

## SOLUTIONS

### Parabola and Strophoid

4026 [1942, 128]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) Construct a triangle $ABC$ knowing $a$, $A$ and given that the median and symmedian from $A$ are perpendicular and parallel to two given directions. (2) Indicate the properties of this special triangle. (3) Let $B'$ and $C'$ be the orthogonal projections of $B$ and $C$ on a variable straight line $AP$ which cuts $BC$ in $P$. The locus of the harmonic conjugate of $P$ with respect to $B'$ and $C'$ is a right strophoid having the vertex $A$ for a double point and tangent to the bisectors of angle $A$.

*Solution by R. Bouvaist, Vincelles, Saône et Loire, France. Construction of the triangle.* The segments $A\mu$, $A\sigma$ being drawn parallel to the given directions, the interior bisector of the angle formed is $A\delta$; the two lines $A\beta$ and $A\gamma$ are drawn symmetric to $A\delta$ making with it the given angle $A/2$. Let $A\mu'$ be the harmonic conjugate of $A\mu$ with respect to $A\beta$ and $A\gamma$, draw on $A\mu'$ the segment $AB'=a/2$; then through $B'$ draw the parallel to $A\mu$ meeting $A\beta$ in $B$; and through $B$ draw the parallel to $A\mu'$ which must meet $A\gamma$ in $C$, giving the desired triangle $ABC$.

*Characteristic properties when $A\mu$ and $A\sigma$ are perpendicular.* Let $A\mu$ and $A\sigma$ be the median and symmedian from $A$, with $c>b$; we then have

$$\frac{B\mu}{\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)} = \frac{A\mu}{\sin B}, \qquad \frac{C\mu}{\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)} = \frac{A\mu}{\sin C},$$

$$\frac{\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)} = \frac{\sin C}{\sin B} = \frac{c}{b} = \frac{\tan\frac{A}{2}+1}{\tan\frac{A}{2}-1}, \qquad \tan\frac{A}{2} = \frac{c+b}{c-b}.$$

*Locus of the harmonic conjugate of $P$ with respect to $B'$ and $C$.* Let $P'$ be the harmonic conjugate of $P$ with respect to $B$ and $C$; the perpendicular from $P'$ to $AP$ meets it in $Q$, then $QC'PB'$ is harmonic. The envelope of $P'Q$ is tangent to $BC$ ($P$ being the orthogonal projection $H$ of $A$ on $BC$), to the median for $BC$ and to the line at infinity ($P$ at the midpoint $\mu$ of $BC$), to the bisectors of angle $A$ ($P$ at the foot of one of these bisectors). The envelope is moreover of the second class, since from a point $P'$ of $BC$ there is one and only one tangent; it is therefore a parabola with the median $A\mu$ as the directrix. The locus of $Q$ is the pedal of this parabola with respect to the point $A$ on its directrix, and the locus is a stophoid; with double point at $A$, the tangents at this point being the bisectors of the angle $A$, and having for real asymptotic direction the median $A\mu$. If $A\mu$ and $A\sigma$ are perpendicular the point $A$ is the intersection of the directrix and the axis of the parabola, and the strophoid is right angled.

*Editorial Note.* See the proposer's indications of a solution 1943, p. 333.

### Triangles and Conics Circumscribed and Inscribed

4127 [1944, 352]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

The straight lines $AG$, $BG$, $CG$, $DG$ drawn through the vertices and centroid $G$ of the tetrahedron $ABCD$ meet again its circumsphere in $A'$, $B'$, $C'$, $D'$, and the planes perpendicular to the respective lines at these latter points determine the tetrahedron $A_1'B_1'C_1'D_1'$. Show that(1) The two tetrahedrons have the same centroid and are hyperbolic. (2) The non-focal axis of the quadric surface with $G$ as a focus inscribed in $A_1'B_1'C_1'D_1'$ is equal to the diameter of the orthoptic sphere of the Monge sphere of the Steiner ellipsoid inscribed in $ABCD$.

*Solution by the Proposer.* The following generalization will be considered.

I. Let $A'$, $B'$, $C'$ be the points in which the straght lines $AP$, $BP$, $CP$ meet again the circumcircle of the triangle $T\equiv ABC$ ($BC=a$, $CA=b$, $AB=c$), where $P$ is arbitrarily chosen. Set $T_1\equiv A_1B_1C_1$, $T_1'\equiv A_1'B_1'C_1'$, the antipedal triangles of $P$ with respect to $T$, $T'\equiv A'B'C'$; let $P'$ be the symmetric of $P$ with respect to the center $O$ of the circle $(O, R)$ the circumcircle of $T$.

The triangles $T$ and $T_1'$ are polar reciprocal with respect to circle $(P)$, center $P$ and radius whose square is the power of $P$ with respect to $(O, R)$. These two triangles, which are orthologic by construction, are also homologic; moreover, $P$ has the same barycentric coordinates with respect to each. These properties are also true for $T'$ and $T_1$.

The triangles $T_1$ and $T_1'$ circumscribe the same conic with foci $P$, $P'$, and the lengths $2\alpha$, $2\beta$ of its axes are such that $\alpha=R$ and $\beta^2=R^2-(OP)^2$. If $P\equiv G$ the centroid of $T$, it is also the centroid of $T_1'$. Thus $T_1$ and $T_1'$ circumscribe a conic with foci $G$ and $G'$, with focal axis $2\alpha=2R$ and the semi axis $\beta$, non-focal, is such that

$$\beta^2 = R^2 - (OG)^2 = (a^2 + b^2 + c^2)/9, \qquad \beta = (a^2 + b^2 + c^2)^{1/2}/3 = \rho\sqrt{2},$$

where $\rho$ is the radius of the Monge circle of the Steiner ellipse inscribed in $T$.

THEOREM. *In a triangle $T \equiv ABC$, the non-focal axis of the conic inscribed in the antipedal triangle of the centroid $G$, one focus being $G$, has the length of the diameter of the orthoptic circle of the Monge circle of the inscribed Steiner ellipse.*

II. The straight lines from the arbitrarily chosen point $P$ to the vertices of the tetrahedron $T \equiv ABCD$ meet again the circumsphere $(O, R)$ of $T$ in $A'$, $B'$, $C'$, $D'$, where $BC = a$ and $DA = a'$, *etc.* Denote by $T_1 \equiv A_1B_1C_1D_1$, $T_1' \equiv A_1'B_1'C_1'D_1'$ the antipedal tetrahedrons of $P$ with respect to $T$, $T'$; by $P'$ the symmetric of $P$ with respect to the center $O$ of $(O, R)$. Then $T$ and $T_1'$ are polar reciprocal with respect to sphere $(P)$ with center $P$ and with the radius whose square is the power of $P$ with respect to $(O, R)$. These two spheres are orthologic by construction, and they are also hyperbolic; moreover, $P$ has the same barycentric coordinates for $T$ and $T_1'$; and these properties are also true for $T'$ and $T_1$.

Also $T_1$, $T_1'$ circumscribe the quadric surface of revolution with foci $P$, $P'$ and the lengths $2\alpha$, $2\beta$ of the axes of the meridian conic are such that $\alpha = R$, $\beta^2 = R^2 - (OP)^2$.

If $P \equiv G$ the centroid of $T$, it is also the centroid of $T_1'$. Thus $T_1$, $T_1'$ circumscribe the quadric surface of revolution with foci $G$ and $G'$, where the focal axis length $2\alpha = 2R$ and the non-focal semi-axis length $\beta$ is such that

$$\beta^2 = R^2 - (OG)^2 = \Sigma[a^2 + (a')^2]/16,$$
$$\beta = \{\Sigma[a^2 + (a')^2]\}^{1/2}/4 = \rho\sqrt{3},$$

where $\rho$ is the radius of the Monge sphere of the Steiner ellipsoid inscribed in $T$. Hence we have

THEOREM. *In a tetrahedron $T \equiv ABCD$, the non-focal axis of the quadric surface of revolution inscribed in the antipedal tetrahedron of the centroid $G$, one focus being $G$, has the same length as the diameter of the orthoptic sphere of the Monge sphere of the inscribed Steiner ellipsoid.*

*Note* 1. In the case of a triangle $T$, the second focus $G'$ of the conic inscribed in the triangles $T_1$ and $T_1'$, symmetric of the point $G$ with respect to the center $O$ of the circumscribed circle, is on the straight line joining the Steiner point to the reciprocal of the orthocenter.[1] Moreover, the point $G'$ is such that if circles are described with diameters $G'A$, $G'B$, $G'C$, the sum of the powers of a vertex of $T$ with respect to two of the circles not passing through that vertex is the same for the three vertices.[2]

*Note* 2. For the hypothesis that $P \equiv G$, the triangle $T' \equiv A'B'C'$ has interesting properties. The construction of a triangle $T \equiv ABC$ knowing the vertices of $T'$ amounts to determining the point $G$ in its plane such that the three vectors carried by $GA'$, $GB'$, $GC'$ of intensities $1/GA'$, $1/GB'$, $1/GC'$ have a zero re-

[1] Mathesis, 1888, p. 178

[2] E. Lemoine, Journal de G. de Longchamps, 1885, p. 219.

sultant. This leads to the determination of the foci of a conic touching the sides of $A'B'C'$ at their midpoints (inscribed Steiner ellipse). It follows also that the foci $S$ and $S'$ of the inscribed Steiner ellipse $(E)$ for $T$ coincide with the centroids of the antipedal triangles $T_1$ and $T_1'$ of the points $S$ and $S'$ with respect to $T$ and there results the equations

$$(SA)^2 \cdot BC \cdot SS_a = (SB)^2 \cdot CA \cdot SS_b = (SC)^2 \cdot AB \cdot SS_c,$$
$$(S'A)^2 \cdot BC \cdot S'S_a' = (S'B)^2 \cdot CA \cdot S'S_b' = (S'C)^2 \cdot AB \cdot S'S_c',$$

where $S_i$ and $S_i'$ are the orthogonal projections of $S$ and $S'$ on $BC$, $CA$, $AB$. For, $SB_1C_1$ and $BCS'$ being similar triangles, we have

$$(SA)^2 \cdot BC \cdot SS_a = SA \cdot BC \cdot SS_a \cdot S'S_a' = SB \cdot CA \cdot SS_b \cdot S'S_b' = SC \cdot AB \cdot SS_c \cdot S'S_a'$$

since $S$ is the centroid of $T_1$ and the products $SS_i \cdot S'S_i'$ are equal to the square of the non-focal semi axis of $(E)$.

*Note* 3. For the hypothesis that $P \equiv G$, the centroid of the tetrahedron $T$, the four vectors $GA', \cdots, GD'$ carry vectors of intensities $1/GA', \cdots, 1/GD'$ which have a zero resultant since the tetrahedrons $T$ and $T_1'$ have the same centroid. But the determination of the points $G$ knowing the points $A', \cdots, D'$[3] is subject not to a quadric surface inscribed in $T'$ but to a surface of the third class, the polar reciprocal of the Cayley cubic surface (locus of the points whose orthogonal projections on the faces of the tetrahedron are coplanar).

### Self Reciprocal Curves

4167 [1945, 400]. *Proposed by R. A. Staal, Student, University of Toronto*

What curves are self-reciprocal with respect to the conic $x^2+y^2=z^2$ (or the circle $x^2+y^2=1$)?

*Note by Howard Eves, Oregon State College.* Associated with problem 4167 is problem E 666 [1945, 218]. A solution for E 666 by Francis Hall has already appeared [Jan. 1946, 40]. To this solution of Hall's, I appended the following editorial Note:

"Using homogeneous coordinates it is easy to show that the curve $f(x\ y, z)=0$ will be self-reciprocal to the circle $x^2+y^2=z^2$ if and only if $f(x, y, z)$ divides $f(\partial f/\partial x, \partial f/\partial y, -\partial f/\partial z)$. Imposing this condition on the curve $x^m y^n = kz^{m+n}$ we readily find the values of $k$ as given above."

*Remark by the Proposer.* The condition for an algebraic curve

$$f \equiv f(x, y, z) = 0$$

to be self-reciprocal with respect to $x^2+y^2=z^2$, seems to be that the polynomial $f$ should divide $f(\partial f/\partial x, \partial f/\partial y, -\partial f/\partial z)$. An instance is

$$f = x^m y^n - kz^{m+n}, \qquad k = \sqrt{m^m n^n/(m+n)^{m+n}}.$$

[3] Thébault, Mathesis, 1939, p. 51 (Sûjet d'étude).

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*The Common Sense of the Exact Sciences.* By W. K. Clifford. New York, Alfred A. Knopf, 1946. 66+249 pages. $4.00.

The original work as planned by Clifford was to have been entitled *The First Principles of the Mathematical Sciences Explained to the Non-Mathematical.* This would have been a clumsy title, but more descriptive of its nature than *The Common Sense of the Exact Sciences*, the title finally adopted in accordance with a preference expressed by Clifford shortly before his death in 1879. He left the manuscript unfinished. The labor of revision and completion was begun by R. C. Rowe and finished by Karl Pearson. The first edition appeared in 1885, and this was followed by a second and a third edition, but the book has now been a long time out of print, so that the present edition is very welcome. It is essentially a reproduction of the third edition, the notable additions being a Preface by Bertrand Russell and an Introduction by the Editor, James R. Newman.

Every teacher of mathematics, particularly elementary mathematics, should read this Preface. I would like to quote the whole of it, because it would make a better review of the book than I can write. However, the following quotation must suffice: "A taste for mathematics, like a taste for music, can be generated in some people, but not in others . . . . Pupils who have not an unusually strong natural bent towards mathematics are led to hate the subject by two shortcomings on the part of their teachers. The first is that mathematics is not exhibited as the basis of all our scientific knowledge, both theoretical and practical: the pupil is not convincingly shown that what we can understand of the world, and what we can do with machines, we can understand and do in virtue of mathematics. The second defect is that the difficulties are not approached gradually, as they should be, and are not minimized by being connected with easily apprehended central principles, so that the edifice of mathematics is made to look like a collection of detached hovels rather than a single temple embodying a unitary plan. It is especially in regard to this second defect that Clifford's book is valuable."

The Introduction gives an interesting account of Clifford as man and as scientist. Here are a few brief facts. He was born at Exeter in England in 1845. He studied at King's College, London, and later at Cambridge. "In a rigidly conventional age he was marked by eccentricities of habit, dress, and opinion." In addition to mathematics, he studied French, German, Spanish, Arabic, Greek, Sanskrit, and hieroglyphics. He also learned Morse code and shorthand, and topped his athletic career hanging by his toes from the cross-bar on the weather-

cock of a church steeple. In 1868 he was elected to a fellowship at Trinity, and in 1871 was appointed professor of applied mathematics at University College, London, Clerk Maxwell being one of those recommending him for the position, on the basis that his researches did not tend to "the elaboration of abstruse theorems by ingenious calculation, but to the elucidation of scientific ideas by the concentration upon them of clear and steady thought." He died of tuberculosis in 1879.

There are five chapters with the following titles: Number, Space, Quantity, Position, Motion. These titles give a rough idea of the scope of the book. There is a good deal of material open to criticism on various grounds, but there runs through the whole work a thread of something which, if not pure gold, looks very like it. Would that a similar thread ran through the textbooks of today! Writers of textbooks should ponder the fact that killing the interest of students is an easy task; its stimulation is a much more subtle thing, and they should not be ashamed to learn from a master of exposition in this field. If the textbook writer is an active research mathematician, as Clifford was, he would do well to investigate Clifford's secret of the interest-grasping simile and the complete absence of talking-down to his readers.

To the modern scientist the book is not likely to yield much, directly. But indirectly it may give food for thought. One would have to be a highly specialized historian of science to fill the details in the scientific backdrop before which Clifford performed his intellectual act. But we may imagine it filled with a dignified figure of Mother Nature, holding two scrolls —one with Euclid's axioms and the other with Newton's laws of motion. Only the very observant had noticed that a little worm had already eaten into Euclid's axioms, and only an iconoclast would play with the idea that Newton's laws might suffer a like indignity. But Clifford was observant and he was an iconoclast. He was moreover an admirer of Riemann, and had translated Riemann's now famous paper "On the Hypotheses which lie at the Bases of Geometry." It was left to Einstein to detect the flaw in Newton's concept of time, and so reduce Newton's laws from a position of divine authority to the level of a useful approximation, but the worm that began this process of decay was Clifford's. Science is such a terribly serious thing that it is a relief to know that Clerk Maxwell had a pet demon and Clifford a pet worm; its name was AB, and anyone interested in knowing more about it will find a sketch of it on p. 194 of the volume under review, and a description of its psychological reactions on the subsequent pages. All of which goes to show that the progress of science owes as much to the men who play with new ideas as it does to those who elaborate "abstruse theorems by ingenious calculation."

The publishers are to be congratulated on producing a volume pleasing to the eye and worthy of its author.

J. L. Synge

*Analytic Geometry and Calculus*. Second Edition. By H. B. Phillips. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 504 pages. $4.50.

In this book analytical geometry and calculus are presented in the form and order in which these subjects are required for courses in science and engineering. The author assumes that it is the calculus that is most used, and introduces analytical geometry only as it is needed to develop the calculus. Chapter I gives a brief but adequate introduction to the ideas of coordinates, functions, graphical representation, limits of sequences and functions including the limit of $\sin \theta/\theta$ as $\theta \to 0$, continuity of functions, the Cauchy criterion of convergence, and uniform continuity. In Chapter II instantaneous speed and the slope of a curve as the limit of the slope of a secant line are studied preliminary to introducing the derivative. The differential is then defined in the following way: The differential of the independent variable is equal to its increment and the differential of the function of the independent variable is equal to the product of its derivative and the increment of the independent variable. There then follows the differentiation of the algebraic functions, and of the sine and the cosine. With this preparation the author takes up the use of the differential in approximation and in determining small errors, derivatives of higher order, implicit functions, maxima and minima, rates of change, velocity and acceleration, mean value theorem, Rolle's theorem, and indeterminate forms. Chapter III deals with integration and summation. The process of finding a function with a given differential is called integration. The definite integral is defined as the limit of a sum, and it is proved that if $F(x)$ is a function such that $dF(x)=f(x)dx$ then the integral of $f(x)$ from $x=a$ to $x=b$ is $F(b)-F(a)$. Integration is then used to obtain areas, volumes, pressures and work. The mean value theorem and the fact that if $F(x)$ is the integral of $f(x)$ from $a$ to $x$ then $dF(x)=f(x)dx$ are proved.

The first three chapters contain, as the author states in the preface, a complete course in the calculus. As to the wisdom of so great a concentration on the calculus at the expense of analytical geometry in the early stages of a course for scientists and engineers, there is room for a difference of opinion. It is a procedure with which the reviewer is in complete agreement. At this stage the students show considerable enthusiasm for the ideas of the calculus. They are finding a use for them in their courses in physics, chemistry, and engineering, and it is a wise policy to exploit this interest.

In Chapters IV, V, and VI the author takes up the straight line, Newton's method, conic sections, transformation of coordinates, graphs of algebraic functions, determinants and their application to linear equations, and trigonometric functions. Chapter VII returns to the calculus with the differentiation and integration of the exponential and logarithmic functions, and from this point on the topics follow fairly closely the established tradition in elementary calculus, with one notable exception. The notation of vector analysis is developed in two and three dimensions, and this is used in finding centroids, in studying velocity and acceleration, tangents and normals to curves in two and three dimensions,

angular velocity, tangent planes and normals to a surface, line integrals and gradients. It is the reviewer's opinion that this innovation is long overdue.

Enough has now been said to indicate that this text is something new in the field of elementary calculus. There is a fresh approach to every topic. Moreover, as one reads the book it becomes clear that the changes from tradition are not made merely for the sake of being different, but because long experience has convinced the author that this is the way to give to scientists and engineers the mathematical training that will be most useful to them. In regard to this the reviewer would raise only one question: Should the student go as far with the calculus as this book carries him without becoming acquainted with elementary differential equations?

R. L. JEFFERY

*A Manual of Operation for the Automatic Sequence Controlled Calculator.* (Annals of the Computation Laboratory of Harvard University, Vol. I.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1946. 561 pages. $10.00.

The Staff of the Harvard Computing Laboratory has undertaken in this Manual, the difficult task of explaining to the uninitiated the construction and use of the elaborate Harvard Sequence Controlled Calculator. As its title implies, the Manual is intended primarily as a set of instructions for operators of the Calculator; but several chapters will be of value both to mathematicians who have problems suitable for machine computation and to designers of automatic computers.

The parts of the Manual which will be of interest to the more extensive class of readers are the first 52 pages, including a brief historical sketch of digital calculators in Chapter I and a description of the Calculator in Chapter II, and the 66-page bibliography of computing machines and techniques which contains over a thousand references. The bibliography is divided into 23 sections dealing with such subjects as Historical Background of Automatic Calculating Machinery, Determinants, Zeros of a Polynomial, Harmonic Analysis, Interpolation, Asymptotic Expansions, Numerical Differentiation and Integration, Ordinary and Partial Differential Equations and Integral Equations.

The Harvard Sequence Controlled Calculator, built by the International Business Machine Corp. under the guidance of Prof. Howard H. Aiken, is a major engineering project. It is an assemblage of calculating and control elements, mounted on racks 8 feet high and totalling 63 feet in length, and weighing about 5 tons. A 4-horsepower motor furnishes its mechanical power through a network of shafts and gears. Sixteen photographs in the Manual show views of the machine as a whole and of various components. One photograph of a computing element of the early (1834) Babbage Difference Engine—a forerunner of modern automatic calculators—contrasts the mechanisms available to designers in the past with those used in the Sequence Calculator.

The Calculator has been operated for nearly two years by the Staff of the Harvard Computing Laboratory which, under the editorship of Lt. Grace M.

Hopper, USNR, has compiled this Manual. As a kind of side line to its regular confidential war work in this period, it has calculated ab initio at least six volumes of mathematical tables one of which—a 230 page table of Modified Hankel Functions—has been published. In these computations the Calculator normally uses 23 digital places but it can, under certain conditions, deal with numbers of 46 or of 12 digits. Prof. Aiken estimates that the machine will do the work of about 100 well equipped manual computers.

Chapter II of the Manual contains a general description of the Calculator and of its major component parts. Functionally, the Calculator contains the components common to almost all automatic computing devices, namely: means for accepting formulas and numerical data; for transferring and storing numbers; for performing standard arithmetic operations, such as addition, multiplication, and so forth, on these numbers; and finally, for presenting the results of the computations.

The storage, addition and transfer of numbers in the Calculator are carried out by methods and devices which are familiar to those who have used the standard International Business Machine equipment, although such familiarity on the part of the reader is not assumed in the Manual. The novel features of the Calculator reside principally in the control of the machine.

The Sequence Controlled Calculator has, like Babbage's Engines and like most automatic computing devices since that time, been conceived as an automatic means for carrying out a sequence of mathematical operations on any set of numbers which may be presented to it. The sequence of operations is derived by a human process of "coding" from mathematical formulas (a power series of a Bessel Function, for example). The control sequence is entered into the machine separately from the numerical data, so that it may be used over and over with many different sets of numerical data.

The numbers which the Calculator is instructed to operate on are coded and the codes are punched either in standard IBM cards or on a continuous tape. Certain constants may be set manually on switches, there being one 10—position switch for each digit to be stored. In all, the Calculator contains two punched-card readers, one "value" tape (exclusive of "interpolation" tapes) and sufficient switches to store 60 numbers of 23 digits each.

Instructions to the machine consist of repetitions of the command "take the number from unit $A$, deliver it to unit $B$ and start operation $C$," the codes for $A$, $B$ and $C$ being punched in the sequence control tape. The unit $A$ may be any one of the sources of numerical data or any one of the 72 adding-storage units in the Calculator. The unit $B$ must, of course, be a unit which, like the adding-storage units, is capable of receiving a number.

Three interpolating units are built into the machine. Each such unit is supplied with a tape and each is capable of interpolating up to the 11th order between the numerical values punched in its tape. Besides these units, electromechanical tables of sin $A$, log $A$ and $10^x$, are permanently wired into the Calculator. Internal sequence controls permit the main control tape to call for

interpolation from any one of these tables. Other operations available are addition, multiplication, division, reading of absolute values, and so forth.

The results of computations are presented automatically, either on IBM punched cards or on one or both of the two electric typewriters with which the Calculator is equipped.

Some form of mathematical check on the computation is essential to reliable operation of the Calculator. A special counter is provided which will stop the machine should the check computation indicate an error in the results.

To provide for flexibility in the location of decimal points and in the number of digits used in multiplication and division, a plugboard has been included in the Calculator. Chapter V of the Manual is devoted to special instructions for using the plugboard.

A number of examples of problems are worked out in Chapter VI. These include the computation and checking of a quartic polynomial, of $(x^2-1)^{-3/2}$ by an iterative process, of an integral, of the solution of a system of linear algebraic equations and other problems.

Mechanisms which will perform the fundamental arithmetic operations involved in computation have been improved and simplified by the intensive study of hundreds of philosophers, scientists and inventors and by the construction and practical use of thousands of models since their crude beginnings in the 17th century, but mechanisms which will carry out the analogous duties for the "programming" of sequences of such operations have not as yet had the benefit of a comparable background. The early stages in the development of any class of mechanisms are characterized by relatively clumsy and groping attempts to satisfy conflicting requirements, and it is no reflection on the ingenuity and capability of the Harvard Computing Laboratory Staff to point out that this generalization applies to the Sequence Control of the Calculator. It is not a simple or easy task to learn to set up the controls for the Calculator, but it is to be hoped that continued development of the machine will lead to simpler and easier procedures in this important part of the field of automatic computation. In fact, the existence of the Calculator should, in itself, furnish both an incentive to devise such improvements and an opportunity to try them out under conditions of actual use. The present methods of coding or translating from mathematical symbols to machine language are given in some detail in Chapter IV, and are illustrated in the examples worked out in Chapter VI. The reader will be impressed by the dissimilarity of the two languages, and will probably conclude that the translation from one to the other had best be left to the experts in the process.

G. R. Stibitz

*Analytische Geometrie der Ebene und des Raumes.* (Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, No. 4; Mathematische Reihe, Band II). By Rudolf Fueter. Basel, Birkhäuser, 1945. 180 pages. Pamphlet bound, 18.50 s. fr.; cloth bound, 22.50 s. fr.

The chapter headings are as follows: Chapter I (pp. 11–51), Point and Line

in the Plane. Chapter II (pp. 52–94), Point, Plane and Line in Space. Chapter III (pp. 95–142), Curves of the Second Degree in the Plane (Conic Sections). Chapter IV (pp. 143–172), Surfaces of the Second Degree.

The book is elementary in character and treats the material which is usually presented in a first course in analytic geometry. In the first twenty-one pages of the text most of the simple tools for the study of plane analytic geometry are presented, such as; rectangular (cartesian) coördinates and polar coördinates, including the relations between them, equations of translations and rotations, determinants and vectors. Even though vectors are introduced no systematic use of them is made later on. It is somewhat disappointing that a third order determinant is expanded by the device of rewriting the first two columns.

In Chapter IV the quadric surfaces are considered with their equations in "standard forms." The problem of reducing the general second degree equation to one of the standard forms is mentioned but not treated.

It is the opinion of the reviewer that the book is well written. The ideas are carefully explained and the discussions are thorough. This is a nice book for the undergraduate to use for practice in reading German. Finally, the typography is excellent.

J. H. TAYLOR

## NEW BOOKS RECEIVED

*Concise Analytic Geometry*. By C. H. Sisam. New York, Henry Holt and Co., 1946. 9+155 pages. $2.00.

*Applied Elasticity*. By John Prescott. New York, Dover Publications, 1946. 6+666 pages. $3.95.

*Higher Mathematics for Students of Chemistry and Physics*. Fourth Edition. By J. W. Mellor, New York, Dover Publications, 1946. 21+641 pages. $4.50.

*Elementary Matrices and Some Applications to Dynamics and Differential Equations*. By R. A. Frazer, W. J. Duncan, and A. R. Collar. New York, Macmillan Co.; Cambridge University Press, 1946. 16+416 pages. $4.00.

*Grundebegriffe der Wahrscheinlichkeitsrechnung*. By A. Kolmogoroff. New York, Chelsea Publishing Co., 1946. 5+62 pages. $2.25.

*Grundlagen der Analysis*. (Foreword translated into English and vocabulary added). By Edmund Landau. New York, Chelsea Publishing Co., 1946. 20+139 pages. $2.50.

*Lectures on the Calculus of Variations*. By G. A. Bliss, Chicago, University of Chicago Press, 1946. 9+296 pages. $5.00.

*Mathematical Methods of Statistics*. By Harald Cramér. Uppsala, Almqvist and Wiksells, 1945; Princeton University Press, 1946. (Princeton Mathematical Series, No. 9). 16+575 pages. $6.00.

*Mathematics of Finance*. By J. A. Northcott. New York, Rinehart and Co., 1946. 10+252 pages. $3.00.

*Practical Electrical Mathematics*. By W. E. Rasch. Boston, D. C. Heath and Co., 1946. 8+360 pages. $2.00.

*Reelle Funktionen. Vol.* 1. *Zahlen, Punktmengen, Funktionen.* By Constantin Carathéodory. New York, Chelsea Publishing Co., 1946. 6+184 pages. $3.25.

*Scientific and Technical Aspects of the Control of Atomic Energy.* (United Nations Department of Public Information.) New York, Columbia University Press, 1946. 5+42 pages. $0.25.

*Tables of Fractional Powers.* Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1946. 489+30 pages. $7.50.

*Trigonometry Refresher for Technical Men.* By A. A. Klaf. New York and London, McGraw-Hill Book Co., Inc., 1946. 10+629 pages. $5.00.

*Vorlesungen über Zahlentheorie aus der Elementaren Zahlentheorie.* By Edmund Landau. New York, Chelsea Publishing Co., 1946. 8+184 pages.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

Honor societies have been in existence for many years, and men and women of high scholastic standing have coveted membership in such societies. Departmental honor fraternities are of rather recent origin, but are now found on most campuses. Two national mathematical fraternities are serving mathematicians in the United States. Their growth and progress have demonstrated the need and value of affiliation of members of a mathematical club at one institution with those of similar organizations at other institutions. Some of the advantages gained by such an affiliation include: a recognized standard in honoring students of unusual ability in mathematics, the encouragement of student participation in undergraduate as well as graduate research, and the bringing together of students and faculty in fraternal relations with those of other campuses.

In an endeavor to acquaint mathematicians with the history and purpose of these two fraternities and to solicit further interest in them, this Department presents in this issue of the MONTHLY some descriptive material about *Pi Mu Epsilon* and *Kappa Mu Epsilon.* If any active mathematical club is interested in affiliating with one of the two societies, it is suggested that the faculty member in charge of the club correspond with the national secretary of the fraternity in which the group is interested.

### PI MU EPSILON

*Pi Mu Epsilon,* the older of the two societies, was founded by Professor E. D. Roe, Jr., a member of the mathematics faculty at Syracuse University. It was incorporated in 1914 under the laws of the State of New York. Professor Roe remained Director General of the fraternity during the remainder of his life.

The purpose of *Pi Mu Epsilon*, as stated by Director General Tomlinson Fort, is "to serve mathematicians on the campuses of our universities by bringing together in a national organization those faculty, graduate students, and undergraduate students who are interested in Mathematics." All chapters hold mathematical meetings similar to those of the usual mathematics clubs.

Many new activities have been sponsored by chapters of *Pi Mu Epsilon:* the holding of mathematical contests, the awarding of mathematical prizes, the sponsoring of mathematical lectures by distinguished speakers, and the installation of mathematics clubs both in their own institutions and in high schools. For a number of years the chapter at New York University sponsored a state-wide mathematics contest among secondary schools with the greatest success. Nothing is secret concerning *Pi Mu Epsilon*; guests are frequently present at meetings and at initiations.

The National Officers of *Pi Mu Epsilon* are:

| | |
|---|---|
| Director General: | Tomlinson Fort, University of Georgia |
| Vice-Director General: | E. H. C. Hildebrandt, Northwestern University |
| Secretary-Treasurer General: | John S. Gold, Bucknell University |
| Councillors General: | George Williams, Oregon State College |
| | C. A. Hutchinson, University of Colorado |
| | C. H. Richardson, Bucknell University |
| | E. R. Smith, Iowa State College |

The name, location, and date of founding of chapters follow:

| | | |
|---|---|---|
| New York Alpha | Syracuse University | 1914 |
| Ohio Alpha | Ohio State University | 1919 |
| Pennsylvania Alpha | University of Pennsylvania | 1921 |
| Missouri Alpha | University of Missouri | 1922 |
| Alabama Alpha | University of Alabama | 1922 |
| Iowa Alpha | Iowa State College | 1923 |
| Illinois Alpha | University of Illinois | 1924 |
| Pennsylvania Beta | Bucknell University | 1925 |
| Montana Alpha | Montana State University | 1925 |
| New York Beta | Hunter College | 1925 |
| Missouri Beta | Washington University | 1925 |
| California Alpha | University of California | 1925 |
| Ohio Beta | Ohio Wesleyan University | 1927 |
| Kentucky Alpha | University of Kentucky | 1927 |
| Nebraska Alpha | University of Nebraska | 1928 |
| Kansas Alpha | University of Kansas | 1928 |
| Pennsylvania Gamma | Lehigh University | 1929 |
| Oklahoma Alpha | University of Oklahoma | 1929 |
| California Beta | University of California | 1930 |
| Pennsylvania Delta | Pennsylvania State College | 1930 |
| Arkansas Alpha | University of Arkansas | 1931 |
| Oregon Alpha | University of Oregon | 1931 |
| Washington Alpha | State College of Washington | 1931 |
| North Carolina Alpha | Duke University | 1932 |
| Washington Beta | University of Washington | 1932 |
| New York Gamma | Brooklyn College | 1933 |

| | | |
|---|---|---|
| Wisconsin Alpha | Marquette University | 1933 |
| New York Delta | New York University | 1933 |
| Georgia Alpha | University of Georgia | 1934 |
| New York Epsilon | St. Lawrence University | 1935 |
| Kansas Beta | Kansas State College | 1935 |
| Ohio Gamma | University of Toledo | 1936 |
| Colorado Alpha | University of Colorado | 1936 |
| New York Zeta | Columbia University | 1937 |
| Oklahoma Beta | Oklahoma Agricultural and Mechanical College | 1938 |
| Oregon Beta | Oregon State College | 1938 |
| Wisconsin Beta | University of Wisconsin | 1939 |
| Louisiana Alpha | Louisiana State University | 1939 |
| Michigan Alpha | Michigan State College | 1940 |
| Arizona Alpha | University of Arizona | 1941 |
| Delaware Alpha | University of Delaware | 1941 |
| Illinois Beta | Northwestern University | 1944 |
| Missouri Gamma | St. Louis University | 1945 |

## KAPPA MU EPSILON

*Kappa Mu Epsilon* was founded by Dr. Kathryn Wyant, a member of the mathematics faculty at Northeastern Oklahoma State Teachers College, in 1931. However, the need for a society in mathematics which would devote all of its interests to undergraduate students seems to have developed simultaneously in many parts of the United States during the years 1929 to 1931. Two colleges in Mississippi, Mississippi State College for Women and Mississippi State College, as well as the University of New Mexico, had shown considerable interest in such a society. These three chapters immediately joined Dr. Wyant in the formation of a national mathematics fraternity of collegiate rank.

The purpose of the fraternity as stated by Professor E. R. Sleight, the National President of Kappa Mu Epsilon, is "to further interest in mathematics and to develop appreciation for the beauty of it, to provide a society for the recognition of unusual ability in mathematics, and to bring them together in fraternal relationship. It encourages undergraduate research and collects information concerning the applications of mathematics."

In the fall of 1941, under the leadership of Professor C. V. Newsom, then National President, *Kappa Mu Epsilon* embarked upon a new enterprise to further undergraduate work in Mathematics. An official journal, *The Pentagon*, was established in order that the fraternity might carry out some of its plans and purposes. It was the belief of the editors that there is a place for a magazine which caters to the needs of college students interested in mathematics, and which features research papers on the level of undergraduate students. These plans and ideals have been realized and in each issue of *The Pentagon* there appear papers by undergraduates.

An initial step toward encouraging college students to attempt something beyond the ordinary routine of class room procedure was taken by the *Michigan*

*Alpha* Chapter of *Kappa Mu Epsilon* at Albion, Michigan. During the Spring of 1940, invitations were sent to the colleges and universities of the state to meet with that chapter for the purpose of organizing a Michigan Undergraduate Association in Mathematics. Papers were solicited for the meeting and a program by eight students from various Michigan institutions was presented. For two years following this initial effort, very successful meetings were held, one at the Michigan State Normal College at Ypsilanti and one at the Central Michigan College of Education at Mount Pleasant. The war made it impossible to continue but plans have been underway for a meeting at Michigan State College.

The National Officers of *Kappa Mu Epsilon* are:

| | |
|---|---|
| President: | E. R. Sleight, Albion College |
| Vice-President: | F. W. Sparks, Texas Technological College |
| Secretary: | E. Marie Hove, Hofstra College |
| Treasurer: | L. F. Ollmann, Hofstra College |
| Historian: | Sister Helen Sullivan, O.S.B., Mount St. Scholastica College |

The name, location, and date of founding of chapters follow:

| | | |
|---|---|---|
| Oklahoma Alpha | Northeastern State College | 1931 |
| Iowa Alpha | Iowa State Teachers College | 1931 |
| Kansas Alpha | Pittsburg State Teachers College | 1932 |
| Missouri Alpha | Southwestern Teachers College | 1932 |
| Mississippi Alpha | State College for Women | 1932 |
| Mississippi Beta | Mississippi State College | 1932 |
| Nebraska Alpha | Wayne State Teachers College | 1933 |
| Illinois Alpha | State Normal University | 1933 |
| Kansas Beta | Emporia State Teachers College | 1934 |
| Alabama Alpha | Athens College | 1935 |
| New Mexico Alpha | University of New Mexico | 1935 |
| Illinois Beta | Charleston State Teachers College | 1935 |
| Alabama Beta | State Teachers College | 1935 |
| Louisiana Alpha | Louisiana State University | 1936 |
| Alabama Gamma | Alabama College | 1937 |
| Ohio Alpha | Bowling Green State University | 1937 |
| Michigan Alpha | Albion College | 1937 |
| Missouri Beta | Warrensburg State Teachers College | 1938 |
| South Carolina Alpha | Coker College | 1940 |
| Texas Alpha | Texas Technological College | 1940 |
| Texas Beta | Southern Methodist University | 1940 |
| Kansas Gamma | Mount St. Scholastica College | 1940 |
| Iowa Beta | Drake University | 1940 |
| New Jersey Alpha | Upsala College | 1940 |
| Ohio Beta | College of Wooster | 1941 |
| Tennessee Alpha | Tennessee Polytechnic Institute | 1941 |
| New York Alpha | Hofstra College | 1942 |
| Michigan Beta | Central Michigan College of Education | 1942 |
| Illinois Gamma | Chicago Teachers College | 1942 |
| New Jersey Beta | Montclair State Teachers College | 1944 |
| Illinois Delta | College of St. Francis | 1945 |
| Michigan Gamma | Wayne University | 1946 |

## NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.*

### POSTDOCTORAL FELLOWSHIP OF SIGMA DELTA EPSILON

Sigma Delta Epsilon, Graduate Women's Scientific Fraternity, regularly offers a postdoctoral fellowship amounting to $1500. Applications for the year, 1947–48, should be submitted before February 1, 1947, to the Fellowship Board authorized to make the award.

Women with the equivalent of a Ph.D. degree, carrying on research, in the mathematical, physical, or biological sciences, who need financial assistance and give evidence of high ability and promise are eligible. During the term of her appointment the appointee must devote the major part of her time to the approved research project, and not engage in other work for remuneration (unless such work shall have received the written approval of the Board before the award of the fellowship).

Application blanks may be secured from Dr. Louise S. McDowell, 28 Dover Road, Wellesley 81, Mass. Announcement of the award will be made early in March.

### PERSONAL ITEMS

Lehigh University makes the following announcements: Assistant Professors F. S. Beale, E. H. Cutler and A. E. Pitcher have returned from leaves of absence; J. O. Chellevold and R. S. Wentworth have been appointed to assistant professorships.

The United States Naval Academy makes the following announcements: Assistant Professors H. C. Ayres, N. H. Ball, A. E. Currier and J. R. Hammond have been promoted to associate professorships; the following have been appointed to associate professorships: Assistant Professor R. P. Bailey of Lafayette College and Assistant Professor L. H. Chambers of Marshall College; the following to assistant professorships: J. M. Holme, J. P. Hoyt, J. F. Milos, Dr. K. L. Palmquist, Dr. J. F. Paydon, Dr. V. N. Robinson, Dr. S. S. Saslaw.

Professor W. E. Anderson of Miami University has retired as head of the department but is continuing to teach on a part-time basis.

Assistant Professor L. Virginia Carlton of Wesleyan College, Macon, Georgia, has been appointed to an assistant professorship at Centenary College.

G. S. Cook has been appointed to an assistant professorship at Colorado School of Mines.

Dr. Paul Erdös has been appointed to a research professorship at Syracuse University.

W. W. Gandy has been appointed to an associate professorship at Northwestern State College, Louisiana.

Mary A. Goins has been appointed to an assistant professorship at Marshall College.

Associate Professors L. H. McFarlan and A. H. Taub of the University of Washington have been promoted to professorships.

Assistant Professor Thirza A. Mossman of Kansas State College has been promoted to an associate professorship.

F. C. Mosteller has been appointed lecturer in the department of social relations of Harvard University.

Professor C. A. Reagan of Friends University, Wichita, Kansas, has been appointed acting president.

J. K. Reckzeh of the University of Kentucky has been appointed to an assistant professorship at State Teachers College, Jersey City, New Jersey.

Professor J. B. Rosenbach of Carnegie Institute of Technology has been appointed assistant head of the department of mathematics.

Professor L. W. Sheridan of the College of Mount St. Vincent has been appointed to an associate professorship at the College of St. Thomas, St. Paul, Minnesota.

Professor J. A. G. Shirk of Kansas State Teachers College, Pittsburg, Kansas, has retired as head of the department but will continue his teaching duties. Professor R. G. Smith has been appointed head of the department.

Dr. K. H. Stahl has been appointed assistant professor of engineering mathematics at the University of Colorado.

Professor W. H. Watson of the University of Saskatchewan has resigned to accept an appointment with the Canadian Government.

Professor J. H. Zant of Oklahoma Agricultural and Mechanical College has been appointed director of instruction of the Okmulgee Branch, newly established to give instruction to veterans.

The following appointments to instructorships are announced:

The University of Buffalo: Mrs. Joan S. Anderson, V. N. Behrns, Mrs. Jeanne J. Dinwoodie, Mrs. Lorraine W. Farber, Lillian Gough, June M. McArtney, Mabel D. Montgomery, N. H. Sampson, F. C. Warner, Mrs. Ina W. Welmers

Lehigh University: L. Benson, W. Hibbard, H. A. Seebald, R. H. Spohn, K. C. Walters, R. W. Young

Mississippi Southern College: John Jones, Jr., J. T. Lewandowski

Union College: H. K. Holt, E. F. Gillette, E. F. Ormsby.

The United States Naval Academy: M. V. Gibbons, E. C. Gras, T. A. Lamke, Joseph Milkman, J. W. Popow, R. W. Rector, J. A. Tierney, E. C. Watters, J. H. White

The University of Colorado (engineering mathematics): George Barnes, D. L. Barrick, F. J. Casey, R. H. Glass, E. W. Grigs, P. F. Hultquist, V. J. Moore, L. W. Rutland, Jr., M. E. Sperline

The University of Maine: Dr. W. B. Caton

Professor Dunham Jackson of the University of Minnesota died November 6, 1946. He was a charter member of the Mathematical Association.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### ANNUAL MEETING OF THE KANSAS SECTION

The thirty-first annual meeting of the Kansas Section of the Mathematical Association of America was held at the Kansas State Teachers College in Emporia on Saturday, April 13, 1946. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics. Mr. Edison Greer, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was one hundred and fifteen including the following thirty-four members of the Association: Sister M. Nicholas Arnoldy, Wealthy Babcock, W. D. Bemmels, Florence Black, L. E. Curfman, Lucy T. Dougherty, Paul Eberhart, Sister Ann Elizabeth, W. H. Garrett, Laura Z. Greene, Edison Greer, J. R. Hanna, J. O. Hassler, A. J. Hoare, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, Sister Jeannette Obrist, O. J. Peterson, P. S. Pretz, G. B. Price, C. B. Read, J. A. G. Shirk, G. W. Smith, R. G. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, E. B. Wedel, J. J. Wheeler, Ferna E. Wrestler.

At the business meeting the following officers were elected for next year: Chairman, C. A. Reagan, Friends University; Vice-Chairman, Sister M. Helen Sullivan, Mount St. Scholastica College; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *Varying definitions of mathematical terms*, by Professor C. B. Read and Professor J. R. Hanna, University of Wichita.

The authors presented a survey of recent representative texts, illustrating situations in which authorities use ambiguous or incomplete definitions of mathematical terms, fail to state limitations or assumptions made, or fail to point out the existence of alternative definitions. Illustrative concepts included mantissa, asymptote, principal values of inverse trigonometric functions, definitions of terms common to mathematics and related fields, order of operations, exponents, and significant figures. Those interested in further details may obtain the complete study by requesting *University Studies, Number 17*, from the University of Wichita.

2. *Mathematics for women*, by Sister M. Helen Sullivan, Mount St. Scholastica College.

The aim of the paper is to convey a basic, educational attitude which the writer feels is missing from the present day philosophy of teaching mathematics. The speaker's thesis is this—that since the majority of women are destined to be homemakers, our approach in the teaching of mathematics in women's institutions must be entirely different from that heretofore employed. We have erred in using text books and other devices that cater to the tastes and

interests of men. Mathematics has much to offer in the development of a well-rounded feminine personality. It is the task of teachers of mathematics in women's schools to employ all the forces of mathematics in the training of women.

3. *Common factors in college and high school methods of teaching mathematics*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler discussed five teaching problems shared in common by college and high school teachers of mathematics that need to be solved for successful teaching: (1) teaching how to interpret in symbols a quantitative situation expressed in words; (2) teaching students to understand the meaning and use of symbols; (3) securing proficiency in the mechanical manipulation of mathematical symbols; (4) developing and utilizating the power of imagination; and (5) teaching for transfer.

4. *College entrance tests in mathematics*, by Professor W. T. Stratton, Kansas State College, and Dean E. B. Stouffer, University of Kansas.

Professor Stratton gave a brief account of the development of the testing program at Kansas State, and presented considerable data to show that the program is well worth while. The plan of procedure followed at present is to have the freshmen tests given during freshmen week, the papers graded, and the results in the hands of the assigners before the students are assigned. Students failing to come up to a certain standard are placed in a non-credit course their first semester without reference to the amount of high school credit they presented. The objectives to be gained by this program are: (1) to reclaim those who have not had a chance in high school to get the subject, and (2) to separate this failing group from the regular classes so that a higher standard of work can be attained, and so as to place the lower group where they may avoid the stigma of failing in case they have the ability to do college work. Dean Stouffer said that beginning with the present academic year the Department of Mathematics of the University of Kansas has used entrance tests to supplement the information on high school records in order to place freshmen in the courses they are prepared to carry successfully. He stated that the number of students who were required to repeat courses does not greatly exceed the number who were permitted to enter courses in advance of those they would normally enter. The plan will be continued and further reports will be given.

5. *Some mathematical considerations of supersonic flight*, by Professor C. B. Tucker, Kansas State Teachers College.

The speaker discussed the methods available to determine the behavior of an airfoil moving at speed greater than sound. The definitions of "shock wave" and "Mach angle" were introduced. Using the necessary assumptions, the coefficients of lift, drag and moment, and the center of pressure were developed for a double wedge airfoil by the Ackeret theory. These coefficients were compared to those of other airfoils and the results of wind-tunnel tests.

6. *Application of mathematical statistics to agricultural experimentations*, by H. C. Fryer, Kansas State College, introduced by the Secretary.

A discussion was given of certain types of data which require a normalizing transformation before the analysis of variance can be used. An experiment involving tests of cattle fly sprays was used for illustration.

7. *Report on board of governors meeting*, by Professor G. W. Smith, University of Kansas.

Professor Smith gave a report of the meetings of the board of Governors of the Association which were held in Chicago in November 1944 and 1945. He reviewed briefly the formation of the Board and explained the nature and the extent of its work.

ANNA MARM, *Secretary*

## ANNUAL MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan at Ann Arbor on Saturday, April 13, 1946. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts and Letters. Morning and afternoon sessions and a luncheon-business meeting were held, at all of which the Chairman, Professor J. W. Bradshaw, presided.

About seventy persons attended the meeting including the following thirty-six members of the Association: H. M. Ackley, J. F. Arena, J. W. Baldwin, W. D. Baten, W. M. Borgman, J. W. Bradshaw, R. V. Churchill, C. J. Coe, A. H. Copeland, C. C. Craig, P. S. Dwyer, C. M. Erikson, C. H. Fischer, K. W. Folley, J. W. Foust, J. S. Frame, V. G. Grove, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, P. S. Jones, Wilfred Kaplan, Theodore Lindquist, E. D. Rainville, C. C. Richtmeyer, E. H. Rothe, L. J. Rouse, T. R. Running, E. R. Sleight, T. H. Southard, G. G. Speeker, B. M. Stewart, R. L. Wilder.

At the business meeting the nominating committee, consisting of Professors H. M. Ackley, W. D. Baten and K. W. Folley, nominated Professor E. E. Ingalls of Albion College as Chairman and Professor L. J. Rouse of the University of Michigan as Secretary-Treasurer, and these nominees were unanimously elected.

At the morning and afternoon sessions the following program of eight papers was presented.

1. *Four great mathematicians*, by Professor E. R. Sleight, Albion, College.

Professor Sleight presented a review of the life, work and influence of four early British mathematicians. This paper has been published in the *National Mathematics Magazine.*

2. *Representation of orbits in the restricted problem of three bodies*, by Professor G. P. Loweke, Wayne University, introduced by Professor A. L. Nelson.

By a suitable transformation the speaker set up a correspondence between the orbits of the infinitesimal body and a one parameter family of surfaces in space, and studied the orbits in the light of this representation. This paper was published as the speaker's doctoral dissertation, University of Berlin, 1936.

3. *A problem concerning orthogonal trajectories*, by Professors Fritz Herzog and C. P. Wells of Michigan State College.

The authors considered two families of curves in the plane, namely $\phi(x, y) = C$ and $\psi(x, y) = C$, the two families being orthogonal, and the functions $\phi$ and $\psi$ both harmonic. The two families are then said to possess property $E$ with respect to the second family if the arc lengths along all curves of the second family are divided proportionally by curves of the first family. The authors showed that if the property holds with respect to the second family, then the property is likewise enjoyed with respect to the first family. They then showed that all families possessing this property fall into four cases, two trivial and two more complicated.

4. *On theorem* 23, *book* 1, *Menelaus Sphaerics*, by Donat Kazarinoff, University of Michigan, introduced by the Secretary.

Mr. Kazarinoff criticized the existing proof of the theorem and offered an amended proof.

5. *Transfinite ordinals*, by Professor K. W. Folley, Wayne University.

Professor Folley's paper was an extension of that of Ben Dushnik published in the *Bulletin of the American Mathematical Society*, vol. 37, 1931. He showed that the Dushnik theorem could be extended to some, but not all, numbers of the second kind.

6. *Conditional invariants and quadric surfaces*, by Professor G. Y. Rainich, University of Michigan.

A conditional invariant of a form (tensor) is an expression in terms of the coefficients of the form (components of the tensor) which remains invariant under a transformation of coordinates if, and only if, certain invariants of the form (tensor) vanish. Professor Rainich showed how such conditional invariants characterize geometric properties of degenerate forms. As a simple example he found the expression for the distance between two parallel lines in terms of the coefficients of the single equation representing the two lines.

7. *Space drawings with a trimetric ruler*, by Professor J. S. Frame, Michigan State College.

The trimetric ruler is an invention of Professor Frame greatly facilitating the drawing of solid figures in parallel perspective. It consists of a scalene triangle of cardboard or plastic with a scale on each edge. Professor Frame developed the theory of the ruler and illustrated its use.

8. *On a certain system of congruences*, by Professor Ben Dushnik, University of Michigan, introduced by the Secretary.

The speaker showed that the problem of solving the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=n$$

in positive integers is equivalent to finding the points of the surface $x^3+y^3+z^3-kxyz=0$, $(k=1, 2, 3, \cdots)$, having positive integral coordinates. He presented several simple examples.

L. J. ROUSE, *Secretary*

## MAY MEETING OF THE WISCONSIN SECTION

The fourteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, on Saturday, May 4, 1946. Sister Mary Felice presided at the morning session, and Professor Morris Marden presided at the afternoon session.

There were sixty-nine in attendance including the following twenty-six members of the Association: L. K. Adkins, K. J. Arnold, R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, R. H. Bruck, H. H. Conwell, Sister Mary Felice, J. D. Fitzpatrick, Fannie Hopkins, R. C. Huffer, J. F. Kenney, S. C. Kleene, C. C. MacDuffee, Morris Marden, A. C. Moeller, G. A. Parkinson, H. P. Pettit, Abraham Spitzbart, P. L. Trump, J. I. Vass, R. D. Wagner and Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, H. P. Evans, University of Wisconsin; Program Committee, L. K. Adkins, LaCrosse State Teachers College, Louise A. Wolf, University of Wisconsin in Milwaukee (Chairman). The election of R. C. Huffer, Beloit College, as governor of Region Number Nine was announced. A committee of the section was authorized to push the cause of mathematical education and to maintain contact with the coördinating committee of the national association. It was voted to hold the next meeting in May, 1947, at the University of Wisconsin, and to invite the Wisconsin Association of Physics Teachers to participate in making program plans.

At the morning session the following papers were presented:

1. *Some aspects of the theory of loops*, by Professor R. H. Bruck, University of Wisconsin.

Professor Bruck discussed the notion of a "generalized ring" $R\equiv R(+, \cdot)$ forming a loop under addition, obeying the usual two-sided distributive law, and such that the non-zero elements of $R$ form a group $G$ under multiplication. He showed that, when the group $G$ is given, the existence of $R$ is equivalent to the existence of a permutation $S$ of $G$ with certain prescribed properties; and that addition in $R$ may be defined in terms of $S$. Conditions were exhibited necessary and sufficient that $R$ be the basis of a coördinate system for a projective geometry. The discourse concluded with an explanation of the known one-

to-one correspondence between finite projective geometrics with $n+1$ points on a line and complete sets of $(n-1)$ mutually orthogonal $n$-rowed latin squares.

2. *A brief survey of meteorology*, by W. R. Jarmain, introduced by the Secretary.

The speaker pointed out that the development of meteorology as applied to weather forecasting has been slow, due to the fact that it required some quite recently invented scientific devices. Not until the twentieth century was the subject on a mathematical basis, and even today, forecasting is still very much an art as well as a science. The importance of Bjerknes' polar front theory in making day to day predictions was emphasized. Five day forecasting is still in its infancy, and considerable research remains to be done in this field. Extraterrestrial influences on the earth's weather were deemed rather unimportant. The speaker indicated that meteorology has made great strides in recent years, and has a bright future.

3. *Social implications of atomic energy*, by Professor Elda Anderson, Milwaukee Downer College, introduced by Professor Ethelwynn R. Beckwith.

At the afternoon session reports were given by members of the educational committee referred to above, indicating some aspects of the status of mathematical education in Wisconsin and activities of the National Committee for the Co-ordination of Studies in Mathematical Education. There followed a discussion in which there was general and enthusiastic participation. It was agreed that the Wisconsin Section should proceed to organize and plan for the purpose of cooperating in all ways feasible in a constructive effort to meet the problems of mathematical education in Wisconsin's secondary schools.

PAUL L. TRUMP, *Secretary*

## ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The fifth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at The Cooper Union, Cooper Square, New York, N. Y., on Saturday, May 4, 1946. Professor H. E. Wahlert, Vice-Chairman of the Section, presided at the morning session; and Professor H. E. Miller, Chairman of the Section, presided at the afternoon session.

There were ninety present, including the following fifty-four members of the Association: R. G. Archibald, Brother Bernard Alfred (Welch), C. B. Boyer, A. D. Bradley, Benjamin Braverman, A. B. Brown, Jewell Hughes Bushey, H. R. Cooley, W. H. H. Cowles, W. H. Fagerstrom, Edward Fleisher, R. M. Foster, Harriet M. Griffin, J. I. Griffin, George Grossman, Frank Hawthorne, R. V. Heath, Morris Hertzig, J. H. Hlavaty, Solomon Hurwitz, L. C. Hutchinson, F. W. John, R. A. Johnson, Aida Kalish, Herman Karnow, Edward Kasner, L. S. Kennison, E. R. Kiely, Nathan Lazar, C. H. Lehmann, Emanuel Levine, H. F. MacNeish, John Mandel, D. May Hickey Maria, Joseph Milkman, F. H. Miller, E. C. Molina, P. B. Norman, L. F. Ollmann, Water Prenowitz, Edward

Rayher, W. D. Reeve, S. G. Roth, Leila Rubashkin, Charles Salkind, Aaron Shapiro, Lao G. Simons, James Singer, E. R. Stabler, H. E. Wahlert, Israel Wallach, Alan Wayne, V. H. Wells, R. C. Yates.

At the opening of the morning session Dean George F. Bateman welcomed the Section to The Cooper Union. At the opening of the afternoon session a brief business meeting was held at which the following officers were elected for the coming year: Chairman, H. E. Wahlert, New York University; Vice-Chairmen, W. H. H. Cowles, Pratt Institute, and Morris Hertzig, Forest Hills High School; Secretary, C. B. Boyer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School.

The program consisted of the following papers:

1. *Cartesian geometry from Fermat to Lacroix*, by Professor C. B. Boyer, Brooklyn College.

The basic principle of coördinate geometry was discovered independently by Descartes and Fermat, but the works of these men differ markedly in emphasis. The speaker therefore proposed that the problem of finding the equations of given loci be designated "analytic geometry in the sense of Descartes," and that the inverse aspect, the study of curves determined by given equations, be referred to as "analytic geometry in the sense of Fermat." It was pointed out that the Cartesian aspect dominated thought for well over a century, but that the point of view of Fermat was represented in England by Newton and Maclaurin and on the Continent by Euler and Cramer. The widely-held opinion that Descartes arithmetized geometry was shown to be inconsistent with the Cartesian goal of exhibiting the geometric "constructibility" of determinate and indeterminate equations. A true arithmetization of geometry was hinted at by Lagrange, but it was finally carried out in 1795–1798 by Monge and Lacroix. Comparing this "analytical revolution" in France with the so-called chemical revolution of the same period, the speaker proposed that the idea of expressing a geometry in algebraic language be designated "analytic geometry in the sense of Monge and Lacroix."

2. *Geometry of ship waves*, by Professor J. J. Stoker, New York University, introduced by Professor H. E. Wahlert.

3. *Evaluating a syllabus in experimental geometry a priori*, by Charles Salkind, Samuel J. Tilden High School.

To replace the present tenth-year course, a group of New York City teachers of mathematics have prepared an instrument titled, "Experimental (non-Regents) Geometry Course." The speaker pointed out that the alleged superiority of the course lies in (1) exhibiting geometry as an elementary illustration of the scientific method (induction); (2) clarifying the complementary roles of experimentation and deduction. He held, however, that an a priori evaluation of the course discloses that it adds to existing burdens without removing any of the fundamental difficulties; and he questioned the validity of the proposed

program for the following reasons. (1) Procedures based *primarily* on measurement are amply provided for in the junior high school. (2) The results of deductive reasoning do *not* gain credence from experimentation and measurement. Contrary evidence exists to show that overindulgence in informal methods *inhibits* the acceptance of deductive results. (3) Scientific induction should be taught to secondary pupils, but by teachers specially qualified to do so.′ (4) The proposed course fails to meet the standards of a properly-planned educational experiment.

4. *Coordinating high school and college mathematics*, by Professor W. D. Reeve, Teachers College, Columbia University.

Professor Reeve's paper appears as the first article in the present issue.

5. *Infinity in art*, by Professor Edward Kasner, Columbia University.

The speaker described the construction and properties of what he called an *infinite Christmas tree.* From the upper end of a vertical unit line segment, two mutually perpendicular segments of length $\frac{1}{2}$ are constructed to make angles of 135° with the unit segment; from the extremities of each of these, two new mutually perpendicular segments of length $\frac{1}{4}$ are similarly drawn at 135° to the half-unit segments; and so on ad infinitum. The distance from the base to any one of the asymptotic limiting points, measured along the trunk and branches, is two units. The class of nodes (or branching points) is denumerable and hence has as its number Cantor's $\aleph_0$; the points of condensation (terminal buds) are non-denumerable and their number is that of the continuum $c$. The speaker suggested the determination of the area of the smallest circumscribing rectangle, the study of the design of nodes and buds, and the construction of an analogous configuration in three dimensions.

C. B. BOYER, *Secretary*

## CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, April, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April 18–19, 1947
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

# The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in November 1944 to Professor R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," published in the *National Mathematics Magazine,* vol. 15 (1941), pp. 331-356.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production. Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 — NUMBER 2


CONTENTS




FEBRUARY 1947

# The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)




EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the Secretary-Treasurer, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the first of the month in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---


Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, \$4 a Year, To Others, \$5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# SOME IDENTITIES IN THE THEORY OF DETERMINANTS*

G. B. PRICE,† University of Kansas

"Cayley once said to me, in conversation, that if he had to give fifteen lectures on the whole of mathematics, he would devote *one* of them to determinants."—Felix Klein in *Elementary Mathematics from an Advanced Standpoint* [**23**, p. 143].

"Open the book where you will and you find yourself in the midst of a live question having vital connections at the present day with other live branches of mathematics. This is no small praise when one recalls the possibilities for barren formalism which the subject of determinants presents."—Maxime Bôcher in his review of Kowalewski's *Determinantentheorie* [**10**, p. 136].

"In fact, the importance of the concept of determinant has been, and currently is, vastly over-estimated. Systems of quations can be solved as easily and neatly without determinants as with, as is illustrated in Chapter I of this Monograph. In fact, perhaps ninety per cent of matric theory can be developed without mentioning a determinant. The concept is necessary in some places, however, and is very useful in many others, so one should not push this point too far."—C. C. MacDuffee in *Vectors and Matrices* [**27**, Introduction, p. v].

**1. Introduction.** Cayley considered determinants one of the central theories in all of mathematics; Bôcher emphasized chiefly their connections with other branches of mathematics; andMacDuffee points out that they are not so essential in some parts of algebra as they have long been considered. Mathematicians today, their judgment ripened by experience, probably would agree that the theory of determinants belongs to the class of tool theories and is not a subject of major interest for itself alone. Furthermore, the mathematics of today, almost completely dominated by algebra, topology, and similar abstract disciplines, presents but few situations in which determinants are needed as tools. The theory of determinants provides many powerful ones, however, and there is every reason to believe that it will yet play an important rôle in further developments in mathematics.

This exposition treats certain fundamental identities, knowledge of which is necessary for any extensive applications of determinants. Most of these identities concern determinants whose elements are themselves determinants. In particular, the paper treats the Laplace expansion of a determinant; the Binet-Cauchy multiplication theorem; Sylvester's theorem of 1839 and 1851; the Sylvester-Franke theorem; the Bazin-Reiss-Picquet theorem; the Cauchy, Jacobi, Franke, and Reiss theorems; and Sylvester's theorem on superdeterminants. A good modern proof of each of these except the first—which is too well known to require proof—will be given. The proofs emphasize matrix methods. The two key results are the Laplace expansion and the Sylvester-Franke theorem, and their proofs go back to first principles. No completely simple and direct proof of the Sylvester-Franke theorem is known, and discovery of one would be an event of real interest.

* An address presented at the invitation of the Program Committee at the Cornell meeting of the Association on August 20, 1946.

† Guggenheim Fellow.

Finally, the early history of these theorems is traced, and references are given to the original papers. Alternative proofs are indicated, and the interrelations of the theorems are pointed out.

**2. Notation and definitions.** The ordered array

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}, \tag{2.1}$$

with elements $a_{ij}$ which are complex numbers, will be called a matrix; it will be indicated also by the briefer notation $(a_{ij})$, $i=1, \cdots, m$; $j=1, \cdots, n$. The transpose $A'$ of $A$ is the matrix whose rows are respectively the columns of $A$. The determinant whose elements are the elements in the square matrix $A$ will be denoted by $|A|$ or $|a_{ij}|$.

Let $M$ and $N$ denote the sets of integers $1, 2, \cdots, m$ and $1, 2, \cdots, n$ respectively. Let $I^{(k)}$ denote a combination of $k$ integers $i_1, i_2, \cdots, i_k$ selected from $M$, and let $I_i^{(k)}$, $i=1, 2, \cdots, C(m, k)$, be the complete set of combinations of $k$ integers that can be formed from the $m$ integers in $M$. Similarly, let $J_j^{(k)}$, $j=1, 2, \cdots, C(n, k)$, be the complete set of combinations of $k$ integers $j_1, j_2, \cdots, j_k$ that can be formed from the $n$ integers in $N$. The integers in both $I_i^{(k)}$ and $J_j^{(k)}$ are arranged in their natural order. It should be emphasized that the numbering of the combinations $I_i^{(k)}$ and $J_j^{(k)}$ is entirely arbitrary, but that a definite order is chosen now which will be used throughout the remainder of the paper. Finally, let $M-I_i^{(k)}$ (respectively $N-J_j^{(k)}$) denote the set of $m-k$ integers ($n-k$ integers), arranged in their natural order, in $M$ (in $N$) but not in $I_i^{(k)}$ (not in $J_j^{(k)}$).

Next, let the submatrix of $A$ formed by the $k$ rows designated by the $k$ integers in $I_i^{(k)}$ be denoted by $A(I_i^{(k)})$. Similarly, let the submatrix of $A$ formed by the $k$ columns designated by the $k$ integers in $J_j^{(k)}$ be denoted by $A(J_j^{(k)})$. Finally, let $A(I_i^{(k)} \cdot J_j^{(k)})$ denote the submatrix of $A$ which is contained in both $A(I_i^{(k)})$ and $A(J_j^{(k)})$. It follows that $A(I_i^{(k)} \cdot J_j^{(k)})$ is a square matrix with $k$ rows and columns.

The $k$th compound of $A$, denoted by $A^{(k)}$, is defined for each integer $k$ such that $1 \leqq k \leqq m$, $1 \leqq k \leqq n$, by

$$A^{(k)} = (|A(I_i^{(k)} \cdot J_j^{(k)})|), \quad i = 1, 2, \cdots, C(m, k); j = 1, 2, \cdots, C(n, k). \tag{2.2}$$

The element in the $i$th row and $j$th column of $A^{(k)}$ is thus the $k$th order determinant $|A(I_i^{(k)} \cdot J_j^{(k)})|$ of a submatrix of $A$.

When $A$ is a square matrix with $n$ rows and columns, $A(N-I_i^{(k)} \cdot N-J_j^{(k)})$ will denote the submatrix of $A$ which is complementary to $A(I_i^{(k)} \cdot J_j^{(k)})$; it is obtained from $A$ by striking out the rows designated by the integers in $I_i^{(k)}$ and the columns designated by the integers in $J_j^{(k)}$. For the sake of completeness and symmetry, $A(N-I_1^{(n)} \cdot N-J_1^{(n)})$ will be defined to be the matrix with the single

element 1. Thus when $A$ is a square matrix with $n$ rows and columns, the $k$th adjoint compound of $A$, denoted by $\operatorname{adj}^{(k)} A$, is defined for each $k$ such that $1 \leqq k \leqq n$ by

$$\operatorname{adj}^{(k)} A = ((-1)^{s_i+s_j} \,|\, A(N - I_i^{(k)} \cdot N - J_j^{(k)}) \,|\,)', \quad i = 1, 2, \cdots, C(n, k); j = 1, 2, \cdots, C(n, k). \tag{2.3}$$

Here $s_i$ and $s_j$ are the sums of the $k$ integers in $I_i^{(k)}$ and $J_j^{(k)}$, respectively. Thus $\operatorname{adj}^{(k)} A$ is a compound matrix whose elements are signed determinants with $(n-k)$ rows and columns. Note the prime which indicates that the matrix in the right member of (2.3) is to be transposed. It may be observed that $\operatorname{adj}^{(1)} A$ is the matrix usually denoted* by $\operatorname{adj} A$ and called the adjoint of $A$.

When $A$ and $B$ are any two square matrices, $A[B(J_i^{(k)})/A(J_j^{(k)})]$, or $A[B_i^{(k)}/A_j^{(k)}]$ for short, will be used to denote the matrix obtained by replacing in order the $k$ columns $A(J_j^{(k)})$ in $A$ by the $k$ columns $B(J_i^{(k)})$ in $B$.

Finally, notation will be required for bordered matrices. Let $R^{(h)}$ and $S^{(h)}$ denote the set of integers $1, 2, \cdots, h$ in $N$. Let $U_i^{(k)}$, $i=1, 2, \cdots, C(n-h, k)$ (respectively $V_j^{(k)}$, $j=1, 2, \cdots, C(n-h, k)$) denote the complete set of combinations of $k$ integers each that can be selected from the sets $N-R^{(h)}(N-S^{(h)})$. Then $A(R^{(h)}+U_i^{(k)} \cdot S^{(h)}+V_j^{(k)})$, $i, j=1, 2, \cdots, C(n-h, k)$, will denote all the submatrices of $A$ that can be formed by bordering $A(R^{(h)} \cdot S^{(h)})$ with $k$ of the remaining rows and columns.

**3. Laplace's expansion.** Let $|A|$ be any $n$th order determinant and $k$ any integer such that $1 \leqq k \leqq n$. Then the Laplace expansion of $|A|$ is

$$|A| = \sum (-1)^{s_i+s_j} \,|\, A(I_i^{(k)} \cdot J_j^{(k)}) \,|\,|\, A(N - I_i^{(k)} \cdot N - J_j^{(k)}) \,|, \tag{3.1}$$

where $s_i$ and $s_j$ are defined in (2.3). The summation extends either over the set $i=1, 2, \cdots, C(n, k)$ with $j$ arbitrary, or over the set $j=1, 2, \cdots, C(n, k)$ with $i$ arbitrary. This result is too well known to require proof here (see [**1**, pp. 78–81], [**9**, pp. 24–26], [**24**, pp. 34–36]).

The following formulas follow from Laplace's expansion:

$$A^{(k)} \operatorname{adj}^{(k)} A = \operatorname{adj}^{(k)} A A^{(k)} = |A| I, \tag{3.2}$$

$$|A^{(k)} \operatorname{adj}^{(k)} A| = |\operatorname{adj}^{(k)} A A^{(k)}| = |A|^{C(n,k)}. \tag{3.3}$$

Here $A$ is any square matrix with $n$ rows and columns and $I$ is the identity matrix.

**4. The Binet-Cauchy multiplication theorem.** The multiplication theorem provides an evaluation of the determinant of the product of two matrices.

THEOREM. *Let $A$ and $B$ be two matrices with $m$ rows and $n$ columns, and let*

---

* The notations $A^{(k)}$ and $\operatorname{adj}^{(k)} A$ are standard; see, for example, Aitken [**1**, pp. 90–91] and Albert [**2**, p. 57]. Aitken, along with many writers in the British Isles, uses *adjugate* where *adjoint* is commonly used in America.

*$AB'$ denote the row by column matrix product of $A$ and $B'$. Then*

$$|AB'| = 0 \qquad m > n, \tag{4.1}$$

$$= |A||B| \qquad m = n, \tag{4.2}$$

$$= \sum_{j=1}^{C(n,m)} |A(J_j^{(m)})||B(J_j^{(m)})| \qquad m < n. \tag{4.3}$$

Although there are at least two standard proofs of this theorem (see [1, pp. 80–81, 85–87], [24, pp. 60–70]), the most elegant one is based on the following obvious identity for partitioned matrices:

$$\begin{pmatrix} I & A \\ O & I \end{pmatrix}\begin{pmatrix} A & O \\ -I & B' \end{pmatrix} = \begin{pmatrix} O & AB' \\ -I & B' \end{pmatrix},$$

where $I$ is the identity matrix and $O$ is the zero matrix. Multiplication by the first factor on the left of this equation replaces each row of the other factor by itself plus certain multiples of other rows. Since these operations do not change the value of a determinant,

$$\begin{vmatrix} A & O \\ -I & B' \end{vmatrix} = \begin{vmatrix} O & AB' \\ -I & B' \end{vmatrix}. \tag{4.4}$$

Next, evaluate these two determinants by means of Laplace's expansion. The determinant on the right in all cases is $(-1)^{mn+n}|AB'|$. The one on the left is clearly $|A|\ |B'|$ if $m=n$, but in the other cases it must be examined more closely. Written out in full, this determinant is

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & b_{11} & b_{21} & \cdots & b_{m1} \\ 0 & -1 & \cdots & 0 & b_{12} & b_{22} & \cdots & b_{m2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & b_{1n} & b_{2n} & \cdots & b_{mn} \end{vmatrix}.$$

If $m>n$, an expansion of this determinant by Laplace's method, using minors of $m$ rows and columns selected from the first $m$ rows, shows that its value is zero; (4.1) follows. Exactly the same expansion is used when $m \leqq n$. Consider a term of the expansion in which the first factor is one of the determinants $|A(J_j^{(m)})|$. The second factor can be expanded by Laplace's method in terms of minors of $m$ rows and columns formed from the last $m$ columns. There is a single term in this expansion, and its value is $|[B(J_j^{(m)})]'|$, or $|B(J_j^{(m)})|$, except perhaps for sign. A calculation of the signs shows that the complete product which

includes the factor $|A(J_j^{(m)})|$ is $(-1)^{mn+n}|A(J_j^{(m)})||B(J_j^{(m)})|$. All terms in the expansion are zero except those for which the first factor is one of the determinants $|A(J_j^{(m)})|$. Thus, when $m \leqq n$, the relation

$$\sum_{j=1}^{C(n,m)} (-1)^{mn+n} |A(J_j^{(m)})| |B(J_j^{(m)})| = (-1)^{mn+n} |AB'|$$

is obtained by equating the evaluations of the two sides of (4.4). Equations (4.2) and (4.3) follow from this relation, and the proof is complete.

An important extension of the multiplication theorem concerns the $k$th compound of the product of two matrices.

THEOREM. *Let $A$ be a matrix with $m$ rows and $n$ columns, and let $B$ be a matrix with $n$ rows and $m$ columns. If $k$ is any integer such that $1 \leqq k \leqq m$ and $1 \leqq k \leqq n$, then*

$$|(AB)^{(k)}| = |A^{(k)}B^{(k)}|, \tag{4.5}$$

$$|AB|^{C(m-1,k-1)} = |A^{(k)}B^{(k)}|. \tag{4.6}$$

The proof of (4.5) follows from the matrix identities

$$\begin{aligned} (AB)^{(k)} &= (|A(I_i^{(k)})B(J_j^{(k)})|) \\ &= \left( \sum_{t=1}^{C(n,k)} |A(I_i^{(k)} \cdot J_t^{(k)})| |B(I_t^{(k)} \cdot J_j^{(k)})| \right) \\ &= (|A(I_i^{(k)} \cdot J_j^{(k)})|)(|B(I_i^{(k)} \cdot J_j^{(k)})|) \\ &= A^{(k)}B^{(k)}. \end{aligned} \tag{4.7}$$

The first equality here results from observing that the element in the $i$th row and $j$th column of $(AB)^{(k)}$ is the determinant of the matrix obtained by multiplying the $i$th combination $A(I_i^{(k)})$ of rows of $A$ into the $j$th combination $B(J_j^{(k)})$ of columns of $B$. The second equality follows from an application of (4.3). The third equality results from the definition of matrix multiplication. The fourth equality results from the definition of the $k$th compound. The proof of (4.5) now follows by equating the determinants of the first and last arrays.

The proof of (4.6) follows from the Sylvester-Franke theorem that $|(AB)^{(k)}| = |AB|^{C(m-1,k-1)}$ (see (6.1) below). Although the proof will not be complete until the Sylvester-Franke theorem has been proved, the result was included here so that the statement of the multiplication theorem would be complete.

The proof of (4.7) is a proof of the following important theorem in matrix theory: the $k$th compound of the product of two matrices is equal to the product of the $k$th compounds of these matrices.

The multiplication theorem has many important applications. For example, the $(m-1)$ measure $(P_1P_2 \cdots P_m)$ of the $(m-1)$ cell determined by the points $P_i$: $(x_{i1}, x_{i2}, \cdots, x_{in})$, $i=1, 2, \cdots, m$, in $n$-space is known [8, pp. 293–296] to be $|AA'|^{1/2}/(m-1)!$, where $A$ is the matrix whose $i$th row is

$$(x_{i1} - x_{m1}, x_{i2} - x_{m2}, \cdots, x_{in} - x_{mn}), \qquad i = 1, 2, \cdots, (m-1).$$

An application of (4.3) yields the formulas of analytic geometry for distance, area, and so on in the following form:

$$(P_1P_2) = \frac{1}{1!}\left[\begin{vmatrix} x_{11} & 1 \\ x_{21} & 1 \end{vmatrix}^2 + \begin{vmatrix} x_{12} & 1 \\ x_{22} & 1 \end{vmatrix}^2 + \cdots + \begin{vmatrix} x_{1n} & 1 \\ x_{2n} & 1 \end{vmatrix}^2\right]^{1/2},$$

$$(P_1P_2P_3) = \frac{1}{2!}\left[\begin{vmatrix} x_{11} & x_{12} & 1 \\ x_{21} & x_{22} & 1 \\ x_{31} & x_{32} & 1 \end{vmatrix}^2 + \begin{vmatrix} x_{11} & x_{13} & 1 \\ x_{21} & x_{23} & 1 \\ x_{31} & x_{33} & 1 \end{vmatrix}^2 + \cdots + \begin{vmatrix} x_{1n-1} & x_{1n} & 1 \\ x_{2n-1} & x_{2n} & 1 \\ x_{3n-1} & x_{3n} & 1 \end{vmatrix}^2\right]^{1/2},$$

and so on. The identity (4.3) in the case $m=2$, $n=3$ is the familiar Lagrange's identity of vector analysis. Again, if $A$ is any matrix whose elements are real, the determinant $|AA'|$ is known **[14, pp. 29–30]** as Gram's determinant. It follows from (4.3) that $|AA'| \geqq 0$. When $A$ is a matrix with two rows, this inequality is Schwarz's inequality. A necessary and sufficient condition that the $m$ vectors which form the $m$ rows of $A$ be linearly independent is $|AA'| > 0$.

**5. Sylvester's theorem of 1839 and 1851.** This theorem gives an important series of representations for the product of two determinants.

THEOREM. *Let $A$ and $B$ be any two square matrices with $n$ rows and columns, and let $k$ be any integer such that $1 \leqq k \leqq n$. Then*

$$(5.1)\qquad \begin{aligned} \sum_{t=1}^{C(n,k)} |A[B(J_t^{(k)})/A(J_i^{(k)})]|\,|B[A(J_j^{(k)})/B(J_t^{(k)})]| &= 0, && i \neq j, \\ &= |A|\,|B|, && i = j. \end{aligned}$$

The most elegant proof of this theorem **[16, vol. II, pp. 73–74]** is obtained by equating corresponding elements in two evaluations of the matrix product $\operatorname{adj}^{(k)} AB^{(k)} \operatorname{adj}^{(k)} BA^{(k)}$. First recall (3.2) and the fact that matrix multiplication is associative. Then

$$\begin{aligned} \operatorname{adj}^{(k)}AB^{(k)}\operatorname{adj}^{(k)}BA^{(k)} &= \operatorname{adj}^{(k)}A\cdot B^{(k)}\operatorname{adj}^{(k)}B\cdot A^{(k)} \\ &= \operatorname{adj}^{(k)}A\cdot |B|\, I\cdot A^{(k)} \\ &= \operatorname{adj}^{(k)}AA^{(k)}\cdot |B|\, I \\ &= |A|\,|B|\, I. \end{aligned}$$

Again,

$$\begin{aligned} &\operatorname{adj}^{(k)}AB^{(k)}\operatorname{adj}^{(k)}BA^{(k)} \\ &\qquad = \operatorname{adj}^{(k)}AB^{(k)}\cdot\operatorname{adj}^{(k)}BA^{(k)} \\ &\qquad = (|A[B(J_j^{(k)})/A(J_i^{(k)})]|)(|B[A(J_j^{(k)})/B(J_i^{(k)})]|) \\ &\qquad = \left(\sum_{t=1}^{C(n,k)} |A[B(J_t^{(k)})/A(J_i^{(k)})]|\,|B[A(J_j^{(k)})/B(J_t^{(k)})]|\right). \end{aligned}$$

In the matrices on the right, the element in the $i$th row and the $j$th column is shown in each case. Equations (5.1) follow by equating corresponding elements in the two evaluations of the matrix product. The proof is complete.

**6. The Sylvester-Franke theorem.** This theorem evaluates $|A^{(k)}|$ in terms of $|A|$.

THEOREM. *Let $A$ be any square matrix with $n$ rows and columns. Then*

$$(6.1) \qquad |A^{(k)}| = |A|^{C(n-1,k-1)} \qquad k = 1, 2, \cdots, n.$$

A proof [24, pp. 94–98] by mathematical induction on $n$ will be given. The theorem is obviously true for $n=2$. It will be shown that it is true for all determinants of order $n$ if it is true for all determinants of order $n-1$.

Consider $a_{nn}$ as a variable and all other elements in $A$ as constants. Then $|A|$ and $|A^{(k)}|$ are functions of $a_{nn}$, and $|A| = Da_{nn}+E$. Here $D$ is the minor of $a_{nn}$ in $|A|$ and $E$ is the determinant obtained by replacing $a_{nn}$ in $|A|$ by 0. There is no loss in generality in assuming that the elements in $A^{(k)}$ are ordered in such a way that $a_{nn}$ occurs only in the submatrix

$$(6.2) \qquad (|A(I_i^{(k)} \cdot J_j^{(k)})|) \qquad i, j = 1, 2, \cdots, r,$$

where $r = C(n-1, k-1)$. Furthermore, $|A(I_i^{(k)} \cdot J_j^{(k)})| = d_{ij}a_{nn}+e_{ij}$, $i, j=1, 2, \cdots, r$, where $d_{ij}$ is a $(k-1)$ rowed minor of $D$ and $e_{ij}$ is a determinant similar to $E$. It follows from the induction hypothesis, namely, the hypothesis that the theorem is true for all determinants of order $(n-1)$, that

$$|d_{ij}| = D^{C(n-2,k-2)} \qquad i, j = 1, 2, \cdots, r.$$

Then the determinant of the matrix in (6.2) is the following polynomial in $a_{nn}$ of degree $r$:

$$(6.3) \qquad D^{C(n-2,k-2)}a_{nn}^{r} + \cdots .$$

Next, observe that

$$(6.4) \qquad (|A(I_i^{(k)} \cdot J_j^{(k)})|) \qquad i, j = r+1, r+2, \cdots, C(n, k),$$

is the matrix in $A^{(k)}$ complementary to the one in (6.2), and that it is the $k$th compound of the matrix in $D$. Then by the induction hypothesis, the determinant of the matrix in (6.4) is $D^{C(n-2,k-1)}$.

Now use Laplace's expansion to expand $|A^{(k)}|$ by minors formed from the first $r$ rows. It follows from results already established that the first term in the expansion is $D^{C(n-2,k-2)+C(n-2,k-1)}a_{nn}^{r}+ \cdots$, a polynomial in $a_{nn}$ of degree $r$. The other terms in the expansion are polynomials in $a_{nn}$ of degree less than $r$. Since $C(n-2, k-2)+C(n-2, k-1) = C(n-1, k-1) = r$, it follows that

$$(6.5) \qquad |A^{(k)}| = D^{r}a_{nn}^{r} + \cdots .$$

Furthermore, by (3.3) and (4.2)

$$
\begin{aligned}
|A^{(k)}|\,|\operatorname{adj}^{(k)}A| &= |A|^{C(n,k)} \\
&= (Da_{nn}+E)^{C(n,k)}.
\end{aligned}
\tag{6.6}
$$

Thus every value of $a_{nn}$ for which $|A^{(k)}|$ vanishes is a zero of $|A|^{C(n,k)}$. But the latter, equal to $(Da_{nn}+E)^{C(n,k)}$, vanishes only for $a_{nn}=-E/D$. Thus all zeros of the polynomial in (6.5) are $a_{nn}=-E/D$, and

$$|A^{(k)}| = D^r(a_{nn}+E/D)^r = (Da_{nn}+E)^r = |A|^r = |A|^{C(n-1,k-1)}.$$

Since the theorem is true for all determinants of the second order, the proof of the theorem follows by a complete induction.

It may be remarked that the proof given above can be varied slightly so that the Fundamental Theorem of Algebra is not required **[24**, pp. 97–98**]**. The proof has assumed $D\neq 0$, but the usual methods can be used to show that the result holds without this restriction.

COROLLARY. *Let $A$ be any square matrix with $n$ rows and columns. Then*

$$|\operatorname{adj}^{(k)}A| = |A|^{C(n-1,k)} \qquad k = 1, 2, \cdots, n. \tag{6.7}$$

The proof follows from (6.6) and (6.1). A more instructive proof, however, is obtained by showing that $|\operatorname{adj}^{(k)}A|=|A^{(n-k)}|$, $k=1, 2, \cdots, (n-1)$, and recalling the convention that $|\operatorname{adj}^{(n)}A|=1$. Assume $1\leqq k\leqq n-1$. From the $i$th row of $|[\operatorname{adj}^{(k)}A]'|$ factor out $(-1)^{s_i}$ for $i=1, 2, \cdots, C(n, k)$. Similarly, from the $j$th column factor out $(-1)^{s_j}$ for $j=1, 2, \cdots, C(n, k)$. Since

$$\sum_{i=1}^{C(n,k)} s_i + \sum_{j=1}^{C(n,k)} s_j = 2\sum_{i=1}^{C(n,k)} s_i$$

and is thus an even number, the value of the determinant is unchanged. The new determinant is $|A^{(n-k)}|$. Then (6.7) follows from (6.1).

**7. The Bazin-Reiss-Picquet theorem.** Recall the meaning of $B[A(J_i^{(k)})/B(J_j^{(k)})]$, or $B[A_i^{(k)}/B_j^{(k)}]$ for short, from section 2.

THEOREM. *Let $A$ and $B$ be any two square matrices with $n$ rows and columns, and let $k$ be any integer such that $1\leqq k\leqq n$. Then*

$$|A|^{C(n-1,k-1)}\,|B|^{C(n-1,k)} = \big|\,|B[A_j^{(k)}/B_i^{(k)}]|\,\big|, \tag{7.1}$$

*where $i, j=1, 2, \cdots, C(n, k)$ and the element shown in the determinant on the right is the one in the ith row and jth column.*

The proof once more is based on a matrix identity. An application of Laplace's expansion shows that

$$\operatorname{adj}^{(k}BA^{(k)} = (\,|B[A(J_j^{(k)})/B(J_i^{(k)})]|\,) \tag{7.2}$$

where $i, j = 1, 2, \cdots, C(n, k)$. The proof is completed by equating the determinants of the matrices on the two sides of (7.2) and applying the multiplication theorem (4.2), the Sylvester-Franke theorem (6.1) and the corollary (6.7) to the left side.

It should be observed that the proof of (7.1) was based on the Sylvester-Franke theorem. There is a second proof **[37]** in the case $k=1$ which, although not so elegant, is instructive. Furthermore, it depends on nothing more than Cramer's Rule and the multiplication theorem. Assume $|B| \neq 0$ and express each column of $A$ as a linear combination of the columns of $B$ so that $A = BC$. Then $|A| = |B|\,|C|$ by the multiplication theorem. Calculate the elements of $C$ by Cramer's Rule and substitute in the last equation; the result is (7.1) in the case $k=1$. The restriction $|B| \neq 0$ can be removed in the usual way. This proof does not require the result $|\text{adj } B| = |B|^{n-1}$, which is the simplest case of the Sylvester-Franke theorem.

Many important relations are corollaries of the identity (7.1). For example, if $A = I$ and $k=1$, then (7.1) becomes

$$|B|^{n-1} = |\text{adj } B|. \tag{7.3}$$

By specializing $A$ and $B$ properly, Jacobi's theorem can be obtained **[48]** from (7.1) in the case $k=1$. If

$$A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} A_1 & O \\ A_2 & I \end{pmatrix},$$

where $A_1 = (a_{ij})$, $i, j = 1, 2, \cdots, h$, then (7.1) with $k=1$ can be simplified to give the following identity, which is the simplest case of Sylvester's theorem on superdeterminants (see **[24**, pp. 76–80, 90–94, 100–101**]** and **[1**, pp. 45–50**]**):

$$|A|\,|A_1|^{n-h-1} = \left|\,|A(R^{(h)} + U_i^{(1)} \cdot S^{(h)} + V_j^{(1)})|\,\right|, \tag{7.4}$$

$i, j = 1, 2, \cdots, n-h$. The element shown in the determinant on the right is the one in the $i$th row and $j$th column. The identity (7.4) is the basis of the standard method of evaluating determinants by reducing them to equal determinants of lower order. If $A_1$ has a single element, then Sylvester's theorem (7.4) evaluates the $n$th order determinant $|A|$ in terms of a determinant of order $(n-1)$, each element of which is a determinant of order 2. If $A_1$ has two rows and columns, the order of $|A|$ is reduced by 2, but each element of the new determinant is a determinant of order 3; and so on.

Many special cases of the identity (7.1) are listed by Cullis **[16**, vol. II, pp. 63–65**]**.

**8. Jacobi's, Franke's and Reiss' theorems.** The theorems known by the names of Jacobi, Franke, and Reiss evaluate the minors in certain compound determinants. Reiss' theorem includes Franke's, which in turn includes Jacobi's; hence, only Reiss' theorem need be proved. The most elegant proof once more is based on certain matrix identities **[1**, pp. 97–103**]**.

THEOREM. *Let $A$ and $B$ be any square matrices with $n$ rows and columns. Then*

$$(8.1)\quad [\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)} = |A|^{h-C(n-1,k)}|B|^{h-C(n-1,k-1)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}].$$

The proof begins with the obvious identity

$$(8.2)\quad \mathrm{adj}^{(k)}A\cdot B^{(k)}\mathrm{adj}^{(k)}B\cdot A^{(k)} = |AB|\,I.$$

The $h$th compounds of the matrices on the two sides of this equation are equal. If (4.7) is applied to the left side written in the form $\mathrm{adj}^{(k)}\,AB^{(k)}\cdot\mathrm{adj}^{(k)}\,BA^{(k)}$, it follows that

$$(8.3)\quad [\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}[\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)} = |AB|^{h}I.$$

Next, it follows from (3.3) that

$$(8.4)\quad [\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}] = |\,\mathrm{adj}^{(k)}A\cdot B^{(k)}|\,I.$$

If $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]$ is a non-singular matrix, it follows from the Sylvester-Franke theorem (6.1) that its $h$th compound is also non-singular; in this case, $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]^{(h)}$ has a reciprocal matrix. From (8.3) and (8.4) it follows that

$$\begin{aligned}|AB|^{-h}&[\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}[\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)}\\ &= |\,\mathrm{adj}^{(k)}A\cdot B^{(k)}|^{-1}[\mathrm{adj}^{(k)}A\cdot B^{(k)}]^{(h)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}].\end{aligned}$$

Multiply the two sides of this equation by $|AB|^{h}$ and then by the reciprocal of $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]^{(h)}$. The resulting relation is

$$(8.5)\quad [\mathrm{adj}^{(k)}B\cdot A^{(k)}]^{(h)} = |AB|^{h}|\,\mathrm{adj}^{(k)}A\cdot B^{(k)}|^{-1}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A\cdot B^{(k)}].$$

Apply the multiplication theorem and the Sylvester-Franke theorem to the two determinants on the right side of (8.5); the identity (8.1) results. It is shown in the usual way that (8.1) holds even if $[\mathrm{adj}^{(k)}\,A\cdot B^{(k)}]$ is a singular matrix. The proof is complete.

If $k=1$ and $A=I$, the identity (8.1) is

$$(8.6)\quad [\mathrm{adj}\,B]^{(h)} = |B|^{h-1}\mathrm{adj}^{(h)}B.$$

The determinant identities obtained by equating corresponding elements in the matrices on the two sides of this identity are known as Jacobi's theorem. If $h=n$, (8.6) becomes (7.3), a result sometimes known as Cauchy's theorem.

If $B=I$, the identity (8.1) becomes

$$(8.7)\quad [A^{(k)}]^{(h)} = |A|^{h-C(n-1,k)}\mathrm{adj}^{(h)}[\mathrm{adj}^{(k)}A].$$

The determinant identities obtained by equating corresponding elements in the matrices on the two sides of this identity are known as Franke's theorem. If $h=C(n,k)$, relation (8.7) becomes the Sylvester-Franke theorem (6.1).

**9. Sylvester's theorem on superdeterminants.** A special case of Sylvester's theorem has been given in (7.4). The theorems proved in the last section can now be used to prove the general theorem.

THEOREM. *Let $A$ be any matrix with $n$ rows and columns, and let $A(R^{(h)}\cdot S^{(h)})$ be the submatrix with $h$ rows and columns, $0 \leqq h \leqq n-1$, in the upper left hand corner. Let $k$ be an integer such that $1 \leqq k \leqq n-h$. Then*

$$(9.1)\quad |A|^{C(n-h-1,k-1)}|A(R^{(h)}\cdot S^{(h)})|^{C(n-h-1,k)} = ||A(R^{(h)}+U_i^{(k)}\cdot S^{(h)}+V_j^{(k)})||,$$

*where $i, j=1, 2, \cdots, C(n-h, k)$ and the element shown in the determinant on the right is the one in the ith row and the jth column.*

To prove the theorem [24, pp. 100–101], consider first the determinant

$$(9.2)\qquad D = |\alpha_{ij}| \qquad i, j = h+1, h+2, \cdots, n,$$

where $\alpha_{ij}$ is the cofactor of $a_{ij}$ in $|A|$. Thus $D$ is a minor of the transpose of adj $A$. Let $E$ denote the determinant which forms the right member of (9.1). By Jacobi's theorem (8.6) any $(n-h-k)$ rowed minor of $D$ is equal to one of the superdeterminants in $E$, properly signed with $+$ or $-$, and multiplied by $|A|^{n-h-k-1}$. Then

$$(9.3)\qquad |A|^{(n-h-k-1)C(n-h,k)}E = D^{C(n-h-1,n-h-k-1)} = D^{C(n-h-1,k)}$$

by the Sylvester-Franke theorem (6.1) since the left side is equal to the determinant of the $(n-h-k)$ rowed minors of $D$. In making this statement, it must be observed that the value of the compound determinant is the same whether it be formed with signed or unsigned minors (see the proof of (6.7)). By Jacobi's theorem (8.6)

$$(9.4)\qquad D = |A|^{(n-h-1)}|A(R^{(h)}\cdot S^{(h)})|.$$

Substitute this value in (9.3); then

$$(9.5)\qquad \begin{aligned}&|A|^{(n-h-k-1)C(n-h,k)}E\\ &\qquad = |A|^{(n-h-1)C(n-h-1,k)}|A(R^{(h)}\cdot S^{(h)})|^{C(n-h-1,k)}.\end{aligned}$$

Since $(n-h-1)C(n-h-1, k)-(n-h-k-1)C(n-h, k)=C(n-h-1, k-1)$, the relation (9.1) follows from (9.5). The proof is complete.

**10. History and the literature.** According to David Eugene Smith [41, p. 273], the Chinese had some idea of the determinant as early as about 1300 A.D. The distinguished Japanese mathematician Seki Shinsuke Kōwa, or Takakazu, (1642–1708), discovered the expansion of a determinant [41, pp. 439–440] in solving simultaneous equations. In the West, the earliest use of the functions we call determinants occurs in correspondence between Leibniz and De L'Hospital. In a letter to the latter dated 28th April 1693 Leibniz explains his methods and results. This correspondence was not published until 1850, and Leibniz's discoveries apparently had no influence on later developments.

The actual development of the theory of determinants dates from the publication of a book by Cramer [15, pp. 59–60, 656–659] in Geneva in 1750. In it we find the familiar Cramer's Rule. Vandermonde laid the first real foundation for the theory in an important paper [47] read January 12, 1771. Laplace [26]

advanced the theory. Gauss [18] used determinants and the term "determinant of a form." Papers by Binet [7] and Cauchy [12], both read to the Institute on November 30, 1812, make the year 1812 one of the most important in the history of determinants. Binet's paper contained important results, but Cauchy gave a masterly treatment of the entire subject. Cauchy, borrowing the word from Gauss, used the term determinant in the same sense in which it is used today. Schweins [38] published many important discoveries in a book in 1825. They were lost, however, and the fact that he had ever written on determinants was not brought to light until 1884. The year 1841 was also one of outstanding importance for determinants. Jacobi, who had already published several papers on the subject, brought together all the known results and added to them [21] and, following Cauchy, fixed the use of the name determinant. In a second paper in 1841 Jacobi treated Jacobians [22]. Finally, in 1841, Cayley in his first published paper [13] introduced the present-day standard notation of vertical bars enclosing a square array to denote a determinant. Many notations had been used before this time, but Cayley's introduction of a notation capable of representing all determinants gave a great impetus to the study of the subject.

Let us turn now to the history of the theorems treated in this exposition. Vandermonde [47] gave the Laplace expansion for a determinant of order $2m$ in terms of minors of order 2 and $(2m-2)$. Laplace [26] gave the general theorem, but Cauchy in 1812 gave the first adequate proof of Laplace's expansion in his powerful treatment [12] of the whole subject of determinants.

Lagrange [25, *Oeuvres*, p. 580] gave a lemma which apparently contains the first known example of the multiplication theorem. Lagrange evaluated the square of a determinant of the third order, but it is not at all certain that he recognized the result as a determinant. Gauss [18, Chap. V] definitely used the multiplication theorem (4.2) in connection with the transformation of ternary quadratic forms. Binet [7] published for the first time the multiplication theorem (4.3). Muir [29, vol. I, p. 109] thinks that Cauchy [12] was the first to publish a satisfactory proof of the multiplication theorem, but that Binet gave the more general form of it. Sylvester [46] gave the multiplication theorem in the general form (4.6). Finally, it may be remarked that Sylvester [45] (the result can be found also in [24, pp. 242–243] and [31, p. 223]) gave the multiplication theorem in a completely different form for the product of determinants each of which has a column of 1's; he used this result to prove the theorem in Cayley's first published paper [13].

The first known case of Sylvester's theorem of 1839 and 1851 (see (5.1) above) was given by Fontaine [5] in 1748, before the founding of the theory of determinants. Bézout [6, pp. 171–187, 208–223] was the next to give examples of these identities, but he considered only determinants of the second and third order. Following Bézout, many others (see Muir [29]) gave instances of the identities (5.1), but Sylvester was the first to give the general theorem and its proof. In [42] Sylvester proved (5.1) for the case $k=1$, and in [44] he gave the

completely general theorem and its proof. Bruno [11] gave a more satisfactory proof. A proof of (5.1) can be obtained by using Laplace's expansion on a properly formed determinant [31, pp. 117–122], but this method is not elegant. Beaver [4] has given a proof of the theorem and certain extensions.

Sylvester's claim to the Sylvester-Franke theorem (6.1) rests on the fact that he published [43] in March, 1851 the general theorem (9.1) that includes (6.1) as a special case. He did not call attention to the special case, however, and he gave no proof of the general theorem. The proof given in section 6 above is Franke's proof [17]. Franke's paper has two famous footnotes added by the editor, Borchardt. In the second footnote Borchardt deduces the Sylvester-Franke theorem (6.1) from $|A^{(k)}|\,|\operatorname{adj}^{(k)} A| = |A|^{C(n,k)}$ (see (3.3) above) and the fact (see [1, pp. 37–38]) that a determinant, considered as a polynomial in its elements, is prime (for this proof, see [1, p. 92] or [31, pp. 178–179]). It is rather remarkable that, for a result as fundamental as the Sylvester-Franke theorem, these two proofs are the only direct ones found in the literature. Other proofs, by no means simple or direct, are known; see for example the one obtained incidentally by Stéphanos [49, pp. 108–110 (there appears to be a consistent error in the exponent in the statement of the theorem)].

Bazin [3] was the first to treat the Bazin-Reiss-Picquet theorem (7.1) (he considered only the case $k=1$). The general theorem (7.1) was given by Reiss [36]. Picquet rediscovered numerous related theorems [34] and gave a comprehensive exposition of the entire subject [35]. Rubini [37] and Hamburger [19] showed that Sylvester's theorem of 1839 and 1851, in the case $k=1$, can be deduced from the Bazin-Reiss-Picquet theorem in the case $k=1$. Zehfuss [48] showed that Jacobi's theorem (8.6) can be deduced from the Bazin-Reiss-Picquet theorem in the case $k=1$. Picquet [35] showed that Sylvester's theorem on superdeterminants (9.1) can be deduced from the Bazin-Reiss-Picquet theorem, but the proof is not very elegant in the general case (this proof in the case $k=1$ is given in section 7 above).

Jacobi's theorem (8.6) was published for the first time by Jacobi [20]. Franke [17] proved Franke's theorem (8.7), and the editor, Borchardt, gave a simpler proof in the first of the famous footnotes. Reiss [36] gave Reiss' theorem.

Sylvester [43] stated Sylvester's theorem on superdeterminants (9.1) in March, 1851, but he gave no proof. Jacobi [22] had used the identity in the special case $k=1$ in connection with Jacobians (see [29, vol. I, pp. 381–385; vol. II, p. 192]). In 1854 Brioschi deduced the special case of Sylvester's theorem from Jacobi's identity (see [29, vol. II, p. 250]). The general theorem remained without proof until Reiss [36] supplied one in 1867. Later, proofs were given by Picquet [35], Scott [39], and others (several proofs are found in [24, pp. 76–80, 90–94, 100–101]).

One might hope to discover certain general methods for establishing identities in the theory of determinants, and some progress has been made in this direction. Muir [28] gave two general principles which he called the "Law of Complementaries" and the "Law of Extensible Minors" (see also [31, pp. 172–

173, 179–182] and [32]). Nanson [32] has given an impressive list of applications of these two principles. For example, the Sylvester theorem on superdeterminants (9.1) becomes almost trivial as the extensional of the Sylvester-Franke theorem (6.1). The latter theorem, however, remains without a simple, direct proof.

Aitken's book [1] contains a good modern treatment of determinants, but it is by no means complete. Bôcher [9] gives a good introduction but only a few of the theorems treated in this paper. Cullis' two volumes [16] (see the review by Shaw [40]) contain a wealth of material, but they form nevertheless a highly unsatisfactory work on determinants. Kowalewski's book [24] has long been a standard reference work on determinants, but it has certain faults (see the review by Bôcher [10]) and it is not complete. For example, Kowalewski does not properly explain and exploit the relationship of determinants to matrices; furthermore, he does not give the theorems in sections 5 and 7 above. The treatise by Muir-Metzler [31] contains a great deal of material, much of it not readily accessible elsewhere, but it nevertheless leaves much to be desired. Pascal's book [33] provides a valuable reference work which includes all the material of this paper and supplies complete references to the original sources. Turnbull [50] treats, among other things, applications of the identities of this paper in the important theory of symbolic invariants.

Sir Thomas Muir, the historian of the theory of determinants, has assembled the record up to 1920 in five volumes [29, 30]. He gives reviews of all papers and books on determinants, and the complete history of the theorems treated in this exposition can be traced in his five volumes.

## Bibliography

1. A. C. Aitken, Determinants and Matrices. Oliver and Boyd, Edinburgh and London, 1939. vii+135 pp.

2. A. A. Albert, Modern Higher Algebra. The University of Chicago Press, Chicago, 1937. ix+319 pp.

3. H. Bazin, Sur une question relative aux déterminants, (Liouville's) Journal de Mathématiques Pures et Appliquées (1), 16, 145–160 (1851).

4. R. A. Beaver, Vanishing aggregates of determinants and their relationships, this MONTHLY, 39, 266–276 (1932).

5. Alexis Fontaine des Bertins, Mémoires donnés à l'Académie Royale des Sciences, non imprimés dans leurs temps. Par M. Fontaine de cette Académie. Paris, 1764. 588 pp.

6. Étienne Bézout, Théorie Générale des Équations Algébriques. Paris, 1779.

7. J. Binet, Mémoire sur un système de formules analytiques, et leur application à des considérations géométriques, Journal de L'École Polytechnique, vol. 9, cahier 16, pp. 280–354, Paris, 1813.

8. Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra. The Macmillan Company, New York, 1941. vii+450 pp.

9. Maxime Bôcher, Introduction to Higher Algebra. The Macmillan Company, New York, 1907. xi+321 pp.

10. Maxime Bôcher, Kowalewski's Determinants (a review of the first edition of [24] in this bibliography), Bulletin of the American Mathematical Society, 17, 120–140 (1910–1911).

11. F. Faà di Bruno, Démonstration d'un théorème de M. Sylvester relatif à la decomposition

d'un produit de deux déterminants, (Liouville's) Journal de Mathématiques (1), 17, 190–192 (1852).

12. Augustin Cauchy, Sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suites des transpositions opérées entre les variables qu'elles renferment, Journal de L'École Polytechnique, vol. 10, cahier 17, pp. 29–112, Paris, 1815. Oeuvres Complètes D'Augustin Cauchy, 2e série, vol. 1, pp. 91–169, Paris, 1905.

13. A. Cayley, On a theorem in the geometry of position, Cambridge Mathematical Journal, 2, 267–271 (1841). Collected Mathematical Papers, vol. 1, pp. 1–4, Cambridge University Press, 1889.

14. R. Courant and D. Hilbert, Methoden der Mathematischen Physik, vol. 1, 2nd edition (1931), Julius Springer, Berlin, xiv+469 pp.

15. Gabriel Cramer, Introduction à l'Analyse des Lignes Courbes Algébriques. Genève, 1750.

16. C. E. Cullis, Matrices and Determinoids. vol. I (1913), xii+430 pp. vol. II (1918), xxiv+555 pp. Cambridge University Press.

17. E. Franke, Ueber Determinanten aus Unterdeterminanten, (Crelle's) Journal für die reine und angewandte Mathematik, 61, 350–355 (1863).

18. C. F. Gauss, Disquisitiones Arithmeticae. Lipsiae, 1801. Werke, vol. 1, Göttingen, 1870. 478 pp.

19. M. Hamburger, Anwendung einer gewissen Determinantenrelation auf die Integration partieller Differentialgleichungen, Crelle's Journal, 100, 390–404 (1887).

20. C. G. J. Jacobi, De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones linearis in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium multiplicium, Crelle's Journal, 12, 1–69 (1834). Gesammelte Werke, vol. 3, pp. 191–268, Berlin, 1884.

21. C. G. J. Jacobi, De formatione et proprietatibus determinantium, Crelle's Journal, 22, 285–318 (1841). Gesammelte Werke, vol. 3, pp. 355–392.

22. C. G. J. Jacobi, De determinantibus functionalibus, Crelle's Journal, 22, 319–359 (1841). Gesammelte Werke, vol. 3, pp. 393–438.

23. Felix Klein, Elementary Mathematics from an Advanced Standpoint: Geometry. Translated by E. R. Hedrick and C. A. Noble. The Macmillan Company, New York, 1939. ix+214 pp.

24. Gerhard Kowalewski, Einfuhrung in die Determinantentheorie einschliesslich der Fredholmschen Determinanten. 2nd edition (1925). Walter de Gruyter & Co., Berlin und Leipzig. vii+304 pp.

25. J. L. Lagrange, Nouvelle solution du problème du mouvement de rotation d'un corps de figure quelconque qui n'est animé par aucune force accélératrice, Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin, année 1773, pp. 85–128. Oeuvres de Lagrange, vol. 3, pp. 577–616, Paris, 1869.

26. P. S. Laplace, Recherches sur le Calcul Intégral et sur le Système du Monde, Histoire de l'Académie Royal des Sciences de Paris, année 1772, 2e partie, pp. 267–376. Oeuvres Complètes de Laplace, vol. 8, pp. 367–477, Paris, 1891.

27. C. C. MacDuffee, Vectors and Matrices. Carus Monograph No. 7. The Mathematical Association of America, 1943. xi+192 pp.

28. Thomas Muir, The law of extensible minors in determinants, Transactions of the Royal Society of Edinburgh, 30, 1–4 (1881).

29. Thomas Muir, The Theory of Determinants in the Historical Order of Development. vol. I, 2nd edition (1906), xi+491 pp. vol. II (1911), xvi+475 pp. vol. III (1920), xxvi+503 pp. vol. IV (1923), xxxi+508 pp. Macmillan and Co., Ltd., London.

30. Thomas Muir, Contributions to the History of Determinants 1900–1920. Blackie & Son Limited, London and Glasgow, 1930. xxiv+408 pp.

31. Thomas Muir (revised by William H. Metzler), A Treatise on the Theory of Determinants. Longmans, Green and Company, New York, 1933. v+766 pp.

32. E. J. Nanson, On certain determinant theorems, Crelle's Journal, 122, 179–185 (1900).

33. Ernesto Pascal, Die Determinanten. (Translated from the Italian by Dr. Hermann Leitzmann) B. G. Teubner, Leipzig, 1900. xvi+266 pp.

34. H. Picquet, (1) Sur le déterminant dont les éléments sont tous les mineurs possibles d'ordre donné d'un déterminant donné. (2) Analyse combinatoire des déterminants. Comptes Rendus . . . Académie des Sciences (Paris). vol. 86. (1), pp. 310–312. (2) pp. 1118–1119.

35. H. Picquet, Mémoire sur l'application du calcul des combinaisons à la théorie des déterminants, Journal de L'École Polytechnique, vol. 28, cahier 45, pp. 201–243, Paris, 1878.

36. M. Reiss, Beiträge zur Theorie der Determinanten. Leipzig, 1867. viii+113 pp.

37. R. Rubini, Formole di trasformazioni nella teorica dei determinanti, Giornale di Matematica, 16, 198–208 (1878).

38. Ferd. Schweins, Theorie der Differenzen und Differentiale, u. s. w. Heidelberg, 1825. vi+666 pp. (Theorie der Producte mit Versetzungen, pp. 317–431.)

39. R. F. Scott, On compound determinants, Proceedings of the London Mathematical Society, 14, 91–102 (1882).

40. J. B. Shaw, Matrices and Determinoids (a review of **[16]** in this bibliography), Bulletin of the American Mathematical Society, 26, 224–233 (1919–1920).

41. David Eugene Smith, History of Mathematics. vol. I: General Survey of the History of Elementary Mathematics. Ginn and Company, Boston, 1923. xxii+596 pp.

42. J. J. Sylvester, On the derivation of coexistence. Part II. Being the theory of simultaneous simple homogeneous equations, Philosophical Magazine, 16, 37–43 (1840). Mathematical Papers, vol. 1, pp. 47–53, Cambridge University Press, 1904.

43. J. J. Sylvester, On the relation between the minor determinants of linearly equivalent quadratic functions. Philosophical Magazine (4), 1, pp. 295–305, 415 (March, 1851). Mathematical Papers, vol. 1, pp. 241–250, 251.

44. J. J. Sylvester, On a certain fundamental theorem of determinants, Philosophical Magazine (4), 2, 142–145 (August, 1851). Mathematical Papers, vol. 1, pp. 252–255.

45. J. J. Sylvester, On Staudt's theorems concerning the contents of polygons and polyhedrons, with a note on a new and resembling class of theorems, Philosophical Magazine (4), 4, 335–345 (1852). Mathematical Papers, vol. 1, pp. 382–391.

46. J. J. Sylvester, On a theorem concerning the combination of determinants, Cambridge and Dublin Mathematical Journal, 8, 60–62 (December, 1852). Mathematical Papers, vol. 1, pp. 399–401.

47. A. T. Vandermonde, Mémoire sur l'élimination, Histoire de l'Académie Royal des Sciences de Paris, année 1772, 2$^e$ partie, pp. 516–532.

48. G. Zehfuss, Zwei Sätze über Determinanten, Zeitschrift für Mathematik und Physik, 7, 436–439 (1862).

49. Cyparissos Stéphanos, Sur une extension du calcul des substitutions linéaires, (Liouville's) Journal de Mathématiques (5), 5, 73–128 (1900).

50. H. W. Turnbull, The Theory of Determinants, Matrices, and Invariants. Blackie and Son Limited, London and Glasgow, 1928. xvi+338 pp.

# INDUSTRIAL EXPERIENCE FOR MATHEMATICS PROFESSORS

K. L. NIELSEN, Louisiana State University*

**1. Introduction.** During the war a great number of professors of mathematics assumed positions in industry and governmental agencies; there they applied their training, knowledge, and experience to the technical problems of modern warfare. The necessities of war made "pure" mathematicians turn to applied mathematics, and caused professors to leave their academic life and be engulfed by the demands of industry. At the conclusion of the war most of these professors returned to the college campus to resume their teaching and research. Have they benefited by their experience? What are the effects of the adventure? Will the universities derive benefits from the experiences gained by their faculty members while they were on leave of absence? Has industry benefited by the presence of these highly skilled scientists?

These are questions which we may well pause to consider and thus hope to obtain the answer to the important question: *Should teachers of mathematics participate in the work of industry and governmental agencies during times of peace?*

To arrive at a consensus of opinion on these questions would necessitate an extensive survey. No one man can give the complete answer. However, some evidence may be derived from personal experience, and it should be worth while to publicize the experience and exhibit some obvious facts from which the reader may draw his own conclusions.

It is the contention of the writer that the answers to the above questions are all *yes* and that much can be derived from having members of a college staff spend some time with an industrial organization. The benefits are equally divided among the individual, the university, and industry. It is his belief that temporary positions in industry should be sought by the younger members of any college staff and should be made available by industry. It is the purpose of this paper to expound this idea by drawing from the author's personal experience and citing examples of existing plans for the temporary employment of college teachers. In particular, the aim is to arouse mathematicians to the existence of, the need for, and the participation in such employment.

**2. Means of temporary employment.** The importance of advanced mathematics to industry was realized by many industrial organizations (noticeably the aircraft, petroleum, and electrical industries†) prior to the war. These organizations had for some time been employing mathematicians as integral members of their research staffs. The employment was both full time and part time. Thus a large eastern aircraft corporation devised a plan by which college faculty members were employed in a temporary capacity. This plan has already been discussed for physics teachers by D. C. Martin [2]. Since it may not have received the attention of mathematicians and since it was the writer's pleasure

* On leave with U. S. Naval Ordnance Plant, Indianapolis, Indiana.

† This paper is not concerned with the type of mathematics used in industry. For a discussion of industrial mathematics see [1].

to be employed under this plan during the summer of 1942, it may be advisable to explain it in some detail.

The plan was to employ a group of college professors and instructors for the summer months, that is, the months during which normally the universities were not operating a full schedule. The group was composed of physicists, mathematicians, chemists, and engineers. No particular attention seemed to have been paid to any specific number of men from each field.

The usual procedure was to assign each man a specific problem for which he was qualified and in which he expressed an interest. The corporation seemed to have a large number of problems from which to choose, and the assignment was made after a conference with members of the permanent engineering staff.

After the problem had been assigned the college instructor was referred to a supervisor, one of the permanent engineers who was familiar with the problem. He was then comfortably located at a desk usually in the group headed by his supervisor or in a group working on related problems. All knowledge and information that the organization had on his particular problem was made available to him, and any physical equipment he needed to carry out his investigations were placed at his disposal. If his problem was an immediate production or design problem he was included in all meetings, and was placed on the mailing list for progress reports. If his problem was of the nature of an independent investigation he was left on his own to carry out his work. A rather good scientific library was located in the same building.

At the end of the summer or whenever his investigations had led to a definite result he was expected to submit a complete report on the progress made. Individual conferences or meetings were instigated at his own request.

Thus he was left to carry on his investigations in a congenial and pleasant atmosphere. He had ample association, both professional and social, with other college instructors and also the permanent engineering staff of the corporation. Depending upon his own personality he was from time to time approached by the junior engineers who requested aid on problems of both an advanced and elementary nature within his field. In reality he was an integral part of the organization while he was there and received excellent treatment by the corporation.

This was indeed a workable plan. The number of participants was steadily increasing (until the war made an interruption), and the corporation seemed anxious to continue it. The writer was employed in a similar plan with a smaller industrial concern in Detroit during the summer of 1943. Undoubtedly there exist other examples with which the author is not familiar. That these plans will be reinstated during the future peace years seems fairly certain.

During and since the end of the war there has been created a new source of temporary employment on practical problems, which in many instances take the form of consultant work for mathematicians. The reference is to the government sponsored research groups established by contract in government laboratories and on university campuses. In the years to come this should prove to be an

important opportunity for part time employment in applied mathematics. Contracts established at universities eliminate the necessity of traveling, which is particularly advantageous to men with families.

**3. Need for participation and benefits obtained by mathematicians.** R. Weller has stated:

"The writer wishes also to comment on the desirability of the interchange of personnel between government and other activities. Such interchange between university faculties has become well accepted and the practice of encouraging faculty members to associate themselves with industry during vacation periods is common, for in such interchange lies a major opportunity for rapid professional development. Some government agencies have initiated procedures whereby summer work for college students is possible with a view to later full time employment. This might well be extended to include faculty members who could be given leave of absence for the purpose, and could also be encouraged with profit by the larger industrial organizations. If both government and industry (and why not universities?) are well informed regarding each other's activities and a common ground of experience is established, much can be done to promote understanding and cooperation" **[3]**.

Although this statement is primarily directed toward engineers it is equally applicable to mathematicians. There seem to be many reasons why mathematicians should participate in temporary industrial positions. They may be summarized in the following statements:

*The added experience should make for a better individual.*

*Mathematicians have an obligation to fulfill to industry.*

*It behooves mathematicians to keep abreast with industrial developments in order to preserve the scientific position of mathematics.*

Elaborations on these statements may be presented in volumes of words. However, it is more important to be precise and to the point in order that the essentials may not be lost in a maze of verbal eloquence. Suffice it therefore to list the following very essential points.

(1) The emphasis on applied mathematics brought about by the war will carry over in our postwar needs. It has become quite apparent that we need to train more and more applied mathematicians who will fit into industry and governmental organizations as professional mathematicians. In order to train these men more efficiently, members of a university staff must be familiar with the mathematical problems of industry.

(2) A constant complaint by our students and by young men in industry is that professors lack the necessary experience from which to draw practical situations for their illustrations. The complaints are entirely too numerous to be unfounded, and we can not continue to ignore them.

(3) It has also been a constant source of argument, that college curricula in mathematics are designed principally for the training of teachers of mathematics and not industrial mathematicians, physicists, or engineers. If present curricula

are changed, they should be changed by men who have had experience in both teaching and industry.

(4) The quality of mathematics teaching can always be improved; the added experience should make for better mathematics teachers.

(5) Research problems in applied mathematics are frequently buried in industry. University professors do not know of the problems and thus do not contribute to the necessary research. This sad situation should be corrected.

The benefits derived by the participating individual are numerous. They were adequately summarized for the physicist by Dr. Martin [2], and apply also to the mathematician:

"Opportunities for teacher:

(1) To learn, at first hand, the types of problems facing industry, the practical applications of physics to their solution, and the limitations, or approximations necessary, in applying physical principles to production methods.

(2) To discover weaknesses in present methods of training physicists and engineers, and thus to improve their own curricula and methods of teaching.

(3) To work with young physicists and engineers, to learn their viewpoints and share experiences with them.

(4) To meet and share experiences with teachers from other institutions.

(5) To learn the uses of physical apparatus in making industrial measurements and tests.

(6) To observe and perform research, which in turn might lead to an interest in some specific research problem that might be continued at the teacher's own institution.

(7) To travel and live in a different section of the country for a few months.

(8) To provide an income for the summer in a congenial and interesting type of work.

(9) To obtain a permanent position with an industrial organization, if a teacher eventually decided that he was better qualified for, and preferred, that type of work."

**4. The need for participation by industry and government agencies.** Primarily, the reason why industry would undertake the employment of college professors would be for the applicable results of their efforts, the so called "dollar value." This of course is the obvious reason and needs no further elaboration. However, there exists some far reaching and underlying results on which we may well dwell. Again we quote Dr. Martin [2]:

"Some advantages to an industrial organization:

(1) A better understanding of, and a more sympathetic attitude towards, the technical and economic problems of industry in general, and of one industry in particular, on the part of the teachers.

(2) Association of their young engineers and physicists with men of higher academic training and wider experience; this should help to remind the

younger men of the importance of continual study and reading in their particular field.

(3) Provide in the future better trained engineers and industrial physicists, as their teachers would be more familiar with the problems of industry.

(4) Provide additional trained men to work on engineering and research problems; short problems could be devised or longer ones initiated, which might be continued at the teacher's own institution.

(5) Provide, incidentally, good advertising for the products of a particular industry."

These are points which are indeed well worth considering and which apply equally well to mathematicians. However, we should like to go further by adding a few more points which are deemed to be pertinent.

The employment of college professors would provide any industrial organization with a personal contact within one or more universities. This contact could result in obtaining recommendations for graduates seeking positions in the particular organization; recommendations that would really mean something since the recommender is now familiar with the organization as well as the prospective employee. This personal contact could result in the "farming out" of long term research programs, research which could not be expected to be borne by any one organization but which, nevertheless, would benefit that organization. There are other ways in which "personal contact" could prove advantageous.

There is one objection that has been raised by industry with which the writer takes exception; namely, that it takes a man at least six months or more in industry before he is of any practical value to the organization, and therefore short time employment is not beneficial. The mere fact that industry hires consultants disproves this statement. Furthermore, the member of a college staff hired is already a highly trained individual in his own field and if his work is limited to his field or neighboring problems, the results of his investigations will prove fruitful.

The plan presented here should be "a gift from the gods" to the small industries who cannot afford to hire permanent specialized scientists such as mathematicians, physicists, chemists, or the like.

**5. Conclusion.** Throughout this paper it has been pointed out that mathematicians and industry would mutually benefit from temporary employment of college mathematics professors in industry. That the university would benefit should be very obvious; if for no other reason, the improved teaching which will result is indeed ample reward to the university. Furthermore, the university has an obligation to industry which in part can be met by making teaching personnel available to industry. Finally, we again point to Weller's statement **[3]**, "If both government and industry (and the university) are well informed regarding each other's activities and a common ground of experience is established much can be done to promote understanding and cooperation."

It is the belief of the writer that universities should recognize the worth of this part-time employment of their faculty members by

(1) obtaining contracts for research to be conducted at the university,

(2) encouraging their faculty members to assume temporary positions in industry, and

(3) recognizing the merits of the plan by suitable promotions based on such employment.

It is hoped from this article that mathematicians now engaged in college teaching will derive some enthusiasm for participating in the solution of industrial problems by obtaining industrial experience, and that mathematicians will look upon a temporary position in industry as an opportunity to become better mathematicians and teachers. If this is too high an aim, it is hoped that mathematicians will at least take cognizance of some of the viewpoints exhibited above.

**References**

1. T. C. Fry, Industrial Mathematics, this MONTHLY, vol. 48, No. 6, Part II, Supplement, June-July, 1941.

2. D. C. Martin, Industrial Experience for College Teachers, American Journal of Physics, vol. 13, No. 1, Feb., 1945, p. 46.

3. R. Weller, Federal Government Employment For Scientific and Technical Workers, Engineering Experiment Station News, The Ohio State University, vol. XVII, No. 5, Dec., 1945, p. 38.

## MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Send material for this department directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

### ON COMPLETING A DETERMINANT

V. L. KLEE, JR., University of Virginia

As is well known, to each pair of integers $a_1$, $a_2$, with greatest common divisor $(a_1, a_2)=d$, there corresponds a pair of integers $k_1$, $k_2$, such that

$$k_1a_1 + k_2a_2 = d.$$

Indeed, by repeated applications of this fact, it is easily shown that if

$$(a_1, a_2, \cdots, a_n) = d,$$

then there are integers $k_1$, $k_2 \cdots$, $k_n$, for which

$$k_1a_1 + k_2a_2 + \cdots + k_na_n = d.$$

Now the fact mentioned above, for $n=2$, can also be stated thus: Given $(a_1, a_2)=d$, we can find $a_1'$ and $a_2'$ so that the determinant

$$\begin{vmatrix} a_1 & a_2 \\ a_1' & a_2' \end{vmatrix}$$

has value $d$. Generalizing in this direction, we state the following theorem:

THEOREM: *If* $(a_{11}, \cdots, a_{1n})=d$, *there are integers* $a_{ji}(j=2, \cdots, n;\ i=1, \cdots, n)$ *such that the determinant* $|a_{ji}|$ *has value* $d$.

To prove this theorem it clearly will suffice to consider only the case $d=1$. We will use the following theorem from linear algebra:*

*If $U$ is a non-degenerate subgroup of the n-membered modulus* $M$, then there is a basis $y_1, \cdots, y_n$ of $M$ and there are positive integers $k_1, \cdots, k_r$, for some $r \leq n$, such that $k_1y_1, \cdots, k_ry_r$ is a basis for $U$.*

Now suppose $(a_{11}, \cdots, a_{1n})=1$, and let $x_1, \cdots x_n$ be an arbitrary basis for $M$. Let $U$ consist of all elements in $M$ of the form $k\sum_1^n a_{1i}x_i$, where $k$ is an integer. Then since $U$ is a non-degenerate subgroup of $M$, we can pick a basis $y_1, \cdots, y_n$ of $M$ having the properties stated in the theorem. But clearly some element $y_i$ must be $\pm\sum_1^n a_{1i}x_i$, and we may assume our indices chosen so that it is $y_1$. Now for $j=2, \cdots, n$, we can write $y_j=\pm\sum_1^n a_{ji}x_i$. Then, since $y_1, \cdots, y_n$ is a basis for $M$, we have, from another well-known theorem, $|a_{ji}| = \pm 1$. Thus, by changing the signs in one row if necessary, we have proved the theorem.

## A PAIR OF RECURSION RELATIONS

S. T. PARKER, University of Cincinnati

**1. Introduction.** Problem E 695 of this MONTHLY, vol. 52, 1945, p. 516, proposed by H. L. Lee, asked for triangles whose sides are integers in arithmetic progression, and whose areas are integers. The problem involved finding integral solutions of the equation

$$x^2 = 3y^2 + d^2.$$

In solving this equation for $d=1$, the author used certain recursion relations, and then adapted them to the solution of the well-known Pellian equation†

$$x^2 = cy^2 + 1, \qquad 0 < c \neq z^2. \tag{1}$$

In the work that follows, $(a, b)$ will represent a solution in integers of (1); thus

$$a^2 = cb^2 + 1. \tag{2}$$

The chief results of this paper are a theorem which we have called Lemma 4, and the following theorem:

THEOREM: *If the number pair* $(a, b)$ *is such that* $b$ *is the smallest (positive) integer satisfying* (2), *and if* $x_0=1$, $x_1=a$, $y_0=0$, $y_1=b$, *then the two recursion relations*

---

* See, for example, Alexandroff and Hopf, Topologie I. By 'modulus' is meant a linear-form modulus with integers as multipliers.

† See Perron, Die Lehre von den Kettenbrüchen, p. 102.

$$\begin{cases} x_n = ax_{n-1} + bcy_{n-1} \\ y_n = bx_{n-1} + ay_{n-1} \end{cases} \qquad n \geqq 1, \tag{3}$$

*and*

$$\begin{cases} x_n = 2ax_{n-1} - x_{n-2} \\ y_n = 2ay_{n-1} - y_{n-2}, \end{cases} \qquad n \geqq 2, \tag{4}$$

*are equivalent.*

**2. Lemmas.** We need several lemmas.

LEMMA 1. *The recursion relations* (3) *give solutions of* (1).

*Proof:* This is easily proved by mathematical induction.

LEMMA 2. *The recursion relations* (4) *give solutions of* (1).

*Proof*: Applying the method of mathematical induction to prove that

$$x_n^2 - cy_n^2 = 1, \tag{5}$$

we are led to the necessity of the relation

$$x_n x_{n-1} - cy_n y_{n-1} = a. \tag{6}$$

This auxiliary condition is satisfied for $n=1$. Assume that (5) and (6) hold for $n=1, 2, \cdots, k-1$. We show first that

$$x_k^2 - cy_k^2 = 1,$$

and then that

$$x_k x_{k-1} - cy_k y_{k-1} = a,$$

and the conclusion follows immediately.

LEMMA 3. *The set of numbers given by* (4) *contains the set given by* (3).

*Proof*: Rewrite (3) in the form

$$\begin{cases} x_{n-1} = ax_{n-2} + bcy_{n-2} \\ y_{n-1} = bx_{n-2} + ay_{n-2}. \end{cases} \tag{7}$$

From (7), using (2), we obtain

$$bcy_{n-1} = ax_{n-1} - x_{n-2},$$

$$bx_{n-1} = ay_{n-1} - y_{n-2}.$$

Relations (4) follow on substituting these results in (3).

LEMMA 4.* *If* $(a, b)$ *is the pair described in the theorem, then the relations* (3) *give all the solutions of* (1).

* I am indebted to Dean W. H. Durfee of Hobart College for the method of proof used in this lemma.

*Proof*: Assume the lemma false. Then there exist pairs of integers $(x, y)$, satisfying (1) and not obtainable from (3). Therefore there must be one pair $(x_m, y_m)$, from this set, for which $y_m$ is the least. Since $(x_0=1, y_0=0)$ occurs in the set (3), $y_m>b$.

Then

$$x_m^2 = cy_m^2 + 1 = y_m^2\left[c + \frac{1}{y_m^2}\right] < y_m^2\left[c + \frac{1}{b^2}\right] = y_m^2 \cdot \frac{a^2}{b^2} \cdot$$

Therefore

$$x_m < \frac{a}{b} y_m.$$

Suppose that

$$x_m \leqq \frac{a-1}{b} y_m.$$

This would yield

$$x_m^2 - cy_m^2 \leqq \left\{\frac{(a-1)^2}{b^2} - c\right\} y_m^2 = \frac{2-2a}{b^2} y_m^2 < 0,$$

since $c>1$ and $b\geqq 1$, and therefore $a>1$.

This contradiction leads to the double inequality

$$\frac{a-1}{b} y_m < x_m < \frac{a}{b} y_m. \tag{8}$$

The formulas inverse to (3) are easily found to be

$$\begin{cases} x_{n-1} = ax_n - bcy_n \\ y_{n-1} = -bx_n + ay_n. \end{cases} \tag{9}$$

On replacing $(x_n, y_n)$ in (9) by $(x_m, y_m)$, we obtain a new pair $(x_{m-1}, y_{m-1})$ which is found to satisfy (1). Moreover, from (8) and (9), we have

$$y_{m-1} > -b\left(\frac{a}{b}\right) y_m + ay_m = 0,$$

$$y_{m-1} < -b\left(\frac{a-1}{b}\right) y_m + ay_m = y_m.$$

Thus

$$y_m > y_{m-1} > 0,$$

and we have found a pair $(x_{m-1}, y_{m-1})$ with a positive $y_{m-1}$ less than $y_m$. If $(x_{m-1}, y_{m-1})$ is a pair given by relations (3), so is $(x_m, y_m)$, by applying (3) to $(x_{m-1}, y_{m-1})$. Therefore $(x_{m-1}, y_{m-1})$ cannot be in our set, and the contradiction proves the lemma.

**3. Proof of the Theorem.** We note that the number pairs given by (3) are also given by (4), from Lemma 3. The inclusion of (4) in (3) follows from considering Lemmas 4 and 2. Thus the two sets of numbers are the same.

**4. Remarks.** (1) An interesting incidental result is that the number pairs given by (3), or (4), also satisfy the relation

$$x_n x_{n-p} = c y_n y_{n-p} + x_p, \qquad n \geqq p \geqq 0.$$

(2) It would seem that the ratio $a/b$, of the numbers $a$, $b$ described in the theorem, is the first convergent $p/q$ of the continued fraction expansion for $\sqrt{c}$ for which $p^2 = cq^2 + 1$. This $p/q$ is the penultimate* convergent of the first or second period, according as the period is even or odd. (If the period is one, $p/q$ is the second convergent.)

## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.*

### THE REDUCED EQUATION OF THE GENERAL CONIC

A. E. JOHNS, McMaster University

EDITOR'S NOTE. The subject matter of this note is familiar to everyone, and has received a satisfactory treatment in a number of standard texts. Among such expositions, that of Professor Johns is outstanding for its elegance and precision, and it is being published for this reason.

**1. Notation.** The general equation of a conic may be written:

$$aX^2 + 2hXY + bY^2 + 2gX + 2fY + c = 0.$$

Let

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

and let $A$, $B$, $C$, $F$, $G$, $H$ be the cofactors of the corresponding small letters in relation $D$. The fundamental invariants under rotation and translation of rectangular axes are: $a+b$; $C = ab - h^2$; and $D$. In addition, when $C=0$ and $D=0$, $A+B = bc + ca - f^2 - g^2$ is also invariant. The discriminating quadratic is:

$$\lambda^2 - (a+b)\lambda + ab - h^2 = 0$$

* See Hall and Knight, Higher Algebra, Ch. XXVIII.

with roots $\lambda_1$ and $\lambda_2$.

**2. Rotation of axes.** The equations of transformation for a rotation through an angle $\theta$ are:

$$X = lx - my; \qquad Y = mx + ly$$

where $l = \cos\theta$ and $m = \sin\theta$; and the new equation of the conic takes the form:

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

where:

$$\begin{aligned} a' &= al^2 + 2hlm + bm^2 & f' &= fl - gm \\ b' &= am^2 - 2hlm + bl^2 & g' &= gl + fm \\ c' &= c & h' &= (b - a)lm + h(l^2 - m^2). \end{aligned}$$

It is always possible to choose a positive acute angle $\theta$ to satisfy the equation: $\tan 2\theta = 2h/(a-b)$ and so make $h' = 0$. For this $\theta$, $l$ and $m$ are both positive. After this rotation through the positive acute angle $\theta$, and a suitable translation if necessary, the equation of the conic takes on one of the following types of reduced forms.

Case 1. When $C \neq 0$, $\lambda_1 x^2 + \lambda_2 y^2 + D/C = 0$.

These are the central conics, including two intersecting lines.

Case 2. When $C = 0$ and $D \neq 0$,

$$\text{(i)} \qquad \lambda_1 x^2 + 2f'y = 0, \quad \text{where} \quad f' = \pm\left(\frac{-D}{\lambda_1}\right)^{1/2},$$

or

$$\text{(ii)} \qquad \lambda_2 y^2 + 2g'x = 0, \quad \text{where} \quad g' = \pm\left(\frac{-D}{\lambda_2}\right)^{1/2}.$$

These are true parabolas.

Case 3. When $C = 0$ and $D = 0$,

$$\text{(i)} \qquad \lambda_1 x^2 + \frac{A + B}{\lambda_1} = 0,$$

or

$$\text{(ii)} \qquad \lambda_2 y^2 + \frac{A + B}{\lambda_2} = 0.$$

These are parallel lines.

The above forms follow directly from the fact that the expressions listed in §1 are invariant.

**3. Discussion.** When a student asks the instructor which root of the discriminating quadratic is $\lambda_1$, or which of the alternative forms in Cases 2 and 3 is correct, or which sign for $f'$ and $g'$ should be chosen, the answer sometimes given is that it does not matter and that either is correct depending on how the new axes are named. However, if $\theta$ is to be a positive acute angle this reply is not correct. The following is an attempt to determine the proper alternatives on the above assumption.

From the equations of §2 above we have when $h'=0$

$$\lambda_1 - \lambda_2 = a' - b' = (a - b)(l^2 - m^2) + 4hlm,$$

and

$$h' = (b - a)lm + h(l^2 - m^2) = 0.$$

So

$$\lambda_1 - \lambda_2 = \left\{\frac{(a - b)^2}{h} + 4h\right\} lm = \left\{(a - b)^2 + 4h^2\right\} \frac{lm}{h} \cdot$$

Since $lm$ and $(a-b)^2+4h^2$ are always positive, $\lambda_1$ and $\lambda_2$ must be so chosen that $\lambda_1-\lambda_2$ has the same sign as $h$.

For the central conics of Case 1 where $C\neq 0$, this rule is sufficient to determine which root is $\lambda_1$, and which is $\lambda_2$ in the reduced form

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{D}{C} = 0.$$

For Cases 2 and 3 where $C=0=ab-h^2$, $a$ and $b$ must always have the same sign. For simplicity, let us suppose that our original equation is written with these coefficients both positive. Then $a+b$ is also positive. When $C=0$, one root of the discriminating quadratic is zero and the other is $a+b$. Two possibilities now arise depending on the sign of $h$. If $h$ is positive, $\lambda_1=a+b$ and $\lambda_2=0$, since $\lambda_1-\lambda_2$ must have the same sign as $h$. So the reduced forms are, respectively,

$$\lambda_1 x^2 + 2f'y = 0 \quad \text{and} \quad \lambda_1 x^2 + \frac{A + B}{\lambda_1} = 0.$$

If $h$ is negative, $\lambda_1=0$ and $\lambda_2=a+b$, since $\lambda_1-\lambda_2$ must now be negative to match $h$ in sign. The reduced forms are now, respectively,

$$\lambda_2 y^2 + 2g'x = 0 \quad \text{and} \quad \lambda_2 y^2 + \frac{A + B}{\lambda_2} = 0.$$

It remains to determine the signs of $f'$ and $g'$ on the understanding that $\theta$ is a positive acute angle.

When $ab=h^2$ and $h'=hl^2+(b-a)lm-hm^2=0$, we have

$$h^2l^2 + (b - a)hlm - abm^2 = 0$$

or

$$(hl - am)(hl + bm) = 0,$$

whence

$$\frac{l}{a} = \frac{m}{h} \quad \text{or} \quad \frac{l}{b} = \frac{m}{-h} \cdot$$

When $h$ is positive,

$$\frac{l}{a} = \frac{m}{h} = \frac{1}{(a^2 + h^2)^{\frac{1}{2}}} = \frac{1}{(a^2 + ab)^{\frac{1}{2}}} \cdot$$

From §2

$$f' = fl - gm = \frac{af - gh}{(a^2 + ab)^{\frac{1}{2}}} \cdot$$

This form, which is equivalent to $\pm\left(\frac{-D}{a+b}\right)^{\frac{1}{2}}$ with the proper sign, shows that the sign of $f'$ is the same as the sign of $af - gh$, or opposite to that of $F$, the cofactor of $f$ in $D$.

When $h$ is negative,

$$\frac{l}{b} = \frac{m}{-h} = \frac{1}{(b^2 + h^2)^{\frac{1}{2}}} = \frac{1}{(ab + b^2)^{\frac{1}{2}}} \cdot$$

From §2

$$g' = gl + fm = \frac{bg - fh}{(ab + b^2)^{\frac{1}{2}}} \cdot$$

This form, which is equivalent to $\pm\left(\frac{-D}{a+b}\right)^{\frac{1}{2}}$ with the proper sign, shows that the sign of $g'$ is the same as the sign of $bg - fh$, or opposite to that of $G$, the cofactor of $g$ in $D$.

**4. Examples:**

EXAMPLE 1. $8x^2 - 4xy + 5y^2 - 36x + 18y + 9 = 0$.

$$\begin{array}{lll} a = 8 & f = 9 & a + b = 13 \\ b = 5 & g = -18 & C = ab - h^2 = 36 \\ c = 9 & h = -2 & D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = -36^2. \end{array}$$

$\tan 2\theta = -4/3$, $\tan \theta = 2$. Choose $\theta$ a positive acute angle.

$$\lambda^2 - 13\lambda + 36 = 0 \quad \text{or} \quad (\lambda - 4)(\lambda - 9) = 0.$$

Since $h$ is negative, choose $\lambda_1 = 4$, $\lambda_2 = 9$, to make $\lambda_1 - \lambda_2$ negative. The reduced equation $\lambda_1 x^2 + \lambda_2 y^2 + D/C = 0$ becomes $4x^2 + 9y^2 - 36 = 0$.

EXAMPLE 2. $x^2-4xy+4y^2+5y-6=0$.

$$\begin{array}{lll} a = 1 & f = \dfrac{5}{2} & a + b = 5 \\[2ex] b = 4 & g = 0 & ab - h^2 = 0 \\[2ex] c = -6 & h = -2 & D = \dfrac{-25}{4} \end{array}$$

$\tan 2\theta=4/3$, $\tan \theta=1/2$, $\lambda^2-5\lambda=0$.
Since $h$ is negative, choose $\lambda_1=0$ and $\lambda_2=5$ to make $\lambda_1-\lambda_2$ negative. $g'=+5^{1/2}/2$ and the positive sign must be chosen since $G$ is negative. The reduced equation is $5y^2+5^{1/2}x=0$.

EXAMPLE 3. $4x^2+12xy+9y^2+2x+3y-42=0$.

$$\begin{array}{lllll} a = 4 & f = \dfrac{3}{2} & a + b = 13 & & \\[2ex] b = 9 & g = 1 & ab - h^2 = 0 & & \\[2ex] c = -42 & h = 6 & D = 0 & & A + B = -\dfrac{13^3}{4} \end{array}$$

$\tan 2\theta=-12/5$, $\tan \theta=3/2$, $\lambda^2-13\lambda=0$.
Since $h$ is positive, $\lambda_1=13$, $\lambda_2=0$ and the reduced form is $13x^2-13^2/4=0$ or $x=\pm 13^{1/2}/2$.

## SIMPLIFICATION OF EQUATIONS OF CONICS

M. T. BIRD, Allegheny College

I should like to extend the discussion of the simplification of equations of conics initiated by Thornton* and continued by Johnston.†

Consider the general equation of the conic in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

It is assumed that $B$ is not zero. The equation of the conic may be written in the alternative form

$$A(x^2 + y^2) + y[Bx - (A - C)y] + Dx + Ey + F = 0. \tag{2}$$

The angles between the pair of intersecting lines

$$y[Bx - (A - C)y] = 0 \tag{3}$$

are bisected by the mutually perpendicular lines

* Simplification of the equations of conics, this MONTHLY, vol. 41, 1934, p. 36.

† Simplification of equations of conics, this MONTHLY, vol. 44, 1937, p. 30.

$$(4)\qquad \pm\, y = \frac{Bx - (A - C)y}{\sqrt{B^2 + (A - C)^2}}.$$

It follows that the lines (4) are axes of symmetry for the degenerate conic (3). This fact may be used to simplify equation (2).

Let the mutually perpendicular lines (4) be written in the normal form

$$(5)\qquad 0 = mx + ny,\ 0 = -nx + my;\quad m > 0,\ n \geqq 0,\ m^2 + n^2 = 1.$$

The quantities $x'$, $y'$ defined by the equations,

$$(6)\qquad x' = mx + ny,\ y' = -nx + my;\quad m > 0,\ n \geqq 0,\ m^2 + n^2 = 1,$$

measure the distances of the point $P(x, y)$ from the mutually perpendicular lines (5). The quantities $x'$, $y'$ constitute coördinates of the point $P(x, y)$ after a rotation of axes, and the equations (6) define the coördinate transformation whose inverse is seen to be

$$(7)\qquad x = mx' - ny',\ y = nx' + my';\quad m > 0,\ n \geqq 0,\ m^2 + n^2 = 1.$$

The symmetry of the degenerate conic (3) about the lines (5) leads to the conclusion that the equation of the degenerate conic (3) will assume the form

$$y[Bx - (A - C)y] \equiv Mx'^2 + Ny'^2 = 0,$$

under the transformation (7). It is readily verified that the expression $x^2+y^2$ is invariant under the transformation (7). Consequently the transformation (7) carries the general equation of the conic (2) into the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

Consider the example

$$9x^2 - 24xy + 16y^2 - 186x - 252y - 63 = 0.$$

Following the method outlined above we have

$$9(x^2 + y^2) - y(24x - 7y) - 186x - 252y - 63 = 0.$$

The new coördinate axes are defined by the equations

$$\pm\, y = (24x - 7y)/25,$$

which may be written

$$24x + 18y = 0, \qquad 24x - 32y = 0.$$

The new coördinates are defined by the equations

$$x' = (4x + 3y)/5, \qquad y' = (-3x + 4y)/5.$$

The transformation

$$x = (4x' - 3y')/5, \qquad y = (3x' + 4y')/5$$

leads to the equation of the conic in the form

$$25y'^2 - 300x' - 90y' - 63 = 0.$$

In this example the given equation may be written

$$(3x - 4y)^2 - 186x - 252y - 63 = 0.$$

The parenthetical expression suggests at once the introduction of the coördinates defined above in order to obtain the equation

$$y'^2 + ax' + by' + c = 0.$$

## AN APPLICATION OF THE REMAINDER THEOREM

R. W. WAGNER, Oberlin College

The writer of this note has long felt that a student's appreciation of the remainder theorem would be enhanced if more applications of the theorem were presented to him. In this note the remainder theorem will be used to establish two well known rules for the divisibility of one number by another. The author has been unable to find this application in the literature and would be glad to know of a reference to this use of the remainder theorem.

The conclusion of the remainder theorem can be written as the identity

$$\frac{p(x)}{x - a} \equiv q(x) + \frac{p(a)}{x - a}\cdot \tag{1}$$

In the present applications, $p(x)$ is to be a polynomial whose coefficients are in the set of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Then $p(10)$ is the ordinary decimal notation for an integer. Also, $p(1)$ is the sum of the digits of the integer.

In the identity (1) assign to $x$ the value 10 and to $a$ the value 1. This leads to the equation

$$\frac{p(10)}{9} = q(10) + \frac{p(1)}{9}\cdot \tag{2}$$

The expression $q(10)$ will represent an integer, though perhaps not in its proper decimal form. Now, if $p(1)$ is divisible by 9, $p(10)/9$ is an integer and $p(10)$ is divisible by 9. By transposing $q(10)$, the converse is similarly established. Therefore, *an integer is divisible by* 9 *if, and only if, the sum of its digits is divisble by* 9.

If both members of equation (2) are multiplied by 3, the result is

$$\frac{p(10)}{3} = 3q(10) + \frac{p(1)}{3}\cdot$$

Again, $3q(10)$ is an integer. Hence, *an integer is divisible by* 3 *if, and only if, the sum of its digits is divisible by* 3.

A rule for the divisibility of an integer by 11 can be derived from the identity (1) by assigning to $x$ the value 10 and to $a$ the value $-1$.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

E 756. *Proposed by G. Pólya, Stanford University*

Show that

$$\begin{vmatrix} a-x & 1 & 0 & 0 & \cdots & 0 \\ \binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\ \binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \end{vmatrix}$$

$$= \binom{a+n-1}{n} - \binom{2a+n-2}{n-1}x + \binom{3a+n-3}{n-2}x^2 - \cdots + (-1)^n x^n.$$

E 757. *Proposed by Kirkland Stewart, College of Puget Sound*

A wedge is cut from a right circular cylinder by an oblique plane passing through a diameter of the base of the cylinder. Find the volume of the wedge using Cavalieri's Theorem.

E 758. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A ***kite*** consists of the area bounded by a major arc of a circle of radius $r$ and the two tangents drawn at the endpoints of the arc. Show that (1) the area of the kite is equal to half the product of its perimeter by the radius $r$, (2) $Og/OG=3/2$, where $g$ and $G$ are the centroids of the perimeter and area, and $O$ is the center of the kite's arc, (3) $Og'/OG'=4/3$, where $g'$ and $G'$ are the centroids of the surface and volume of the solid of revolution obtained by revolving the kite about its axis, (4) the plane through $G'$ perpendicular to the axis bisects the lateral area of the solid, (5) the volume of the solid is equal to one third the product of its surface by the radius $r$.

E 759. *Proposed by Theodore Running, Ann Arbor, Michigan*

Show that $x^n-(x-a)^n$ can be expressed as the difference of two squares in at least one way for all positive integral values of $n$ and $x$, $a$ odd and less than $x$, not counting the obvious way when $n$ is even.

We may let $a$ be even provided $n$ and $x$ are both odd.

E 760. *Proposed by C. D. Olds, San Jose State College*

If

$$u_k = \frac{2\cdot 6\cdot 10 \cdots (4k-2)}{2\cdot 3\cdot 4\cdot 5 \cdots (k+1)},$$

find the value of

$$u_n + u_1u_{n-1} + u_2u_{n-2} + \cdots + u_{n-1}u_1 + u_n.$$

## SOLUTIONS

### A Property of a Quadrilateral

E 715 [1946, 157]. *Proposed by L. M. Kelly, University of Missouri*

Suppose $ABCD$ is a proper plane convex quadrilateral and $P$ a point exterior to this plane. Consider the four tetrahedra $PABC$, $PABD$, $PACD$, $PBCD$. If $PH$ is the shortest of all the altitudes of these four tetrahedra, show that $H$ must be interior to $ABCD$. (By a proper quadrilateral is meant one with no three vertices linear.)

*Solution by the Proposer.* We shall use the following:

LEMMA. Let $A$ and $B$ be two points of a plane $q$, $P$ a point not on $q$, and $H$ the foot of the perpendicular from $P$ to $q$. If $C$ is a point of $q$ such that the half ray $HC$ intersects the trace of $AB$ in say, $X$, so that $C$ is between $H$ and $X$, then the perpendicular from $C$ to the plane $PAB$ is shorter than $PH$.

The proof is easy. We might note that $X$ may be the point at infinity on $AB$.

THE MAIN THEOREM. Let $ABCD$ be the quadrilateral, and suppose $DC$ intersects $AB$ in $M$ and $BC$ intersects $AD$ in $N$ so that $ABM$ and $BCN$ are the respective orders. This is clearly just a question of notation. Now consider the possible exterior positions of $H$. If it is on the "exterior side" of $AB$, then the half ray $HB$ would surely intersect one of the traces $AD$ or $DC$ in a point $X$ such that $B$ lies between $H$ and $X$, and the distance from $B$ to one of the planes $PAD$ or $PDC$ would be less than $PH$. Similarly an examination of the "exterior sides" of the other three sides of the quadrilateral reveals a contradiction.

### A Poristic Construction

E 713 [1946, 157]. *Proposed by Joseph Rosenbaum, The Milford School, Conn.*

Find a euclidean construction for a non-regular pentagon which has both a circumcircle and an incircle.

*Solution by the Proposer.* It is known that the condition on two circles that they be respectively the circumcircle and the incircle of a pentagon is

$$(R^2 - d^2)^3 + 2rR(R^2 - d^2)^2 - 4r^2R^2(R^2 - d^2) - 8r^3Rd^2 = 0, \tag{1}$$

where $R$, $r$, $d$ are the two radii and distance between the centers (Jacobi's *Collected Works*, VI, p. 277 *ff*).

Equation (1) has the parametric solution

$$R = 2pq^2,$$
$$r = (q - p)(q + p)^2,$$
$$d = 2p(p^3q + p^2q^2 - pq^3)^{1/2}.$$

Now by taking a pair of otherwise arbitrary values for $p$ and $q$, with the restriction that they be positive and

$$q > p > (\sqrt{5} - 1)q/2,$$

the quantities $R$, $r$, $d$ will be real and positive, and furthermore are constructible by compasses and straight-edge, and by Poncelet's porism any one of the corresponding set of pentagons can be constructed, and because $d \neq 0$, the pentagon will be non-regular.

W. B. Clarke produced an interesting symmetrical non-regular pentagon having an incircle and a circumcircle. To obtain the pentagon inscribe an equilateral triangle $ABD$ in a circle, and then locate $C$ and $E$ on the circle such that $\measuredangle BAC = \measuredangle ABE = 40°$. The pentagon $ABCDE$ has an incircle with center at the intersection of the diagonals $AC$ and $BE$. This pentagon, however, is not constructible with euclidean tools.

**Two Applications of Casey's Theorem**

E 720 [1946, 220]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

At the vertices of an equilateral triangle three equal circles are drawn externally tangent to the circumcircle. Show that one of the three tangents to these equal circles, from any point whatever on the circumcircle, is equal to the sum of the other two.

E 728 [1946, 333]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the vertices $A$, $B$, $C$ of an equilateral triangle as centers draw the circles $(A)$, $(B)$, $(C)$ which are concurrent at the center $O$ of the triangle, and then draw an arbitrary circle $(D)$ passing through $O$. Show that the length of one of the common tangents to the circles $(A)$ and $(D)$, $(B)$ and $(D)$, $(C)$ and $(D)$ is equal to the sum of the lengths of the other two.

*Combined solution by Rufus Crane, Ohio Wesleyan University.* These two problems are special cases of Casey's Theorem in the geometry of the circle, which states a relation between the lengths of the common tangents to four circles touching a fifth:

$$t_{12}t_{34} \pm t_{13}t_{24} \pm t_{14}t_{23} = 0.$$

In E 720 the four circles are the three equal circles $S_1$, $S_2$, $S_3$ tangent to the circumcircle, and the point on the circumcircle considered as a point-circle $S_4$. Then $t_{12} = t_{13} = t_{23}$, each being equal to the distance between centers of the equal circles.

In E 728 the four circles are the circles $(A)$, $(B)$, $(C)$, $(D)$, while the fifth circle to which they are tangent is the point-circle $O$. Here $t_{AB}=t_{AC}=t_{BC}$ as in E 720. Identifying $A$, $B$, $C$, $D$ with 1, 2, 3, 4 we have, in each problem

$$t_{34} \pm t_{24} \pm t_{14} = 0.$$

E 720 was also solved by E. F. Allen, Paul Brock, W. E. Byrne, W. B. Clarke, L. M. Kelly, B. R. Leeds, H. R. Leifer, D. W. Matlock, Norman Miller, C. C. Oursler, C. L. Perry, P. A. Piza, Joseph Rosenbaum, P. D. Thomas, R. H. Urbano, and the proposer.

Most solvers gave a straightforward trigonometrical solution. Byrne employed inversive geometry; Thomas offered a synthetic solution employing the theorems of Stewart and Ptolemy; Urbano used analytic geometry. Several solvers noted that the theorem is also true when the circles are internally tangent to the circumcircle and not greater than the circumcircle. In addition, Rosenbaum generalized to an arbitrary triangle with the conclusion that the product of the tangent to one of the equal circles and the side of the triangle opposite the point of contact of that circle with the circumcircle is equal to the sum of the other two corresponding products. As another generalization he noted that if the radii $r_1$, $r_2$, $r_3$ of the three circles are such that $(R+r_1):(R+r_2):(R+R_3)=a_1^2:a_2^2:a_3^2$, where $R$ is the circumradius and $a_i$ are the sides of the triangle, then the conclusion of E 720 continues to hold.

E 728 was also solved by Paul Brock, L. M. Kelly, P. A. Piza, R. H. Urbano, and the proposer.

It is to be noted that both E 720 and E 728 are generalizations of the classic relation $MA+MB=MC$, where $M$ is any point on the minor arc $AB$ of the circumcircle of the equilateral triangle $ABC$.

Casey's Theorem may be found in Coolidge, *Treatise on the Circle and the Sphere*, p. 37, Lachlan, *Modern Pure Geometry*, pp. 244–250, or Johnson, *Modern Geometry*, pp. 121–126.

### A Property of $x^3+y^3=z^3$

E 729 [1946, 333]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Let $x$, $y$, $z$ be three real positive numbers such that $x^3+y^3=z^3$, and set

$$\lambda = \sqrt[3]{3(x+y)}/(\sqrt[3]{z-x}+\sqrt[3]{z-y}).$$

Show that

$$(\sqrt[3]{2}+\sqrt[3]{4})/2 < \lambda < \sqrt[3]{3}.$$

*Solution by E. P. Starke, Rutgers University.* Because of the symmetry of the problem in $x$ and $y$, we may expect extreme values of $\lambda$ to occur when $x=0$ (or $y=0$) and when $x=y$. After interchanging $x$ and $y$ if necessary, we may take $x\leqq y<z$. For the cases mentioned we compute easily

$$\lambda = \sqrt[3]{3} \quad \text{and} \quad \lambda = (\sqrt[3]{2}+\sqrt[3]{4})/2,$$

respectively. To show that there are no other extreme values of $\lambda$, and hence that the given inequalities must hold (since $\lambda$ is a continuous function of $x, y, z$), it seems easiest to substitute

$$x + y = a^3, \qquad z - x = b^3, \qquad z - y = c^3, \qquad a, b, c \text{ positive.}$$

We obtain without difficulty

$$\lambda = \sqrt[3]{3}\, a/(b + c), \qquad a^3 - b^3 - c^3 = 2\sqrt[3]{3}\, abc.$$

By eliminating $a$ between these two equations and substituting $w = c/b$ we find

$$(1) \qquad (1 + w)^2(\lambda^3 - 3) = 3w(2\lambda - 3).$$

If we put $d\lambda/dw = 0$, we obtain from (1) the condition for extreme $\lambda$ as

$$(2) \qquad 2(1 + w)(\lambda^3 - 3) = 3(2\lambda - 3).$$

Upon eliminating $w$ between (1) and (2) we have

$$(2\lambda - 3)(\lambda^3 - 3)(4\lambda^3 - 6\lambda - 3) = 0.$$

The first factor cannot vanish since it corresponds to $w = -1$. The other factors give $w = 0$ and $w = 1$, whence $x = 0$ and $x = y$, as treated above. Now $x = 0$ is excluded by hypothesis, but $x = y$ is not. Hence $\lambda$ can actually attain its lower bound, but not its upper, and the conclusion of the theorem should be corrected to read

$$(\sqrt[3]{2} + \sqrt[3]{4})/2 \leqq \lambda < \sqrt[3]{3}.$$

Also solved by Murray Barbour, P. A. Piza, and the proposer.

Since, to five decimal places, $(\sqrt[3]{2}+\sqrt[3]{4})/2 = 1.42366$ and $\sqrt[3]{3} = 1.44225$, we note that $\lambda$ is quite restricted. The proposer gave the example $x = 10$, $y = \sqrt{82} - 2$, $z = \sqrt{82} + 2$, $\lambda = 1.425$ (approximately).

### The Interior Bisectors and the Medial Triangle

E 730 [1946, 333]. *Proposed by J. H. Butchart, Arizona State College*

The interior angle bisectors of a triangle meet the noncorresponding sides of the medial triangle in six points which lie in pairs on the lines joining the points of tangency of the inscribed circle.

*Solution by L. M. Kelly, University of Missouri.* Consider a triangle $ABC$, the midpoints of the respective sides being $A_1$, $B_1$, $C_1$. Let the interior bisector of angle $A$ meet $A_1C_1$ in $A_2$, and that of $B$ meet $B_1C_1$ in $B_2$. Join $A_2B_2$ and call the intersections of $A_2B_2$ with $AC$ and $BC$ respectively $B_3$ and $A_3$. We propose to show that $B_3$ and $A_3$ are points of contact of the inscribed circle. If true, the theorem of the problem would follow at once.

We observe that $C_1B_2$ is parallel to $BC$, and hence $BC_1B_2$ is isosceles. Thus $B_2C_1 = \frac{1}{2}c = A_2C_1$. Furthermore, since $B_1B_3$ is parallel to $A_2C_1$, triangle $B_2B_1B_3$ is isosceles, and $B_1B_3 = B_1B_2 = \frac{1}{2}c - \frac{1}{2}a$. Thus $B_3C = \frac{1}{2}(b + c - a) = s - a$, and $B_3$ is a point of contact of the inscribed circle.

$$\cos v, \qquad \sin v, \qquad \phi' - u\phi'',$$

where the primes indicate differentiation with respect to $u$. Show that the developables of the congruence intersect the revolute in its lines of curvature.

## SOLUTIONS

### Planes Bisecting the Volume and the Area of a Tetrahedron

3879 [1938, 389]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that it is possible to determine a plane section, limited by three faces of a tetrahedron which divides both the surface and the volume into equal parts. Show that the plane passes through the center of the inscribed sphere.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.* Take oblique cartesian space axes $Dx$, $Dy$, $Dz$ along the edges $DA$, $DB$, $DC$ of the given tetrahedron $DABC$. Let $\lambda$, $\mu$, $\nu$ denote the angles $yDz$, $zDx$, $xDy$, respectively, and let

$$ux + vy + wz + 1 = 0$$

be the equation of the required plane section. If $V$ denotes the volume and $r$ the inradius of the given tetrahedron, and $\omega$ the sine of the solid angle $D-ABC$, we have the relations

$$\omega/uvw = -3V, \tag{1}$$

$$(\sin \lambda)/vw + (\sin \mu)/wu + (\sin \nu)/uv = 3V/r, \tag{2}$$

whence we find

$$(ur \sin \lambda)/\omega + (vr \sin \mu)/\omega + (wr \sin \nu)/\omega + 1 = 0.$$

If $DX$, $DY$, $DZ$ are the edges of the trihedral angle supplementary to $D-xyz$, and if $\alpha$, $\beta$, $\gamma$ denote the angles $xDX$, $yDY$, $zDZ$, respectively, then

$$\omega = \sin \lambda \cos \alpha = \sin \mu \cos \beta = \sin \nu \cos \gamma$$

and we have

$$ur/\cos \alpha + vr/\cos \beta + wr/\cos \gamma + 1 = 0.$$

Since $(r/\cos \alpha, r/\cos \beta, r/\cos \gamma)$ are the coördinates of the incenter of the tetrahedron, this last equation shows that the plane section must pass through the incenter.

There are, then, an infinite number of solutions associated with the vertex $D$ of the tetrahedron. These solutions are given by the tangent planes of a cone having for vertex the incenter of the tetrahedron and circumscribed about the two third class surfaces (1) and (2). Similar infinitudes of solutions are obtained for the other vertices of the tetrahedron.

*Editorial Note.* In contrast to the fourfold infinitude of solutions for the tetrahedron we find that the analogous problem for the triangle possesses, in general, only six solutions, all passing through the incenter of the triangle.

**Law of the Mean for Integrals**

4173 [1945, 463]. *Proposed by Herbert Robbins, United States Naval Academy, Annapolis*

Let $a<b$ be given numbers and let $f(t)$ be defined, continuous, non-negative, and strictly increasing for $a\leqq t\leqq b$. By the law of the mean for integrals, for every $p>0$ there will exist a unique number $x_p$, $a\leqq x_p\leqq b$, such that

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt.$$

Find $\lim_{p\to\infty}x_p$.

*Solution by M. J. Norris, United States Naval Academy, Annapolis.* Let $\epsilon$ be a given number such that $0<\epsilon<(b-a)/2$. Since $f(t)$ is strictly increasing, $f(b-\epsilon)/f(b-2\epsilon)>1$. Hence there exists a positive integer $P$ such that for $p>P$,

$$\left[\frac{f(b-\epsilon)}{f(b-2\epsilon)}\right]^p > \frac{b-a}{\epsilon}$$

or

$$f^p(b-\epsilon) > \frac{b-a}{\epsilon} f^p(b-2\epsilon).$$

We have also

$$\int_a^b f^p(t)dt > \int_{b-\epsilon}^b f^p(b-\epsilon)dt,$$

and therefore

$$\frac{1}{b-a}\int_a^b f^p(t)dt > \frac{1}{b-a}\int_{b=\epsilon}^b f^p(b-\epsilon)dt = \frac{1}{b-a}\cdot\epsilon\cdot f^p(b-\epsilon) > f^p(b-2\epsilon).$$

Hence $x_p$ cannot satisfy

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt$$

unless $x_p>b-2\epsilon$. It follows that there exists a $P$ such that, whenever $p>P$, $x_p>b-2\epsilon$. Hence $\lim_{p\to\infty}x_p=b$.

Solved also by R. C. Buck, and the Proposer.

Buck's solution makes use of a standard theorem, Hardy, Littlewood, Pólya, *Inequalities*, p. 143. $f(x_p)=\mathfrak{M}_p(f;\ a,\ b)$ tends, as $p\to\infty$, toward the essential maximum of $f(t)$. Thus, since $f(t)$ is strictly monotonic, and continuous, and has a continuous inverse, $\lim f(x_p)=f(b)$, and $b=\lim x_p$.

**Boolean Rings**

4174 [1945, 463]. *Proposed by Irving Kaplansky, Harvard University*

Stone has called a ring "Boolean" if all its elements satisfy the equation $x^2=x$. Show that a ring in which $x^2=\pm x$ is either Boolean or the direct sum of a Boolean ring and the Galois field of three elements.

*Solution by M. F. Smiley, United States Naval Academy.* Let $B$ be a ring satisfying $x^2 = \pm x$. If $x^2 = x$, then $x^3 = x^2 = x$; while if $x^2 = -x$, then $x^3 = -x^2 = x$. Hence always $x^3 = x$ for $x \in B$. From $(2x)^3 = 2x$, we obtain $6x = 0$ for every $x \in B$. Let $B_2$ and $B_3$ be the ideals for which $2x = 0$ for $x \in B_2$ and $3x = 0$ for $x \in B_3$. Clearly $B_2$ is a Boolean ring and $B$ is the direct sum of $B_2$ and $B_3$. If $B_3 \neq (0)$, let $z \in B_3$, $z \neq 0$. There is, then, an element $x \in B_3$ for which $x^2 = x$ and $x \neq 0$, for we may take $x = z$ if $z^2 = z$ and $x = -z$ otherwise. Now if $y \neq 0$ and $y \in B_3$, then either (a) $y^2 = y$, or (b) $y^2 = -y$. Case (a). By direct computation $(x+y)^2 + (x-y)^2$ has the value $-x-y$, while our basic assumption permits $\pm(x+y) \pm(x-y) = \pm 2x$ or $\pm 2y = \mp x$ or $\mp y$. Since both $x$ and $y$ are non-zero, we conclude that $y = x$. Case (b). Here $(x+y)^2 + (x-y)^2$ has the value $y - x$. Consequently, since both $x$ and $y$ are non-zero, we have $y = -x$. It follows that $B_3 = [0, x, -x]$, as was to be proved.

Solved also by M. J. Norris, Olga Taussky and John Todd, and the Proposer.

Taussky and Todd remark that this ring is commutative, and that its components $B_2$ and $B_3$ are orthogonal. The Proposer proves the former in his abstract, *Bulletin of the American Mathematical Society*, January 1945.

### Intersections of a Quadric and a Tetrahedron

4178 [1945, 522]. *Proposed by N. A. Court, University of Oklahoma*

If among the twelve points of intersection of a quadric surface with the edges of a tetrahedron there are six points located on the six edges, such that the three lines joining the points on opposite edges are concurrent, the remaining six points of intersection also have this property.

*Note.* This is the converse of a known proposition (see, for example Baker, *Principles of Geometry*, vol. 3, p. 54, ex. 17, 1923): The transversal is drawn from a point $O$ to each pair of opposite joins of four points $A$, $B$, $C$, $D$, so giving rise to six points. Six other points, one on each join, are similarly obtained from another point $O'$. The twelve points lie on a quadric.

*Solution by Howard Eves, Oregon State College.* We shall employ the lemma: *Given a tetrahedron $A_1A_2A_3A_4$ and six points $A_{12}$, $A_{13}$, $A_{14}$, $A_{23}$, $A_{24}$, $A_{34}$, point $A_{ij}$ lying on the edge $A_iA_j$, then a necessary and sufficient condition for $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ to be concurrent in a point $P$ is that the sets $A_2A_{34}$, $A_3A_{24}$, $A_4A_{23}$; $A_1A_{34}$, $A_3A_{14}$, $A_4A_{13}$; $A_1A_{24}$, $A_2A_{14}$, $A_4A_{12}$; $A_1A_{23}$, $A_2A_{13}$, $A_3A_{12}$ be respectively concurrent in points $P_1$, $P_2$, $P_3$, $P_4$.*

The necessary part of the lemma is readily established by considering projections of the entire figure from the vertices onto the opposite faces. This same projection procedure shows that the sufficiency condition insists that $A_1P_1$, $A_2P_2$, $A_3P_3$, $A_4P_4$ are either concurrent or else constitute a hyperbolic set of lines. The second possibility, however, is eliminated by the fact that $A_1P_1$ $A_2P_2$ must intersect, since both lie in the plane $A_1A_{34}A_2$. This guarantees that $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ are also concurrent.

We now establish the given theorem for the case where the quadric is a

sphere, noting that the general case follows from this one by projection. Let the sphere cut the edge $A_iA_j$ in the two points $A_{ij}$, $B_{ij}$, and assume that $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ are concurrent in a point $P$. Then, by the lemma, the sets $A_2A_{34}$, $A_3A_{24}$, $A_4A_{23}$; $\cdots$ are concurrent in points $P_1$, $P_2$, $P_3$, $P_4$. It therefore follows that (see, for example, art. 227 in Johnson's *Modern Geometry*) $B_2B_{34}$, $B_3B_{24}$, $B_4B_{23}$; $\cdots$ are concurrent in points $Q_1$, $Q_2$, $Q_3$, $Q_4$. Hence, by the lemma again, $B_{12}B_{34}$, $B_{13}B_{24}$, $B_{14}B_{23}$ are concurrent in a point $Q$.

Solved also by the Proposer who refers to a theorem in his *Modern Pure Solid Geometry*, p. 119, art. 348, and also to a theorem in the *Nouvelles Annales de Mathématiques*, 1842, p. 403, and due to O. Terquem, cofounder and editor.

### Coplanar Radical Axes

3946 [1940, 115]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$, and an arbitrarily chosen point $P$: (1) Show that the radical planes of an arbitrary sphere through $P$ with each of the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, circumscribing the tetrahedrons in the parentheses, meet respectively the planes of the faces $BCD$, $CDA$, $DAB$, $ABC$, in four lines lying in a plane. (2) If an arbitrary plane cuts the planes of the faces $BCD$, $CDA$, $DAB$, $ABC$ in the straight lines $\Delta_a$, $\Delta_b$, $\Delta_c$, $\Delta_d$, show that the planes $(P, \Delta_a)$, $(P, \Delta_b)$, $(P, \Delta_c)$, $(P, \Delta_d)$ cut the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, respectively, in four circles of a single sphere through $P$.

*Solution by R. Bauvaist, Vincelles, Saône-et-Loire, France*

(1) Let $S$ be the sphere $(ABCD)$ and $\Sigma$ the arbitrary sphere through $P$. If the radical plane of $S$ and $\Sigma$ intersects the plane $BCD$ in the line $\Delta_a$, then every point of $\Delta_a$ has the same power with respect to $S$, $\Sigma$, and the sphere $(PBCD)$. It follows that the four lines $\Delta_a$, $\Delta_b$, $\Delta_c$, $\Delta_d$ are the four lines in question, and they all lie in the radical plane of $S$ and $\Sigma$.

(2) Conversely, $\pi$ being the plane containing $\Delta_a$, $\Delta_b$, $\Delta_c$, $\Delta_d$, there exists a sphere $\Sigma$ passing through $P$ and having $\pi$ as radical plane with sphere $S$. This sphere $\Sigma$ passes through the points of intersection of $\Delta_a$, $\Delta_b$, $\Delta_c$, $\Delta_d$ with the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, whence the four circles of intersection of the four spheres with the planes $(P, \Delta_a)$, $(P, \Delta_b)$, $(P, \Delta_c)$, $(P, \Delta_d)$ all lie on $\Sigma$.

### Row Sums of a Matrix

4168 [1945, 400]. *Proposed by Henry Scheffé, University of California at Los Angeles*

The matrix $X=(x_{ij})$ is real $m\times n$ with $m\leqq n$. Let the $i$th row sum be $y_i = \sum_j x_{ij}$. The elements $x_{ij}$ may vary subject to the conditions (1) $XX'=I$, (2) $y_1=y_2=\cdots=y_m$. Write $y_i=y$. Show that the maximum and minimum values of $y$ are $\sqrt{n/m}$ and $-\sqrt{n/m}$ respectively.

*Solution by R. C. Buck, Harvard University.* Let $H_k$ be the $1\times k$ matrix $(1, 1, \cdots, 1)$. If $A$ is an $n\times n$ matrix, then $\sum(A)$, the sum of all the entries

of $A$, is given by $H_nAH_n'$. The condition that the sum of each of the rows of $X$ is $y$ may be expressed by $H_nX'=yH_m$. Let $A=X'X$. Then

$$\sum(A) = H_n(X'X)H_n' = (yH_m)(yH_m') = my^2.$$

Since $XX'=I$,

$$AA' = X'X(X'X)' = X'XX'X = X'IX = A.$$

Thus,

$$my^2 = \sum(A) = \sum(AA') = H_nAA'H_n'$$

so that

$$my^2 = (H_nA)H_n' = (H_nA)(H_nA)'.$$

Let $H_nA=(x_1, \cdots, x_n)$ be a point of $n$-space. Our equation then becomes

$$my^2 = \sum_1^n x_i = \sum_1^n x_i^2,$$

and $\sum_1^n(x_i^2-n_i)=0$ is the equation of a sphere, center $(\frac{1}{2}, \cdots, \frac{1}{2})$, passing through the origin. $\sum_1^n x_i^2$ is the square of the distance from $P$ to the origin, and is greatest when $P$ is diametrically opposite to the origin, this value being $n$. Thus $my^2 \leqq n$ and $|y| \leqq n/m$.

The restriction $m \leqq n$ seems unnecessary.

### Regular Polygons

4086 [1943, 391: 1944, 480]. Corrected. *Proposed by Paul Erdös, University of Michigan*

Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon and let $O$ be any point in its interior. Show that at least one of the angles $A_iOA_j$ satisfies the relation:

$$\pi\left(1 - \frac{1}{2n+1}\right) \leqq A_iOA_j \leqq \pi.$$

*Solution by C. R. Phelps, Rutgers University.* The stated proposition is true for any number of sides, even or odd, and hence will be shown for a regular $n$-gon.

Let $A_1$ be a vertex such that $OA_1 \leqq OA_j$ for all $j \neq 1$. The lines $A_1A_j$ divide the interior of the polygon so that $O$ lies in, or on the boundary of, some triangle $A_1A_kA_{k+1}$. If $O$ lies on such a boundary, the theorem follows trivially: $A_1OA_k=\pi$. Otherwise, $O$ lies inside the triangle $A_1A_kA_{k+1}$, and the angles $A_1OA_k<\pi$ and $A_1OA_{k+1}<\pi$. Also, since $OA_1 \leqq OA_j$, angles $OA_kA_1 \leqq OA_1A_k$ and $OA_{k+1}A_1 \leqq OA_1A_{k+1}$. But $OA_1A_k+OA_1A_{k+1}=\pi/n$. Thus $A_1OA_k+A_1OA_{k+1}=2\pi-(OA_1A_k+OA_1A_{k+1})-(OA_kA_1+OA_{k+1}A_1) \geqq 2\pi-\pi/n-\pi/n=2\pi(1-1/n)$.

Therefore, either $A_1OA_k \geqq \pi(1-1/n)$ and the theorem is proved, or else $A_1OA_k<\pi(1-1/n)$ and, by subtraction, $A_1OA_{k+1} \geqq \pi(1-1/n)$.

Solved also by Robert Steinberg and the Proposer.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*College Algebra*. By M. W. Keller. Boston, Houghton Mifflin Co., 1946. 12+471 pages. $3.00.

In examining a text like Professor Keller's, one looks, first, for clarity and rigor in the subject matter itself and, second, for a certain liveliness and pedagogical insight in the manner of presentation. (Or perhaps I should list the qualities in reverse order!) Either attribute without the other, in an elementary text, is largely lost. And, considering the tremendous possibilities of the two, I think that new elementary texts still can be greeted with eagerness and serious judgement.

Mr. Keller's *College Algebra* has many unique and fine characteristics to recommend it. The contents include, in addition to the standard, basic topics of college algebra, chapters on Theory of Equations (including Horner's Method and solutions for the general cubic and general quartic), Determinants (including expansion by minors and solutions to $n$ homogeneous linear equations in $n$ unknowns), Mathematics of Finance, Probability and Insurance, Partial Fractions, and Infinite Series. There is no unusual overlapping of topics or anticipation of more advanced mathematics courses.

The clarity of the discussion is at once impressive. There is no hesitation in giving even more detailed explanations than are customary. Indeed, even *incorrect* manipulations are anticipated and pointed out. (Whether psychologists would disagree with the display of wrong cancellations, I do not know—but no one who has had much experience teaching college algebra would disagree!) The presentation of the long division process for polynomials is certainly self-sufficient; the distinction between dependent and inconsistent linear equations is made simple and straightforward; and, particularly, I like the development of mathematical induction through specific examples. The diagrams are streamlined and simple, yet not lacking in essential detail.

Rigor, as well, is given conscientious attention, with the result that several customary pitfalls are neatly avoided. For instance, in rules for operations with exponents, the generalization from positive rational integral to fractional exponents is careful and explicit. The necessity of the check in the solution of fractional equations is emphasized. And, all the way through, pointed distinctions are drawn between general proofs and specific examples.

But it is in the pedagogical insight of the book that Mr. Keller hits his highest stride. It is certainly a pleasure to discover, already written out, things which instructors inevitably find themselves writing out, over and over again. For instance, the derivation of the synthetic division process from long division is given fully, step by step. And as an introduction to the law $b^n \div b^m = b^{n-m}$, the terms in

$5^5 \div 5^2$ are written out completely. Then, when logarithms are presented (not, I regret, immediately after exponents), the logarithms appear, in the first few examples, actually written as exponents on 10. This is indeed refreshing! And there are other such welcome innovations.

It is against the most intangible quality of all—the liveliness and persuasiveness of the presentation—that I levy my only adverse criticism. Particularly in the somewhat stilted beginning of each chapter is lost a real chance for capturing the student's interest. The initial remarks that are made have precision and orderliness—qualities which would attract the graduate mathematics student; but to the undergraduate, they would seem lifeless and uninteresting. (I would not, for this reason, shrink for one moment from the use of mature mathematical vocabulary.) It is difficult for the beginning student to understand the general statement of what is to come unless he first sees specific examples of it. By introducing earlier in the various chapters such interesting examples as occur later, the author could both stimulate initial interest in each topic and give even more pointed indication of the immense and sweeping applicability of the science. The reviewer may appear finical, but this is no trivial matter; it is the "small spark of inspiration, which makes all the difference in the world."

However, let not this one objection detract from the value of the book. All in all, it is a unique and excellent newcomer to the catalogue of college algebra texts.

A. F. STREHLER

*College Mathematics; A General Introduction.* By C. H. Sisam. New York, Henry Holt and Co., 1946. 13+561 pages. $3.50.

As stated in the preface "this text presents the customary first-year course in college algebra, trigonometry, and analytic geometry, together with the notation and elementary processes and applications of the differential and integral calculus."

Except for the calculus there is ample material in the book for a separate course in each of these subjects, although in places the treatment is necessarily brief; the needs of these separate courses, or of a combined course, are admirably met. The derivative is introduced in the chapter on tangents and normals and the discussion is carried through the study of maxima and minima. Integration is covered in the same chapter, through its application to finding area. The average student, it would appear, might find difficulty with this section.

The book does not offer a "unified" course. For the most part the subjects are presented in distinct units, but when it is thought desirable, a topic may be presented in various parts. For example, polar coordinates are introduced after the treatment of the circle, and continued at this stage through the general polar equation of a circle; the polar equation of a conic comes later, in the chapter in conic sections; and then in the chapter on the graph of an equation, polar coordinates are resumed in this connection. This division seems to make the topic fit more naturally into the scheme of things. Further, in this vein, this same

chapter on the graph of an equation comes after that on the calculus, and use is made of slopes, maxima and minima, *etc.*

A highly noteworthy feature of the book is the accuracy of the definitions and explanations. To mention a few of the former, there are the definitions of rational number, given early, of the equation of a locus, and of an extraneous root of an equation, all stated explicitly. For the latter may be mentioned the section on computations involving zero, the statement of the terminology of proportions, and the discussion of inverse trigonometric functions. These, and many others, should reduce the number of students' questions to a minimum.

The exercises are sufficient in number, of varying complexity for selection by the instructor, with answers furnished for the odd-numbered ones, and those for the even-numbered to be printed separately.

There are approximately one hundred pages of tables, with explanations for their use.

A. SPITZBART

*Scientific, Medical, and Technical Books Published in the United States of America 1930–1944; A Selected List of Titles in Print with Annotations.* Edited by R. R. Hawkins, Washington, National Research Council, 1946. 15+1114 pages. $20.00.

"This volume contains a selected list of medical, scientific and technical books by citizens of this country and Canada published within the limits of the continental United States since 1930—books still in print and available for both domestic and foreign distribution." Approximately six thousand works are described, including 424 on mathematics, 211 on physics and 60 on astronomy. The complete title, table of contents, publisher and price of each book is supplemented by a brief descriptive note. For the most part these annotations are written with care and restraint to emphasize significant or unusual features of a book and also to indicate any special use or purpose. The roster of recent mathematical books is quite complete, although a few well known titles are not listed—perhaps because they were temporarily out of print when the volume was compiled. Noteworthy features are an author index, a subject index and a handy directory of publishers.

B. H. COLVIN

## NEW BOOKS RECEIVED

*Introduction to College Mathematics.* By C. V. Newsom. New York, Prentice-Hall, Inc., 1946. 8+344 pages. $4.65 trade price, $3.50 text price.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### CLUB REPORTS, 1945-46

#### Pi Mu Epsilon, University of Georgia

The Georgia Alpha Chapter of *Pi Mu Epsilon* has been most fortunate in having Dr. Tomlinson Fort, Director General, *Pi Mu Epsilon* Fraternity, as Head of the Mathematics Department of the University of Georgia. Dr. Fort and Dr. D. F. Barrow, Faculty Director of the local chapter, have worked together in securing several very prominent mathematicians for *Pi Mu Epsilon* programs.

*Fermat's last theorem,* by Dr. Archibald Henderson of the University of North Carolina—a chapel program.

*Fundamental theorem of arithmetic,* by Dr. H. S. Vandiver of the University of Texas.

*Applications of geometry,* by Dr. J. M. Thomas of Duke University.

*Applied mathematics,* by Dr. R. G. D. Richardson of Brown University.

Topics discussed by members of the local chapter were:

*Euler's transformation,* by Dr. T. Fort and Dr. D. F. Barrow.

*Theory of numbers,* by Dr. R. J. Levit.

*Nomographs,* by Mr. William Burke.

*Graphical solution of spherical triangles,* by Dr. D. F. Barrow.

*Mathematics for naval air cadets,* by Mr. P. A. De Vore.

*Development of scientific organizations with particular emphasis on mathematical organizations,* by Dr. T. Fort.

Two initiation and social meetings were held in December and March at Dr. Fort's home.

Sponsored by *Pi Mu Epsilon,* the Pythagorean Freshman Club was organized in January. The purpose is to promote an appreciation of mathematics.

Officers for 1945–1946 were: President, Catherine Littlejohn; Vice-President, Jean Hannmack; Secretary-Treasurer, Bob Jones; Corresponding Secretary, Iris Callaway; Faculty Director, Dr. D. F. Barrow. Officers for 1946–1947 are: President, Marquilla Stuckey; Secretary-Treasurer, Frances Hammond; Corresponding Secretary, Iris Callaway; Faculty Director, Dr. R. J. Levit.

#### Kappa Mu Epsilon, Albion College

The *Michigan Alpha* Chapter of *Kappa Mu Epsilon* had many interesting meetings during the past year. One in particular centered about the history of Albion's *Mathematics Club.* For roll call each member gave or read a letter containing an autobiography of some woman member of this organization who had, since graduation, done graduate work.

Other topics for roll call were tricks or puzzles in mathematics, contributions

made to trigonometry, and what each member personally wished to do with his background in mathematics.

Two movies constituted one program, and papers were presented at other meetings on the following subjects:

*The use of logarithms in addition and subtraction*
*Colorimetry*
*The V-T fuse*
*Women in mathematics*
*Casting counters and counting boards*
*Philosophy of mathematics*
*Scales of notation*
*Nomographs.*

Two veterans, Robert Maynard and Phil Sawyer, returned to join the group composed of Janis Barker, Marion Bunte, Ruth Helzer, Harriette Leonard, Audrey McPherson, Charles Parkhurst, Audrey Schuett, Mary Shattuck, Amy Thomas, Dr. E. E. Ingalls, and Dr. E. R. Sleight.

The new members initiated during the year were: Jean Moffett, Irwin Weber, Nell Barton, Olive Conway, Faye Engstrom, William Hopkins, Dorothy Manley, John Nizon, Lucille Richardson, Shirley Searles, Eleanor Harper, and Mrs. Myrtle Shattuck.

Officers during the year were: President, Mary Shattuck; Vice-President, Harriette Leonard; Secretary-Treasurer, Marion Bunte; Program committee member, Janis Barker.

### Mathematics Club, University of Virginia

There were five meetings of the elementary *Mathematics Club* with the following speakers and subjects:

*Mathematical puzzles,* by W. R. Utz
*Some problems in number theory,* by E. E. Floyd
*History and properties of the number* $\pi$, by E. V. N. Goetchius
*Some problems in maxima and minima,* by P. B. Buck.

### Kappa Mu Epsilon, Chicago Teachers College

The programs and events recently completed began with an orientation meeting at which refreshments were served. At subsequent meetings the following topics were discussed:

*Use of the slide rule*
*Nomography,* by Dr. J. S. Georges, of Wright Junior College
*Non-euclidean geometry,* by Dr. J. Sachs
*Spherical trigonometry,* by Miss Mary Kirkpatrick
*Must a mathematician teach?,* by Professor J. H. Zant of Oklahoma A. & M. College. Eighty-three persons were present at this interesting meeting.

Thirty-three people attended a dinner party at which kodachrome slides and movies of the "Rockies" and the Northwest were shown. The high light of another meeting was the presentation of a token to Dr. J. T. Johnson upon his

retirement from the mathematics department. At the initiation party twenty-three new members were initiated into *Illinois Gamma* Chapter of *Kappa Mu Epsilon.*

At present each member is preparing to present a problem in which he is especially interested. These problems are those not usually found in class room work, are not too difficult, and may be understood by the membership. Art students have been interested in making posters advertising the meetings and frequently have succeeded in portraying novel ideas concerning certain aspects of mathematics. Articles are being prepared by some of the student members for publication in *The Pentagon.*

The officers for the year 1946–47 are: President, Cloda Augelli; Vice-President, Joseph Duffy; Secretary, Mary Therese Graham; Treasurer, Dolores Grien; Faculty Sponsor, Professor J. J. Urbancek.

### Mathematics Club, Oberlin College

The Oberlin College Mathematics Club held nine regular meetings in 1945–46. Refreshments were served at each meeting after which a paper was presented by a student. Each paper was submitted to a faculty committee to be judged for the Mary Emily Sinclair prize. The following papers were presented:

*Scales of notation*—Artha Jean Burington
*Schwarz paradox*—Jerry Howald
*Space filling curves*—Mary Wright
*Continued fractions*—Charlotte Kessler
*The Apollonius circle problem*—Rodney Hood
*Pascal's theorem and airplane design*—Margaret Waugh
*Solving maximum and minimum problems without calculus*—Mary Kinsman
*Probability and pi*—Rosalind Monastersky and Ruth Berger
*Rollers*—Frank Marzocco.

In addition to the regular meetings two special meetings were held. Professor and Mrs. J. F. Randolph invited the club to their home for a Christmas party. After a social hour, Jerry Howald and Rodney Hood demonstrated space curves by intersecting string models with planes or cones of light from a projector. In the spring a banquet was held; upon that occasion the members solved mathematical puzzles, and Dr. R. S. Burington of Washington, D. C., discussed the role of equivalence in pure and applied science. Dr. Burington used illustrative material from his experience as an applied mathematician in industry and in governmental work.

The officers for the year were: President, Rodney Hood; Vice-President, Mary Kinsman; Secretary-Treasurer, Charlotte Peters; Social Chairman, Margaret Waugh; Publicity Chairman, Jerry Howald; Faculty Adviser, Professor J. F. Randolph.

## NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.*

### THE NATIONAL MATHEMATICS MAGAZINE

The *National Mathematics* magazine has been revived, and will be published as a bi-monthly, except in June-July. It will feature expository articles designed to acquaint its readers with modern researches in specialized fields.

Immediate publication of research papers in pure and applied mathematics is planned. To make this possible from the standpoint of time, refereeing will be replaced by critiques from editors and others published in a department devoted to this purpose, with answers by authors, as is done in some of the engineering magazines. Professor H. V. Craig of Texas will have charge of this department. To make it financially possible authors will have their choice of awaiting their turn or paying for extra pages subject to possible return of their money.

The magazine will be under the management of D. H. Hyers of U.S.C., Glenn James of U.C.L.A., and A. D. Michal of California Institute of Technology. It will be financed by subscriptions of $3.00 per year and sponsor-contributions. These should be sent to Glenn James, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, California.

### DUTCH MATHEMATICIANS

The following deaths are reported: Drs. M. J. Belinfante and M. M. Biedermann of Amsterdam, Professor Otto Blumenthal of the Technical School of Aachen (since 1939 a refugee in Delft), Professor Julius Wolff and Dr. L. W. Nieland of the University of Utrecht, Dr. R. Remak (formerly Privatdozent at the University of Berlin and since 1939 a refugee in Amsterdam), Professor G. Schaake of the University of Groningen. All but the last died in consequence of German activities during the occupation.

The following have retired: Professors J. A. Schouten and J. F. Schuh of the Technical School of Delft, Professor W. van der Woude of the University of Leiden, Professor J. Barrau of the University of Utrecht.

At Amsterdam University: Professor J. G. van der Corput of the University of Groningen and Professor D. van Dantzig of the Technical School of Delft have been appointed to professorships, Dr. E. W. Beth has been appointed to an associate professorship of logic and the philosophy of science; Drs. G. H. A. Grosheide and J. Haantjes have been appointed to lectureships at the Free University of Amsterdam. Professor G. Mannoury has been awarded the honorary degree of Ph.D.

At the Technical School of Delft: Drs. N. G. de Bruyn, S. C. van Veen and C. Visser have been appointed to professorships.

At Groningen University, Drs. J. H. C. Gerretsen and C. S. Meyer have been promoted to professorships.

At the University of Utrecht Dr. H. Freudenthal has been appointed to a professorship.

The Mathematical Centre, an institution for the promotion of pure and applied mathematics, was founded in Amsterdam on February 11, 1946.

## NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The twenty-fifth annual meeting of the N.C.T.M. will be held at Chalfonte-Haddon Hall, Atlantic City, New Jersey, on February 28 and March 1, 1947. The program may be found in the January number of *The Mathematics Teacher*.

## HARRY BATEMAN RESEARCH FELLOWSHIP

The California Institute of Technology announces the creation of a research fellowship in pure mathematics to be known as the Harry Bateman Research Fellowship. Candidates should have obtained the Doctorate, or expect to receive it prior to the beginning of the academic year, 1947–1948. The award will be based on promise of independent research in any field of pure mathematics. The recipient will devote the major part of his time to research. In addition he will be expected to teach one advanced course in mathematics. The stipend will be $3000 for the academic year. The appointment is normally for one year, but may be renewed for a second year. Application blanks may be obtained from the Dean of the Faculty, California Institute of Technology, Pasadena 4, California, and must be returned to the same address before March 15, 1947. The appointment will be announced by April 1, 1947.

## PERSONAL ITEMS

Professor L. V. Ahlfors of the University of Zurich has been appointed to a professorship at Harvard University.

Professor A. A. Albert of the University of Chicago will be a visiting professor at the University of Brazil in 1947.

Assistant Professor R. H. Bardell of the University of Wisconsin in Milwaukee has been promoted to an associate professorship.

Dr. Stefan Bergman has been appointed a lecturer at Harvard University.

Professor Emeritus Felix Bernstein of New York University has been appointed lecturer at Triple-Cities College, Syracuse University, Endicott, New York.

Assistant Professor M. A. Biot of Columbia University has been appointed to a professorship at Brown University.

Associate Professor A. W. Boldyreff of the University of Arizona has been appointed to an associate professorship at Wittenberg College.

Professor J. A. Caparó of Notre Dame University has retired with the title of professor emeritus in electrical engineering.

Dr. J. B. Coleman has been appointed acting associate professor at the University of Georgia.

Dr. R. W. Cowan of the University of Alabama has been promoted to an assistant professorship.

Assistant Professor H. R. C. Dieckmann of Occidental College has been appointed to an assistant professorship at San Jose State College.

Professor H. H. Ferns of the University of Saskatchewan has been appointed head of the department of mathematics.

Dr. A. D. Fialkow has been appointed an adjunct professor at Brooklyn Polytechnic Institute.

Walter Fleming has been appointed lecturer at the University of Manitoba.

Adjunct Professor Mary L. Foster of the University of South Carolina has been appointed to an associate professorship at Henderson State College, Arkadelphia, Arkansas.

N. D. Griffin of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. C. C. Grove has been appointed a special lecturer at the University of Pennsylvania.

F. C. Hall has been appointed to an assistant professorship at Manhattan College.

Assistant Professor F. E. Hohn of Guilford College, Guilford, North Carolina, has been appointed to an assistant professorship at the University of Maine.

Associate Professor Grace M. Hopper of Vassar College has resigned to accept a position in the Cruft Research Laboratory at Harvard University.

Dr. A. S. Householder has accepted a position as principal physicist at the Clinton Laboratories, Monsanto Chemical Company, Knoxville, Tennessee.

Dr. Waclaw Kozakiewicz of the University of Saskatchewan has been promoted to an assistant professorship.

Professor C. C. MacDuffee of the University of Wisconsin has been appointed visiting professor of mathematics at the University of Puerto Rico.

Assistant Professor A. B. Mewborn of the University of Arizona has been appointed to an associate professorship at the postgraduate school of the United States Naval Academy.

Associate Professor R. S. Pate of the University of South Carolina has been appointed to a professorship at Michigan State Normal College, Ypsilanti, Michigan.

Dr. Echo D. Pepper has been appointed to an assistant professorship at the University of Illinois.

Associate Professor H. R. Phalen of the College of William and Mary has been promoted to a professorship.

Assistant Professor Adrienne S. Rayl of the Birmingham Center of the University of Alabama has been promoted to an associate professorship.

Professor Marcel Riesz of the Lunds Universitets Matematiska Institution, Lund, Sweden, has been appointed visiting lecturer at the University of Chicago.

Assistant Professor R. F. Rinehart of Case School of Applied Science has been promoted to a professorship.

Professor W. H. Roever of Washington University has retired with the title emeritus.

Senior Professor R. E. Root of the Postgraduate School of the United States Naval Academy has retired with the title of professor emeritus.

Dr. O. H. Schmidt has been appointed librarian of the Danish Technical Library, Copenhagen, Denmark.

L. R. Shobe, Technical Instructor at General Motors Institute, has been appointed head of the department of mathematics at State Teachers College, Bemidji, Minnesota.

W. H. Simons has been appointed to a lectureship at the University of British Columbia.

Associate Professor M. F. Smiley of Lehigh University has been appointed to an associate professorship at Northwestern University.

R. E. Smith of Allegheny College has been appointed to an associate professorship at the College of William and Mary.

Assistant Professor S. S. Smith of the University of Utah has been promoted to an associate professorship.

Associate Professor Andrew Sobczyk of Oregon State College has been appointed Chief of the Mathematical Research Branch, Watson Laboratories, Cambridge, Massachusetts.

Dr. Ruth W. Stokes has been appointed to an assistant professorship at Syracuse University.

Dr. Mildred M. Sullivan of Queens College, Flushing, New York, has been promoted to an assistant professorship.

Associate Professor P. M. Swingle of the New Mexico College of Agriculture and Mechanical Arts has been appointed to a professorship at the University of Miami.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### ANNUAL MEETING OF THE NEBRASKA SECTION

The twenty-second meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska College of Medicine in Omaha, on Saturday, May 4, 1946. Professor F. S. Harper, Chairman of the Section, presided.

The attendance was twenty, including the following thirteen members of the Association: E. M. Berry, A. K. Bettinger, W. C. Brenke, C. C. Camp, H. M. Cox, J. A. Daum, J. M. Earl, M. G. Gaba, C. B. Gass, F. S. Harper, J. F. Heyda, Sigurd Mundhjeld, and Lulu L. Runge.

At the business meeting the following officers were elected for the coming year: Chairman, Ralph Hull, University of Nebraska; Vice-Chairman, C. B. Gass, Nebraska Wesleyan University; Secretary, Lulu L. Runge, University of Nebraska.

The following papers were presented:

1. *On a certain definite integral*, by Professor M. A. Basoco, University of Nebraska.

This paper was ready by title.

2. *Some special types of generating functions of polynomial systems*, by Professor W. C. Brenke, University of Nebraska.

In this paper Professor Brenke discussed types of generating functions which arise when the function $f(xt)$ in $g(x, t)=e^t f(xt)$ is subjected to special restrictions. It was shown how this generating function gives rise to a broad class of polynomials $Y_n(x)$ among which are the generalized Laguerre polynomials, as shown in his paper in this MONTHLY, vol. 52, 1945. That none of the other classical orthogonal polynomials are among the $Y_n(x)$ was shown by proving that their generating functions can never take the form $e^t f(xt)$.

3. *Integral solutions (with the restrictions) of the equation* $x^2+y^2=z^2$, by Dr. E. M. Berry of Nebraska State Teachers College at Chadron.

The speaker dealt with solutions of the equation $x^2+y^2=z^2$ which satisfy one of the equations $|y-kx|=h$, $|z-kx|=h$, where $k$ is any integer, or a fraction whose denominator is 2, and $h$ is a particular integer.

4. *A fundamental equation in the mathematics of finance*, by Professor F. S. Harper, University of Nebraska.

It was pointed out that the assumptions that money is productive and that capital must not be impaired give rise to a fundamental equation of value that is used to unify the study of many of the topics in the Mathematics of Finance, and which gives a clearer understanding of the difficult subject of depreciation.

5. *A simple method for celestial navigation*, by Professor O. C. Collins, University of Nebraska, introduced by the Chairman.

It was shown by two methods which avoid all measurement (and correction) of angles, how position has been found to within one mile on land. A single measured time interval determines uniquely both celestial coordinates of the zenith. Great circles used as lines of position are plotted by selection from precomputed data, and shown to yield an unambiguous fix. One of the two methods is available for trial in flight or at sea, with the use of a bubble sextant. The other method calls for the design of a new instrument if it is to be used off the ground.

6. *Mathematics with the air corps*, by Professor Ralph Hull, University of Nebraska.

This paper was read by title.

7. *Significant differences in grades*, by Professor C. C. Camp, University of Nebraska.

The speaker found that two students with averages of 90.624 and 90.151 for Phi Beta Kappa did not differ statistically (5% level). This seems to refute current conceptions such as the principle that persons with grade averages of 88.1 or higher should be elected and those with 87.6 or less should not be elected. It would be necessary to consider the $\sigma_m$'s of such students in order to determine the difference required. It might be nearer 2%, and from the study would probably be at least 1%. Quiz averages of 92.25 and 84.125 were found to be not significantly different (5% level) for eight quizzes. The minimum difference required for students with such $\sigma_m$'s would be 8.28. For classes of least variability in individual quiz grades the minimum significant difference in quiz or semester grade was found to be about 5. This leads logically to a grading system with nine passing marks if 60 is the lowest.

8. *The Euler-Maclaurin formula*, by the same author.

This formula for approximate integration when carried to first-derivative terms was recommended as being more accurate than Simpson's Rule. For summing slowly converging series, but starting with about the tenth term, this formula (reversed) is extremely practical. Alternating series with terms grouped in pairs can likewise be evaluated.

9. *The measurement and analysis of the mathematical attainment of the returning G.I.*, by Professor H. M. Cox, University of Nebraska.

Preregistration guidance examinations which are required of new students at the University of Nebraska have been designed so as to measure ability to study and interpret materials in the specific fields. Certain of the examinations prepared by the United States Armed Forces Institute have been used for this purpose. In mathematics, a special examination was prepared by representatives from the Mathematics Department, the Chemistry Department, the Bureau of Instructional Research, and other departments; this examination tests differentially performance in mathematics at several levels of preparation. Although the scores on the mathematics examination would seem to be depressed for those veterans with two or more years absence from the class room,

nevertheless very significant differences exist between groups of men with varying numbers and patterns of high school courses.

LULU L. RUNGE, *Secretary*

## SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association was held at the George Washington University, Washington, D. C., on Saturday, May 11, 1946, with a morning session, luncheon, and an afternoon session. Professor J. B. Scarborough, Chairman of the Section, presided at both sessions.

The attendance was fifty including the following thirty-seven members of the Association: Archie Blake, T. A. Botts, Herman Branson, Randolph Church, Abraham Cohen, J. F. Daly, J. A. Duerksen, F. D. Faulkner, E. J. Finan, B. H. Gere, B. C. Getchell, Michael Goldberg, R. A. Good, D. W. Hall, M. A. Hyman, F. E. Johnston, Sidney Kaplan, L. M. Kells, W. D. Lambert, O. E. Lancaster, A. E. Landry, M. H. Martin, Carol V. McCamman, T. W. Moore, W. H. Norris, Jr., O. J. Ramler, J. N. Rice, R. E. Root, J. B. Scarborough, E. D. Schell, Vivian E. Spencer, Hillel Spitz, J. H. Taylor, Marian M. Torrey, J. A. Ward, G. T. Whyburn, Clement Winston.

The officers elected at the business meeting of the Section were as follows: Chairman, W. K. Morrill, Johns Hopkins University; Secretary-Treasurer, E. J. Finan, Catholic University; Members of the Executive Committee, G. A. Hedlund, University of Virginia and Dr. Archie Blake, Aberdeen Proving Ground. It was agreed to hold the next meeting on Saturday, December 7, 1946, at Johns Hopkins University.

The first three of the following papers were read at the morning session. The remaining two were read at the afternoon session. Dr. Deming's paper was read at the invitation of the Section.

1. *On a poristic system of triangles*, by Professor O. J. Ramler, Catholic University.

In his paper, Professor Ramler featured the use of isotropic coördinates to show the ease and brevity with which one could develop some of the results of Weill, Servais, and Goormaghtigh in connection with loci of notable points associated with triangles inscribed in a circle and circumscribed to a conic.

2. *An integral equation of a general metabolizing system*, by Professor Herman Branson, Howard University.

For the purpose of this paper, a general metabolizing system is one in which materials are being produced, consumed, transported, and/or stored. A mathematical description of such a system leads to an integral equation of the "Faltung" type. It is shown through examples that this equation gives interesting results when applied to first order chemical reactions, and that it is especially effective in discussing certain problems arising in biological research connected

with isotopes. The speaker emphasized the rich descriptive material in biology and chemistry for a course in integral equations.

3. *Early history of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America*, by Professor R. E. Root, Post Graduate School, U. S. Naval Academy.

Professor Root gave a brief account of the founding of the Mathematical Association in 1915 and of the Maryland-Virginia-District of Columbia Section in 1917. The matter was presented by quotations from letters found in an old file. Letters quoted were mostly from E. R. Hedrick and A. Cohen.

4. *On the structure of a cluster*, by Professor R. A. Good, University of Maryland.

The ring is a special case of a more general algebraic system called the cluster. Some important subsets contained in a cluster are its derived ring and its ideals, including the annihilator ideal. Numerous properties of clusters were illustrated by an example of a cluster of order eight.

5. *On the sampling problem in the observation of the election in Greece*, by Dr. W. Edwards Deming, Bureau of the Budget.

E. J. FINAN, *Secretary*

## ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association was held at Carleton College in Northfield, Minnesota, on Saturday, May 11, 1946. Two sessions were held in the forenoon, one at luncheon, and one in the afternoon. Professors Harry Nelson, C. H. Gingrich, The Reverend G. L. Winkelmann, and Professor C. S. Carlson presided at the respective sessions.

Sixty-one persons attended the meeting, including the following twenty-two members of the Association: N. R. Amundson, K. H. Bracewell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, Gladys Gibbons, Sister Mary Seraphim Gibbons, C. H. Gingrich, H. W. Godderz, H. E. Hartig, C. M. Jensen, W. D. Munro, H. E. Nelson, J. M. H. Olmsted, G. C. Priester, F. J. Taylor, H. L. Turrittin, K. W. Wegner, and The Reverend G. L. Winkelmann.

The following officers were elected for the coming year: Chairman, K. H. Bracewell, Hamline University; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, W. H. Bussey, University of Minnesota; W. D. Munro, University of Minnesota; C. S. Carlson, St. Olaf College.

Copies of the *Memorandum to the Secretaries of the Sections of the Mathematial Association of America* from the Committee for the Coordination of Studies in Mathematical Education were passed out to those attending the first morning session. At the business session a resolution was passed instructing the Chairman of the Section to appoint a committee to cooperate with the previously mentioned committee. The newly elected Chairman appointed the following committee for this purpose: K. W. Wegner, Chairman, W. L. Hart, E. J. Camp.

By invitation of the Executive Committee, Professor Robert H. Cameron delivered an address at the second morning session. The title of his address was *A Class of Non-linear Integral Equations which Can Be Solved by Formal Procedures.* Professor Cameron investigated the quadratic convolution equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s-t)f(t)p(s)dtds + p_0\int_{-\infty}^{\infty} f(x-t)f(t)dt + 2\int_{-\infty}^{\infty} f(x-t)q(t)dt + 2q_0 f(x) + r(x) = 0$$

by means of a Fourier transforms. A short résumé of the necessary Fourier transform theory was given as an introduction, and a number of Fourier transforms were calculated. The special case of the above equation

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt + 2f(x) = g(x)$$

was introduced and solved completely for a few particular choices of the function $g(x)$. A case in which this equation has no absolutely integrable solution was observed. Finally, the general equation was attacked by the methods used in solving the simpler one, and a formal solution was obtained. Conditions were given under which it is known that an absolutely integrable solution must exist and be unique.

In addition to the address by Professor Cameron, the following eight papers were presented:

1. *Rational values of trigonometric functions*, by Professor J. M. H. Olmsted, University of Minnesota.

This was a discussion of a paper by Professor Olmsted published in this MONTHLY, vol. 52, 1945, pp. 507–508.

2. *Upper and lower bounds for certain exponential sums*, by Professor H. L. Turrittin, University of Minnesota.

This paper dealt with the problem of computing the greatest lower bound and the least upper bound of the function

$$F(t) = \sum_{n=1}^{n} c_n \cos(P_i(t)), \qquad -\infty < t < \infty$$

where the $c$'s are the real constants and the $P$'s are polynomials with real coefficients. A scheme was outlined for shifting the problem from the transcendental to the algebraic domain by using an extension of a Hardy-Littlewood theorem on Diophantine approximations.

3. *Note on the circular clamped plate with an eccentric point load*, by Professor N. R. Amundson, University of Minnesota.

The speaker considered the derivation of the Green's function for a circle for the biharmonic equation by analogy with the Green's function for Laplace's equation for the circle. Applications were made to circular thin elastic plates.

4. *Some classroom techniques*, by Professor K. W. Wegner, Carleton College.

The speaker emphasized the importance of including among the objectives of any college mathematics course the following two: (1) increasing the ability of the student to read scientific material, (2) increasing the ability of the student to make precise mathematical statements. The lecture technique does not contribute toward the achievement of these objectives. A technique was described which has been successful in attaining the desired results as well as the other objectives of the course.

5. *A set of equivalent bases for topology*, by Mr. Monroe Donsker, University of Minnesota, introduced by Professor J. M. H. Olmsted.

Taking as a starting point the axiomatic system based on the closure operation which was constructed by Kuratowski, the speaker showed that certain of the topological operations which are definable in terms of the closure operation can be used as bases for constructing axiomatic systems which would be equivalent to the system based on closure. For each such operation the axioms necessary for equivalence were exhibited and the general problem of the equivalence of two axiomatic systems were discussed.

6. *Boundary values of analytic functions*, Professor S. E. Warschawski, University of Minnesota, introduced by Professor N. R. Amundson.

The following extension of a theorem of F. Riesz in *Mathematische Zeitschrift*, vol. 18 (1923) was proved: Let $f(z)$ be a function analytic in the interior of a closed rectifiable Jordan curve $C$. Suppose that $C_n$, $n=1, 2, \cdots$, is a sequence of closed rectifiable Jordan curves in the interior of $C$ which "converge" to $C$ as $n \to \infty$ (that is, for every $\epsilon>0$, all the points of all $C_n$ with a sufficiently large index $n$ are at a distance less than $\epsilon$ from $C$). If for a $p>0$ and all $n$, the integrals $\int_{C_n}|f(z)|^p ds \leqq M$, where $M$ is independent of $n$, then $f(z)$ possesses boundary values for nontangential approach almost everywhere on $C$ and $\int_C |f(z)|^p ds$ exists. The proof was accomplished by mapping $C$ and $C_n$ conformally on the unit circle.

7. *The design of computing control systems*, by W. D. Munro, University of Minnesota.

A few of the general aspects of the design of computers for use in bombing, and so forth, were discussed from the point of view of replacing the variables in the problem by the variables in an equivalent system (for example, an electric network). Design of a hypothetical bombsight based on idealized conditions was used as an illustration.

8. *Geometric constructions of the third and fourth degree*, by Jacob Bearman, University of Minnesota, introduced by the Secretary.

The Nicomedes trisection of the angle and the Vieta construction for the duplication of the cube are examples of "insertion" constructions. These are special cases of the general problem of inscribing in a given conic a chord of given length, subject to the restriction that the chord produced shall pass through a given point not necessarily on the conic. The special and general problems lead to quartic equations with coefficients integral rational functions of the coefficients of the equation of the conic and of the cosine or tangent of the angle formed by the coordinate axes. The solutions of the quartics determine the "angular coefficient" of the lines satisfying the given conditions.

From the basic nature of the angle trisection and the Vieta construction, it follows that any cubic or quartic equation can be solved geometrically by ruler and compasses and at most two insertion constructions; and, conversely, any construction involving only ruler and compasses and insertion constructions leads algebraically to equations of degree not higher than the fourth.

L. E. BUSH, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.
Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, April 19, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST, Vancouver, British Columbia, April 10–11, 1947
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April 18–19, 1947
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 3


CONTENTS




MARCH 1947

# The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)





EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the Secretary-Treasurer, W. B. Carver, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the first of the month in which the change becomes effective.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)





Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

## THE APPROXIMATION OF NUMBERS AS SUMS OF RECIPROCALS

H. E. SALZER,* New York, N. Y.

**1. Introduction.** Every positive real number $x<1$ admits a unique expansion of the form

$$x=\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots, \tag{1}$$

where $a_1$, $a_2$, $a_3$, and so forth, are integers so chosen that after $i$ terms, when the sum $x_i$ has been obtained, $a_{i+1}$ is the least integer such that $x_i+1/a_{i+1}$ does not exceed $x$. This expansion will be denoted as the *R-expansion of the number x.* The approximation as far as $x_i$, is obviously closer to $x$ than $1/(a_i^2-a_i)$, since

$$x-x_i<\frac{1}{a_i-1}-\frac{1}{a_i}$$

by the choice of $a_i$. Thus it is apparent that in every $R$-expansion, $a_{i+1}=a_i^2-a_i+\epsilon_i$, where $\epsilon_i$ is a positive integer.

Every expansion of the form (1), where $a_{i+1}=a_i^2-a_i+\epsilon_i$, $\epsilon_i\geqq 1$ (apart from the question as to whether it is the $R$-expansion of some number), converges. In particular, the $R$-expansion of $x$ converges to $x$. As an indication of the rapidity of convergence, note that every such expansion is dominated by the expansion beginning with $a_1=2$ and $\epsilon_i=1$ for $i\geqq 1$, that is,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\cdots.$$

*This latter series can be shown to converge to* 1.

Proof: Let $1/a_i$ be a particular addend. If it has the property that $1/(a_i-1)$ in its place would have made $x_i$ equal to 1, then the next addend will also have that property for $x_{i+1}$. The reason is that the next addend is $1/(a_i^2-a_i+1)$ and if it were $1/(a_i^2-a_i)$, then

$$\frac{1}{a_i}+\frac{1}{a_i^2-a_i}=\frac{1}{a_i-1},$$

and thus by the hypothesis of the induction, the next addend will have that property, and hence all the addends, because it is true for $1/a_1$ and $1/a_2$. But by the property of the addends which was just established,

$$1-x_i=\frac{1}{a_i-1}-\frac{1}{a_i}=\frac{1}{a_i^2+a_i},$$

which is less than any preassigned $\eta$ for all $i>$ some $i_0$; so $x_i\to 1$.

---

* Mathematical Tables Project, National Bureau of Standards.

**2. The principal theorems.**

THEOREM I. *Every infinite expansion of the form* (1) *where* $a_{i+1}=a_i^2-a_i+\epsilon_i$, *where* $\epsilon_i>1$ *for infinitely many values of* $i$ (*this condition on the* $\epsilon_i$ *is also necessary*), *is actually the R-expansion of the number* $x$ *to which it converges.**

Proof: If the theorem were not true, compare the above mentioned expansion of $x$, say $(R')$, with the $R$-expansion of $x$, say $(R)$, and let $1/a_i$ be the first term of $(R')$ that differs from the corresponding term of $(R)$. By the definition of the $R$-expansion for $x$, $1/a_i$ must be less than that corresponding term, say $1/(a_i-b)$. But $x-x_{i-1}$, the remainder after $i-1$ terms in the $R$-expansion, satisfies

$$x - x_{i-1} \geqq \frac{1}{a_i - b} \geqq \frac{1}{a_i - 1} > \frac{1}{a_i} + \frac{1}{a_i^2 - a_i + \epsilon_i} + \cdots, \tag{2}$$

this last inequality being obtained by applying the inductive reasoning of the preceding paragraph to the condition that there is always an $\epsilon_j>1$ for $j>i_0$ due to the infinitude of $\epsilon_i>1$ in $(R')$. But this is a contradiction, since

$$x - x_{i-1} = \frac{1}{a_i} + \frac{1}{a_i^2 - a_i + \epsilon_i} + \cdots .$$

If there is an expansion of the type (1) with $\epsilon_i=1$ for all $i\geqq i_0$, it is not the $R$-expansion of any number; but it will agree with the $R$-expansion of the number to which it converges, up to the term before $1/a_{i_0}$. If the expansion did not agree, it would disagree at a place $j$ where there is a subsequent $\epsilon_i>1$, and again there would arise the same contradiction as before, since, in its remainder expression (the extreme right member of (2) with $i$ replaced by $j$), the presence of a single $\epsilon_i>1$ makes it less than $1/(a_j-1)$ and hence less than $x-x_{j-1}$. Then $1/a_{i_0}$ plus all the succeeding terms will equal $1/(a_{i_0}-1)$ which will also be the last addend in the terminating $R$-expansion.

For any rational number $a/b$ there is the very natural question as to whether its $R$-expansion might continue indefinitely, just as

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots .$$

That such is not the case is contained in the fundamental theorem which follows.

THEOREM II. *The R-expansion of any rational number must terminate.*

Proof: Consider $a/b$ beginning with $1/a_1+\cdots$, so that

$$\frac{a}{b} - \frac{1}{a_1} = \frac{aa_1 - b}{ba_1} .$$

---

* For a finite expansion of the form (1) to be the $R$-expansion, the condition $a_{i+1}=a_i^2-a_i+\epsilon_i$, $\epsilon_i\geqq 1$, is necessary and sufficient.

From the choice of $a_1$,

$$\frac{a}{b} < \frac{1}{a_1 - 1}, \qquad aa_1 - a < b, \qquad 0 < aa_1 - b < a,$$

so that, disregarding cancellation which only helps the argument, the first remainder $a'/b'$ has $a' < a$. Repetition of this reasoning on $a'/b'$ leads to the next remainder $a''/b''$ where $a'' < a'$, and the decreasing sequence of positive numerators in the succession of remainders must end in a 1. (If the last remainder is assumed to be $p/q$ in lowest terms with $1 < p < q$, one could continue with $1/([q/p]+1) + \cdots$, thereby contradicting the fact that $p/q$ is the last remainder.) Hence every rational number has a terminating $R$-expansion and every non-terminating $R$-expansion must represent an irrational number.

As an immediate corollary to Theorems I and II, a convergent infinite series such as

$$\frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^8} + \cdots + \frac{1}{3^{2^n}} + \cdots$$

cannot be equal to a rational number. In fact, no infinite series of the type $\sum_{n=1}^{n=\infty} a_n^{-b_n^n}$, where $a_n$ and $b_n$ are any non-decreasing sequences of integers, each greater than or equal to 2, can equal a rational number.

**3. Alternative methods.** Besides the $R$-expansions, there are similar alternative methods of expanding any $x$, $0 < x < 1$, as a sum of reciprocals. Thus to $x_i$, instead of $1/a_{i+1}$, one might add either $1/a_{i+1}$ or $1/(a_{i+1}-1)$, depending upon which gives a smaller value to $|x - x_{i+1}|$. Such an expansion, denoted as the $\overline{R}$-expansion, would involve an irregular sequence of signs in its terms, since $1/a_j$ would be subtracted from $x_{j-1}$ whenever $x_{j-1} > x$. The absolute value of the remainder in stopping at $1/a_j$ is not greater than $1/2a_j(a_j - 1)$. If $\pm 1/a_i$ is a particular term in the $\overline{R}$-expansion, the next term is always $\pm 1/(2a_i^2 - 2a_i + \epsilon_i)$, where $\epsilon_i \geqq 0$. This shows that the $\overline{R}$-expansion is even more rapidly convergent than the $R$-expansion. The $\overline{R}$-expansion is unique except for certain rational numbers where the last two terms might be either

$$\pm \frac{1}{a_i} \pm \frac{1}{2a_i(a_i - 1)} \quad \text{or} \quad \pm \frac{1}{a_i - 1} \mp \frac{1}{2a_i(a_i - 1)}\cdot$$

Further comparison of the $\overline{R}$-expansion with the $R$-expansion reveals another important difference. Unlike Theorem I for the $R$-expansion, every expansion where $\pm 1/a_i$ is followed by $\pm 1/\{2a_i(a_i-1)+\epsilon_i\}$, $\epsilon_i \geqq 0$, is not necessarily an $\overline{R}$-expansion ("an" instead of "the" to cover the cases where there is no unique $\overline{R}$-expansion) of the number to which it converges. Among the many trivial illustrations of this point are the following: **I.** $a_1 = 1$, all $\epsilon_i = 1$ (not even convergent); **II.** $a_1 = 1$, $\epsilon_1 = 2$, $\epsilon_2 = 0$; **III.** The first two terms may be $1/2 + 1/4 + \cdots$, whereas the $\overline{R}$-expansion might begin with 1. For a nontrivial instance consider $111/264$ when it is expressed as $1/3 + 1/12 + 1/264$,

where obviously $1/a_i$ is followed by $1/\{2a_i(a_i-1)+\epsilon_i\}$, $\epsilon_i \geqq 0$, where $a_1=3$, $\epsilon_1=\epsilon_2=0$. But the $\overline{R}$-expansion of 111/264 is $1/2-1/13-1/381+\cdots$.

The most slowly convergent $R$-expansion is

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\cdots,$$

where the first five terms give an approximation that differs from the true value by more than $3\times 10^{-7}$. The most slowly convergent $\overline{R}$-expansion is

$$\frac{1}{2}\pm\frac{1}{4}\pm\frac{1}{24}\pm\frac{1}{1104}\pm\frac{1}{2435424}\pm\cdots,$$

where the first five terms come nearer to the true value than $1\times 10^{-13}$.* Another good illustration for the purpose of comparison is afforded by the first four terms in the $R$- and $\overline{R}$-expansions for the decimal part of $\pi$, where the advantage of the $\overline{R}$-expansion is evident: Thus

$$(R)\quad \pi = 3+\frac{1}{8}+\frac{1}{61}+\frac{1}{5020}+\frac{1}{1285\ 41455}+\cdots,$$

with an error greater than $6\times 10^{-18}$, while

$$(\overline{R})\quad \pi = 3+\frac{1}{7}-\frac{1}{791}-\frac{1}{3748629}+\frac{1}{15164\ 89608\ 87729}-\cdots,$$

with an error less than $1\times 10^{-30}$.

There is a theorem for the $\overline{R}$-expansion that is similar to Theorem II. It is:

THEOREM III. *Every rational number has a terminating $\overline{R}$-expansion.*

Proof:

$$\frac{a}{b}-\frac{1}{a_1}=\frac{a_1a-b}{ba_1}.$$

If

$$\frac{a}{b}>\frac{1}{a_1},\quad\text{then}\quad \frac{a}{b}<\frac{1}{a_1-1}$$

as in Theorem II, and the next remainder $a'/b'$ has $a'<a$. However, if $a/b<1/a_1$, since $1/(a_1+1)<a/b$ (otherwise $1/(a_1+1)$ would be closer to $a/b$ than $1/a_1$), it follows that $b<aa_1+a$, and $aa_1-b>-a$. In view of the fact that $aa_1-b<0$, $|a'|<a$.

Still another efficient method of expanding in reciprocals, which leads to an alternating series, is to write

---

* It should be noted that the expansion, $1-1/4\pm$ same terms as above, converges equally "slowly."

$$x = \frac{1}{\left[\frac{1}{x}\right]} - r_1, \qquad r_1 = \frac{1}{\left[\frac{1}{r_1}\right]} - r_2, \text{ and so on.}$$

The accuracy of the $i$th approximation is similar to that of the $R$-expansions, namely, $1/(a_i^2 - a_i)$. Again, every rational number $a/b$ is seen to have a finite expansion, since

$$\frac{a}{b} - \frac{1}{a_1} = \frac{aa_1 - b}{ba_1} < 0,$$

and from the fact that

$$\frac{1}{a_1} > \frac{a}{b} > \frac{1}{a_1 + 1},$$

we obtain $-a < aa_1 - b < 0$, so that $|a'| < a$.

In the $R$-expansion of a number, stopping at $1/a_i$ leads to an error of approximately $1/a_i^2$ for $a_i$ only moderately large. When those first $i$ terms are combined into a single fraction, the unreduced denominator is seen to be $k_i a_i^2$, where $k_i$ is considerably less than 1. In every case that $x_i$ is written as $p/q$, the closeness to $x$ is less than $1/q$, which makes the successive approximations of an $R$-expansion similar to decimal approximations in the sense that the error in a decimal approximation is usually about a half-unit in the last place. Thus, the $R$-expansion cannot give rise to what occurs in the convergent series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

where a fraction of enormous size will yield only 1% accuracy. An approximation $p/q$ obtained from an $R$-expansion is not as uniformly good as $p'/q'$, obtained from a simple continued fraction where the closeness of approximation is $\sim 1/q'^2$. But the $R$-expansion converges much more rapidly, and it is often unnecessary to combine the separate reciprocals into a fraction with a denominator larger than the last $a_i$. Similar remarks hold for the $\bar{R}$-expansions, but even more favorably.

**4. Applications.** A useful application of the $R$ and $\bar{R}$-expansions is in solving for the positive roots of algebraic (and also transcendental) equations, by a well-known variant of the Newton-Raphson method. By using the $R$- and $\bar{R}$-expansions less digital work is involved, and the result is exhibited in neat form as a sum of a few reciprocals of integers. It is of interest to compare the use of the $R$- and $\bar{R}$-expansions with a similar process described by T. A. Pierce in his paper, *On an Algorithm and its Use in Approximating Roots of Algebraic Equations*, this MONTHLY (36) 1929, pp. 523–525. Pierce discusses the expansion

$$x = \frac{1}{p_1}(1 - \frac{1}{p_2}(1 - \frac{1}{p_3}(1 - \cdots$$

where at each stage $p_i$ is determined as the largest integer that will permit the succeeding factor in parentheses to be less than 1. His algorithm has the advantage of involving smaller numbers than the use of the $R$- or $\bar{R}$-expansion, but when written as a sum of reciprocals it obviously cannot converge as rapidly as either the $R$- or $\bar{R}$-expansion, due to the fact that each denominator is restricted to being a multiple of the preceding.

For purpose of comparison, consider the same equation that is in Pierce's paper, namely,

$$x^3 - 5x + 2 = 0.^*$$

(For reference, its real root between 0 and 1, to 38 decimals, is .41421 35623 73095 04880 16887 24209 69807 857.) Pierce's method of approximation gives successively

$$x \equiv x_1 = \tfrac{1}{2}(1 - x_2),\dagger \qquad - x_2^3 + 3x_2^2 + 17x_2 - 3 = 0,$$

from which

$$x_2 = \tfrac{1}{5}(1 - x_3), \qquad x_3^3 + 12x_3^2 - 452x_3 + 64 = 0,$$

from which

$$x_3 = \tfrac{1}{7}(1 - x_4), \qquad - x_4^3 + 87x_4^2 + 21977x_4 - 111 = 0,$$

from which

$$x_4 = \frac{1}{197}(1 - x_5).$$

The first four terms obtained by Pierce's method, that is,

$$\frac{1}{2}\left(1 - \frac{1}{5}\left(1 - \frac{1}{7}\left(1 - \frac{1}{197}\right)\right)\right),$$

give an approximation that is correct only to within 4 units in the 7th decimal place. Pierce erred in giving .41421 *3564* instead of .41421 *3198* as the fourth approximation, and thus incorrectly claimed that the fourth approximation is correct to within 2 units in the 9th decimal place. A further application of Pierce's process leads to

$$x_5^3 + 17136x_5^2 - 8529\ 39668x_5 + 4286128 = 0, \qquad x_5 = \frac{1}{199}(1 - x_6),$$

and the fifth approximation, .41421 35623 73142, due to the exceptionally small value of $x_6$, just happens to agree with the true value of the root of 13 decimal places.

Application of the $R$-expansion method leads to

---

* Notice that this equation is different from that well-known and "overworked" test equation $x^3-2x-5=0$, whose real root has been calculated to 152 decimals.

† Here $x_i$ is used differently than in the text preceding this example.

$$x \equiv x_1 = \tfrac{1}{3} + x_2$$

since $x$ is between $1/2$ and $1/3$; then

$$27x_2^3 + 27x_2^2 - 126x_2 + 10 = 0$$

from which

$$x_2 = \frac{1}{13} + x_3, \qquad 59319x_3^3 + 73008x_3^2 - 266643x_3 + 1054 = 0$$

from which

$$x_3 = \frac{1}{253} + x_4, \qquad 9606283\ 17363x_4^3 + 119\ 37026\ 24829x_4^2$$
$$- 430\ 86992\ 40846x_4 + 197\ 46514 = 0$$

from which

$$x_4 = \frac{1}{218201} + x_5.$$

The fourth approximation here, namely,

$$\frac{1}{3} + \frac{1}{13} + \frac{1}{253} + \frac{1}{218201},$$

is accurate to within 2 units in the 11th decimal place (compare with Pierce above). Another step gives

$$x_5 = \frac{1}{6\ 13235\ 43802} + x_6,$$

which leads to a fifth approximation that is correct to within 2 units in the 22nd decimal place (compare again with Pierce above). Work can be saved in obtaining the last desired term (also in Pierce's method) by noting that it is not necessary to find more than the linear part of the last equation, and even there it is not necessary to retain more than a certain number of significant figures in the calculation. Thus if, in the present example, one did not wish to go beyond $x_4$, instead of the above cubic equation in $x_4$, with such large coefficients, it would have sufficed to know only

$$\cdots - 430\ 870 \times 10^7 x_4 + 197\ 465 \times 10^2 = 0.$$

In entirely similar fashion, application of the $\overline{R}$-expansion would yield a fourth approximation of

$$\frac{1}{3} + \frac{1}{12} - \frac{1}{408} - \frac{1}{450\ 832},$$

which differs from the root by less than 2 units in the 12th decimal place (com-

pare with both Pierce and the $R$-expansion), and the fifth approximation which would be obtained from the fourth by subtracting 1/62 70135 66048, would be accurate to within 1 unit in the 24th decimal place (compare again with both Pierce and $R$-expansion).

Either the $R$- or $\overline{R}$-expansion can be employed to find a complex root of an equation, merely by applying it to the root's real and imaginary part, each considered separately.

## GEODESIC PERSPECTIVITIES UPON A SPHERE*

JOHN DE CICCO, Columbia University

**1. Conformal perspectivities.** Recently Kasner and the author developed a proof of the following proposition in the real and imaginary domains:

*If a surface admits a conformal perspectivity upon a given sphere from a given point, then the surface is also a sphere (or plane) homothetic to the original with respect to the point of perspectivity* [1].

That is, the point of perspectivity and the two centers of the spheres are collinear, and the distances of the two centers from the point of perspectivity are proportional to the two radii. Moreover, the spheres are inscribed in the tangent cone (real or imaginary) of the original sphere with vertex at the point of perspectivity.

It is possible to deduce as a corollary from the above theorem the converse of Ptolemy's theorem on stereographic projection which states that the only conformal perspectivities upon a plane are Ptolemy's stereographic projection, and the obvious limiting case of a parallel plane [2].

**2. Gnomic projection.** The following result has also been proved previously:

*If more than $3\infty^1$ geodesics of a surface are projected into straight lines on a plane under a perspectivity, then all geodesics project into straight lines and the surface is a sphere (and the obvious limiting case of any plane, parallel or not), with center at the point of perspectivity* [3].

Surfaces may be classified into the following four distinct classes according to the number of geodesics which are projected into straight lines on a plane by a perspectivity:

1. The non-ruled surfaces. At most $\infty^1$.
2. The ruled surfaces excluding the quadrics. There are always $\infty^1$ (the rulings) and at most $2\infty^1$.
3. The quadrics excluding the gnomic projections of spheres. There are always $2\infty^1$ (the two systems of rulings) and at most $3\infty^1$.
4. The gnomic projections of spheres; and planes. All $\infty^2$.

* Presented to the American Mathematical Society, April, 1946.

**3. The fundamental theorem.** In the present article, we shall find all surfaces for which there exist perspectivities upon a sphere such that all the geodesics correspond to the great circles. We shall prove the following result.

FUNDAMENTAL THEOREM. *If a surface admits a perspectivity upon a sphere from a given point such that all the geodesics of the surface correspond to the great circles of the sphere, then the surface is a sphere* (*or plane*) *homothetic to the original with respect to the point of perspectivity.*

This includes the characterization of gnomic projection which is stated above, as a special case.

**4. Part I of the proof.** Let $(x, y, z)$ denote cartesian coördinates of a point in space. Take the center of perspectivity as the origin $(0, 0, 0)$. No loss in generality is suffered by assuming the given sphere $S$ to be of unit radius and its center on the $z$-axis, for any other surface which is a geodesic perspectivity of $S$ is one also of any other sphere homothetic to $S$ with respect to the origin. Accordingly the equation of our sphere $S$ may be written as

$$X^2 + Y^2 + (Z - c)^2 = 1. \tag{1}$$

For the surface $\Sigma$, we shall use cylindrical coordinates $(\rho, \theta, z)$, where $x=\rho \cos \theta$, $y=\rho \sin \theta$. Then $z=f(\rho, \theta)$; and $p=\partial z/\partial\theta$, $q=\partial z/\partial\rho$, $r=\partial^2 z/\partial\theta^2$, $s=\partial^2 z/\partial\theta\partial\rho$, $t=\partial^2 z/\partial\rho^2$.

The perspective image $(X, Y, Z)$ on the sphere $S$ of a point $(\rho, \theta, z)$ on the surface $\Sigma : z=f(\rho, \theta)$, is

$$X = \frac{\rho \cos \theta}{\lambda}, \qquad Y = \frac{\rho \sin \theta}{\lambda}, \qquad Z = \frac{z}{\lambda}, \tag{2}$$

where $\lambda$ is a function of $(\rho, \theta)$ defined by the quadratic equation

$$a\lambda^2 - 2cz\lambda + (\rho^2 + z^2) = 0; \qquad a = c^2 - 1. \tag{3}$$

For our further work, it is advantageous to find the total derivatives of $\lambda$ with respect to $\theta$. The first order total derivative $\lambda'$ is given by

$$(a\lambda - cz)\lambda' + (z - c\lambda)(p + q\rho') + \rho\rho' = 0. \tag{4}$$

The second order total derivative $\lambda''$ is defined by

$$\begin{aligned}(a\lambda - cz)\lambda'' + a\lambda'^2 - 2c(p + q\rho')\lambda' + (p + q\rho')^2 + \rho'^2 \\ + (z - c\lambda)(r + 2s\rho' + t\rho'^2) + (\rho + zq - cq\lambda)\rho'' = 0.\end{aligned} \tag{5}$$

By means of (3) and (4), it is found that this may be written as

$$\begin{aligned}(a\lambda - cz)\lambda'' + \left[\frac{1}{z^2 - a\rho^2}\right][(z - \rho q)\rho' - \rho p]^2 \\ + (z - c\lambda)(r + 2s\rho' + t\rho'^2) + (\rho + zq - cq\lambda)\rho'' = 0.\end{aligned} \tag{6}$$

From (1) and (2), it is seen that the great circles of the sphere $S$ are defined by

$$A\rho\cos\theta + B\rho\sin\theta + C(z - c\lambda) = 0. \tag{7}$$

After eliminating $(A, B, C)$ by differentiation, we find

$$\begin{aligned} r + 2s\rho' + t\rho'^2 + q\rho'' - c\lambda'' - \frac{2\rho'}{\rho}(p + q\rho' - c\lambda') \\ + (z - c\lambda)\left(1 + \frac{2\rho'^2}{\rho^2} - \frac{\rho''}{\rho}\right) = 0. \end{aligned} \tag{8}$$

By (3), (4), and (6), this simplifies into the form

$$\begin{aligned} \frac{\lambda}{\rho}(z - \rho q)\rho'' - \lambda(r + 2s\rho' + t\rho'^2) \\ + \left[\frac{c}{z^2 - a\rho^2}\right][(z - \rho q)\rho' - \rho p]^2 - \frac{2\lambda\rho'}{\rho^2}[(z - \rho q)\rho' - \rho p] \\ + c\rho^2 - z\lambda = 0. \end{aligned} \tag{9}$$

The linear-element $d\sigma$ of the surface $\Sigma : z = f(\rho, \theta)$, is

$$d\sigma^2 = (\rho^2 + p^2)d\theta^2 + 2pqd\theta d\rho + (1 + q^2)d\rho^2. \tag{10}$$

The geodesics are given by the equation

$$\begin{aligned} [\rho^2(1 + q^2) + p^2]\rho'' = (p\rho' - \rho^2 q)(r + 2s\rho' + t\rho'^2) \\ + \rho[2(1 + q^2)\rho'^2 + 3pq\rho' + (\rho^2 + p^2)]. \end{aligned} \tag{11}$$

To discover the common geodesics, we have to eliminate $\rho''$ between (9) and (11). This leads to a cubic equation in $\rho'$. As a consequence, we may state the following result:

*In general, there are at most* $3\infty^1$ *geodesics of a surface* $\Sigma$ *which are projected into great circles of a sphere* $S$ *under a perspectivity* **[4]**.

Let more than $3\infty^1$ geodesics be projected into great circles. The cubic equation in $\rho'$ must be identically zero. Upon placing the coefficient of $\rho'^3$, equal to zero, we find

$$\lambda(z - \rho q)pt = 0. \tag{12}$$

Neither $\lambda$ nor $(z - \rho q)$ can be zero. For otherwise we merely find a cone, real or imaginary, with vertex at the origin. In this case, the perspectivity is degenerate.

Therefore in all cases, we must have $pt = 0$.

**5. Part II of the proof.** *The only surfaces* $\Sigma$ *which satisfy the conditions of our fundamental theorem and for which* $p \neq 0$ *are planes.* By (12), we find $t = 0$. Hence any such surface $\Sigma$ must be of the form

$$z = \rho\alpha(\theta) + \beta(\theta); \quad p = \rho\alpha_\theta + \beta_\theta \neq 0, \ q = \alpha, \ r = \rho\alpha_{\theta\theta} + \beta_{\theta\theta}, \ s = \alpha_\theta, \ t = 0. \tag{13}$$

Upon setting the coefficients of $\rho'^2$ in (9), where $\rho''$ is given by (11), equal to zero, and noting that $z-\rho q\neq 0$, we obtain

$$(14)\qquad 2\lambda p(s\rho - p)(z^2 - a\rho^2) + c\rho^2(z - \rho q)[p^2 + \rho^2(1 + q^2)] = 0.$$

First of all, let us suppose that $\lambda$ is rational. The discriminant $z^2-a\rho^2 =(\alpha^2-a)\rho^2+2\alpha\beta\rho+\beta^2$, of the quadratic equation (3), must be a perfect square in $\rho$. This is so only when $a\beta=0$. But if $\beta=0$, the surface is a cone with vertex at the origin. Hence $\beta\neq 0$ and it follows that $a=0$, that is, $c^2=1$. This means that the sphere $S$ passes through the origin. Since $c^2=1$, it follows by (3) and (13) that the condition (14) may be written in the form

$$(15)\qquad \begin{aligned}\beta_\theta(\alpha\rho + \beta)(\alpha_\theta\rho + \beta_\theta)[(\alpha^2 + 1)\rho^2 + 2\alpha\beta\rho + \beta^2]&\\ - \beta\rho^2[(\alpha_\theta^2 + \alpha^2 + 1)\rho^2 + 2\alpha_\theta\beta_\theta\rho + \beta_\theta^2] &= 0.\end{aligned}$$

This is an identity in $\rho$. Upon setting the term independent of $\rho$ equal to zero, we have $\beta^3\beta_\theta^2=0$. Since $\beta\neq 0$, we find that $\beta=z_0$, a non-zero constant.

The preceding equation reduces to the condition, $\alpha_\theta^2+\alpha^2+1=0$. Thus we obtain the solutions, $z=i\rho\cos(\theta-\alpha)+z_0$, where $\alpha$ is a constant. In cartesian coordinates, these are the imaginary planes, $z=i(x\cos\alpha+y\sin\alpha)+z_0$, tangent to the null sphere, $x^2+y^2+(z-z_0)^2=0$. All the straight lines of any one of these planes are not the perspective images of all the great circles of our sphere $S$ with respect to the origin. Thus the case where $\lambda$ is rational does not yield any solutions whatsoever.

The only case that remains to be considered in this section is where $\lambda$ is irrational. By (14), we deduce the conditions

$$(16)\qquad s\rho - p = 0,\qquad c[p^2 + \rho^2(1 + q^2)] = 0.$$

From (13), we find that $\beta=z_0$, a non-zero constant. If $c\neq 0$, we get the imaginary planes discussed above. Hence it follows that $c=0$ and $\beta=z_0\neq 0$. Of course, this means that the sphere $S$ must have its center at the origin.

Upon setting both the coefficient of $\rho'$ and the term independent of $\rho'$ in the cubic identity obtained by eliminating $\rho''$ from (9) and (11) equal to zero, we obtain

$$(17)\qquad \alpha_\theta(\alpha_{\theta\theta} + \alpha) = 0,\qquad \alpha(\alpha_{\theta\theta} + \alpha)z_0 + \rho(\alpha_{\theta\theta} + \alpha)(\alpha_\theta^2 + \alpha^2 + 1) = 0.$$

The second is an identity in $\rho$. From it, we deduce that all possible solutions satisfy the condition, $\alpha_{\theta\theta}+\alpha=0$. Hence all the solutions are: $z=\rho(A\cos\theta+B\sin\theta)+z_0$. These are all the planes, $z=Ax+By+z_0$.

Therefore we have succeeded in proving that the gnomic projections are the only ones which satisfy the conditions of our theorem and for which $p\neq 0$.

**6. Part III of the proof.** *The only solutions of our problem for which* $p=0$, *are the spheres* (*and planes* $z=a$ *constant*) *homothetic to the sphere* $S$ *with respect to the point of perspectivity.* Since $p=0$, the cubic identity in $\rho'$ obtained as a result

of eliminating $\rho''$ from equations (9) and (11), yields the equations

$$
\begin{aligned}
t\lambda(zq + \rho)(z^2 - a\rho^2) &= c\rho(1 + q^2)(z - \rho q)^2, \\
q\lambda(zq + \rho) &= c\rho^2(1 + q^2).
\end{aligned} \tag{18}
$$

Either of the conditions $\lambda = 0$, or $z^2 = a\rho^2$, leads to a cone (real or imaginary), with vertex at the origin. Henceforth $\lambda \neq 0$ and $z^2 - a\rho^2 \neq 0$.

If $1 + q^2 = 0$, then $zq + \rho = 0$. This again yields the imaginary cone with vertex at the origin. Thus $1 + q^2 \neq 0$.

If $c = 0$ and $zq + \rho \neq 0$, then $q = 0$. In this case, the sphere $S$ has its center at the origin and our surfaces $\Sigma$ are the planes, $z =$ a constant. Again we have obtained a gnomic projection.

If $c = 0$ and $zq + \rho = 0$, the sphere $S$ has its center at the origin and the surfaces $\Sigma$ are spheres concentric with $S$.

The remaining case to be discussed is that in which $\lambda \neq 0$, $z^2 - a\rho^2 \neq 0$, $1 + q^2 \neq 0$, $c \neq 0$, and of course $p = 0$ From (18), we derive the equation

$$
\frac{t}{q}(z^2 - a\rho^2) = \frac{1}{\rho}(z - \rho q)^2. \tag{19}
$$

To solve this equation, introduce the substitutions

$$
\rho^2 + \phi(z) = 0, \qquad q = -\frac{2\rho}{\phi_z}, \qquad t = -\frac{2}{\phi_z} + \frac{4\phi\phi_{zz}}{\phi_z^3}. \tag{20}
$$

Substituting these into (19), we find

$$
2(z^2 + a\phi)\phi_{zz} - a\phi_z^2 - 4z\phi_z + 4\phi = 0. \tag{21}
$$

Differentiating this with respect to $z$, we get $\phi_{zzz} = 0$. Thus $\phi$ is a quadratic polynomial in $z$. Substitute this into (21). By (20), we find that our surfaces $\Sigma$ must necessarily be of the form

$$
\rho^2 + \alpha z^2 + 2\beta z + \gamma = 0, \tag{22}
$$

where $(\alpha, \beta, \gamma)$ are constants satisfying the condition

$$
\gamma(a\alpha + 1) - a\beta^2 = 0. \tag{23}
$$

Substitute (22) into the second of equations (18). We find that

$$
[z(1 - \alpha) - \beta]\lambda = c[(\alpha - 1)(\alpha z^2 + 2\beta z) + (\beta^2 - \gamma)]. \tag{24}
$$

It is noted that the conditions $\alpha = 1$, $\beta = 0$, can not hold simultaneously, for otherwise $\gamma = 0$, and we would get an imaginary cone with vertex at the origin.

Since the coefficient of $\lambda$ in (24) can not be identically zero, we can solve for $\lambda$ and substitute the result into the quadratic equation (3). This yields the following identity of fourth degree in $z$:

$$
\begin{aligned}
&ac^2[(\alpha - 1)(\alpha z^2 + 2\beta z) + (\beta^2 - \gamma)]^2 \\
&\qquad + 2c^2z[z(\alpha - 1) + \beta][(\alpha - 1)(\alpha z^2 + 2\beta z) + (\beta^2 - \gamma)] \\
&\qquad - [z(\alpha - 1) + \beta]^2[(\alpha - 1)z^2 + 2\beta z + \gamma] = 0.
\end{aligned} \tag{25}
$$

*We shall prove that* $\alpha = 1$. Let us suppose the contrary. Substitute $z = -\beta/(\alpha - 1)$ into the preceding equation. We find $ac^2[\beta^2 - (\alpha - 1)\gamma] = 0$. If $a \neq 0$, we obtain by this and from (23) that $\beta = \gamma = 0$. Thus we have found a cone with vertex at the origin. Hence if $\alpha \neq 1$, then $a = 0$. By (23), we deduce that $\gamma = 0$. If $\alpha \neq 1$, we have $a = 0$ and $\gamma = 0$. Thus we can divide out of (25) the factor $z\ [z(\alpha - 1) + \beta]$. The result is

$$2[(\alpha - 1)(\alpha z^2 + 2\beta z) + \beta^2] - [z(\alpha - 1) + \beta][(\alpha - 1)z + 2\beta] = 0. \tag{26}$$

Upon placing the coefficient of $z$ in this, equal to zero, we find $(\alpha - 1)\beta = 0$. Since $\alpha \neq 1$, we see that $\beta = 0$. Thus $\beta = \gamma = 0$ and we have found a cone with vertex at the origin.

The above argument shows that $\alpha = 1$ in all cases. Since $\alpha = 1$, we find from (23) that the value of $\gamma$ is

$$\gamma = \frac{a\beta^2}{c^2}. \tag{27}$$

Upon substituting $\alpha = 1$ and this value of $\gamma$ into (25), we find that (25) is identically zero. Thus by (22), we discover that our surfaces $\Sigma$ must be the spheres

$$x^2 + y^2 + (z + \beta)^2 = \frac{\beta^2}{c^2}, \tag{28}$$

homothetic to our sphere $S$. Moreover by (25), the magnifying function $\lambda$ is the constant $\lambda = -\beta/c$.

**7. Conclusion.** The above completes the proof of our fundamental theorem. The distinction between the result on conformal perspectivities and that of geodesic perspectivities should be noticed. In a conformal perspectivity, a point on the sphere $S$ can correspond to either one of the two images on the homothetic sphere $\Sigma$, whereas in our new theorem, a point on the sphere $S$ can correspond only to the single homothetic image on the second sphere $\Sigma$.

**References**

1. Kasner and De Cicco, Conformal perspectivities upon a sphere. This is a forthcoming paper in the Revista de Matematecas de la Universidad Nacional de Tucuman, Argentina. In the same volume, see an interesting extension to differential projective geometry by Terracini.

2. Kasner and De Cicco, Converse of Ptolemy's theorem on stereographic projection, Proceedings of the National Academy of Sciences, Volume 31, 1945.

3. Kasner and De Cicco, Converse theory of gnomic and equiareal perspectivities, Proceedings of the National Academy of Sciences, Volume 32, 1946, pp. 16–19.

4. Kasner, The problem of partial geodesic representation, Transactions of the American Mathematical Society, Volume 7, 1906, pp. 200–206.

## APPROXIMATING SUMS

M. A. ZORN, Indiana University

**1. Introduction.** The approximating sums for the definite integral of a function may be formed largely without reference to the character of the path. We propose to determine a necessary and sufficient condition for their convergence in the case of integrands which are analytic functions of a complex variable.

**2. Necessary conditions.** We denote the interval $0 \leqq t \leqq 1$ by $I$; any finite sequence $t_0, \cdots, t_n$, with $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_n = 1$, determines a composition $\Delta$. We write $\Delta t_i = t_i - t_{i-1}$, $i = 1, \cdots, n$, and $|\Delta; t| = \mathit{maximum}_{i=1,\cdots,n}\, \Delta t_i$ (the *mesh* of $\Delta$). A path is given by an arbitrary complex function $g(t)$ on $I$; the point set $g(I)$ we denote by $C$. We define $z_i$ as $g(t_i)$, for a given path and decomposition, and $\Delta z_i$ as $z_i - z_{i-1}$, $i = 1, \cdots, n$; we shall use the symbol $|\Delta; z|$ for the maximum of the numbers $|\Delta z_i|$. Summation from 1 to $n$, the latter being determined by $\Delta$, is indicated by $\sum$; it will be useful to have a special symbol for $\sum |\Delta z_i|^2$, for which we introduce $\|\Delta; z\|$. Intermediate points $z_i'$ on the path are obtained by forming $g(t_i')$, with $t_{i-1} \leqq t_i' \leqq t_i$, for $i = 1, \cdots, n$.

The *admissible* integrands will be single-valued complex functions $f(z)$, defined and differentiable on an open set $G$ which contains the set $C$; the set $G$ may vary with the integrand.

As usual, we say that $\lim \sum f(z_i')\Delta z_i$ exists, if there is a complex number $J$ such that for $e > 0$, a $d > 0$ exists, with $|\sum f(z_i')\Delta z_i - J| < e$ whenever $|\Delta; t| < d$.

We state first the following lemma.

LEMMA 1. *A necessary and sufficient condition for the existence of the limit of the approximating sums $\sum f(z_i')\Delta z_i$ is the existence of the limit for special intermediate points, where each $t_i'$ has one of the values $t_{i-1}$ or $t_i$.*

For the proof, it suffices to point out that by virtue of the identity

$$\sum f(z_i')(z_i - z_{i-1}) = \sum f(z_i')(z_i - z_i') + \sum f(z_i')(z_i' - z_{i-1}),$$

a general approximating sum is equal to a special approximating sum with a mesh which is certainly not larger.

THEOREM 1. *In order that the limit of the approximating sums exists for the (admissible) integrand $f(z) = z$ it is necessary that* $\lim \sum |\Delta z_i|^2 = 0 \equiv \lim \|\Delta : z\| = 0$.

With a subset $S$ of the set $1, \cdots, n$ we associate the special selection: $z_i' = z_{i-1}$ if $i$ is in $S$; otherwise, $z_i' = z_i$. We shall presently attach, in specified ways, to each $\Delta$ a set $S(\Delta)$; summation over this subset will be indicated by $\sum'$.

The difference between $\sum f(z_i)\Delta z_i$ and $\sum f(z_i')\Delta z_i$ becomes, in our case, $\sum z_i \Delta z_i - \sum z_i' \Delta z_i = \sum (z_i - z_i')\Delta z_i = \sum'(\Delta z_i)^2$. If the limit of the approximating sums exists, this sum must tend towards zero; and the same will hold for its real and imaginary parts, $\sum' \operatorname{Re}((\Delta z_i)^2)$ and $\sum' \operatorname{Im}((\Delta z_i)^2)$.

We now choose $S(\Delta)$, in four ways; namely, as the set of indices for which respectively $\operatorname{Re}((\Delta z_i)^2) > 0$, $\operatorname{Re}((\Delta z_i)^2) < 0$, $\operatorname{Im}((\Delta z_i)^2) > 0$, and $\operatorname{Im}((\Delta z_i)^2) < 0$.

Combination of the resulting limit relations yields

$$\lim \sum [\,|Re((\Delta z_i)^2)| + |\operatorname{Im}((\Delta z_i)^2)|\,] = 0;$$

since the terms of this sum dominate $|(\Delta z_i)|^2$, we obtain the desired result: $\lim \sum |\Delta z_i|^2 = 0$.

An immediate corollary is the following.

LEMMA 2. *If the limit of the approximating sums exists for all admissible integrands, the function $g(t)$ is continuous.*

When the quantity $|\Delta; z| \equiv \max_{i=1}^{n} |\Delta z_i|$ is less than or equal to the square root of $\|\Delta; z\|$, it must tend toward zero with $|\Delta; t|$. Since every $t$-interval of $I$ can be contained in a decomposition whose mesh is the length of the interval, we obtain uniform continuity of $g(t)$. The set $C$ will thus have to be compact; this we shall use in the next section.

**3. Sufficient conditions.** Turning to the sufficiency question, we verify first that for each admissible $f(z)$ and each compact set $C$ there is an $r>0$ and an $M$ such that for any $p$ on $C$ the inequality $|z-p| \leqq r$ implies that $z$ is in $G$ and that $|f(z)| \leqq M$. For if this were not so, we could produce, for $i=1, 2, \cdots$, points $p_i$, $z_i$ with $p_i$ on $C$, $|p_i - z_i| \to 0$, and either $z_i$ outside of $G$ or $|f(z_i)| > i$. Because of the compactness of $C$ we could then arrange it so that the sequence $p_i$ converges to a point $p$ of $C$; the sequence $z_i$ would have to converge to the same point. Since $G$ is open, the points $z_i$ lie ultimately in $G$; since $f(z)$ is continuous at $p$, the numbers of $f(z_i)$ will approach $f(p)$ and their absolute values will stay bounded.

So far we have used only the continuity of the integrand; let us now make use of the property of differentiability. For each $p$ on $C$ we consider the function $(f(z)-f(p))/(z-p)$. It is bounded near $p$ and may therefore be continued into $p$, by the theorem on removable singularities; we thus obtain a function which is differentiable for $|z-p| \leqq r$. For $|z-p| = r$, its absolute value is at most $2M/r$; since such a function has its maximum on the boundary, we get $f|(z)-f(p)| \leqq 2M/r$ for $|z-p| \leqq r$. Thus we have established the following result.

LEMMA 3. *For a compact set $C$ an admissible integrand has the following property, (property $X$): there exist an $r>0$ and an $L$ such that for $p$ on $C$, $|z-p| < r$ implies $|f(z)-f(p)| \leqq 2L|z-p|$.*

In the next lemma we employ the Cauchy integral (along straight segments or polygonal lines, for continuous integrands). From the properties of admissible integrands only the continuity and property $X$ come into play. The set $C$ is again assumed to be compact.

LEMMA 4. *If a function $f(z)$ is continuous on $G$ and has property $X$, then $|\Delta; z| < r$ implies*

$$\left| \sum f(z_i)\Delta z_i - \sum \int_i f(z)dz \right| \leqq L\|\Delta; z\|. \tag{1}$$

Here the integral $\int_i$ is understood to be taken along the straight segment from $z_{i-1}$ to $z_i$; it exists since the integrand is continuous there. For the sum of the integrals, which is the integral along a polygonal path, we write $\int_\Delta$.

For the proof, we consider $f(z_i)\Delta z_i - \int_i f(z)dz$, which is equal to $\int_i(f(z_i) - f(z))dz$. Using property $X$ we obtain

$$\left|\int_i (f(z_i) - f(z))dz\right| \leqq \int_i |f(z_i) - f(z)|\, d|z|$$

$$\leqq 2L\int_i |z_i - z|\, d|z| = L|\Delta z_i|^2.$$

The inequality (1) is an immediate consequence.

We pass on to an observation which is essentially a matter of elementary topology in the plane; the approximating sums are not directly involved.

LEMMA 5. *If the function $g(t)$ is continuous and the integrand $f(z)$ admissible then there exists an $e>0$ such that all integrals $\int_\Delta f(z)dz$ with $|\Delta; t| < e$ are equal.*

To prove this we choose $e$ so small that $|\Delta; t| < e$ implies $|\Delta; z| < r$. This is possible since $g(t)$ is uniformly continuous. Since two decompositions of mesh $< e$ have a common subdecomposition of mesh $< e$, it suffices to show that passing to a subdecomposition does not change the integral. This in turn will be taken care of if we can do it in the special case where only one new intermediate value $t^*$ is inserted between (say) $t_i$ and $t_{i-1}$. We have then $|t^* - t_{i-1}| < e$ and $|t^* - t_i| < e$, thus also $|z^* - z_{i-1}|$, $|z^* - z_i| < r$. These inequalities show that the triangle $z_{i-1}$, $z^*$, $z_i$ lies in the circle $|z^* - z| < r$, in which $f(z)$ is differentiable.

The difference between the two integrals is readily expressed as an integral over the boundary of the triangle; it vanishes, by Cauchy's integral theorem, and this completes the proof of the lemma.

Let us designate by $J$ the common value—or limit—of these integrals $\int_\Delta$ for $|\Delta; t| < e$. From (1) we obtain

$$|\textstyle\sum f(z_i)\Delta z_i - J| \leqq L\|\Delta; z\|, \tag{2}$$

which is valid whenever $|\Delta; t| < e$.

What happens if we choose other intermediate points? We may, as was pointed out in Lemma 1, restrict ourselves to the case where the intermediate points are special; that is, $z_i' = z_i$ or $z_{i-1}$. For $|\Delta; t| < e$, the property $X$ of admissible integrands yields if $g(t)$ is continuous, the inequality

$$|\textstyle\sum f(z_i)\Delta z_i - \sum f(z_i')\Delta z_i| \leqq 2L\|\Delta; z\|. \tag{3}$$

Indeed, the left member does not exceed $\sum|f(z_i) - f(z_{i-1})|\,|\Delta z_i|$, which, if $|\Delta; z| < r$, is dominated by $\sum 2L|z_i - z_{i-1}|\,|\Delta z_i| = 2L\|\Delta; z\|$.

The combination of (2) and (3) yields

$$|\textstyle\sum f(z_i')\Delta z_i - J| \leqq 3L\|\Delta; z\| \tag{4}$$

which is seen to hold if (I) $g(t)$ is continuous, (II) the integrand is admissible, and (III) $|\Delta; t| < e$.

**4. Recapitulation.** In view of Theorem 1, Lemmas 1 and 2, and equation (4), we may finally present the answer to our question:

THEOREM 2. *A necessary and sufficient condition for the convergence of the approximating sums belonging to all admissible integrands is that* lim $\|\Delta; z\|$ *be equal to zero.*

ILLUSTRATIVE EXAMPLE. The path defined by $g(t) = t + it \sin(1/t)$, $0 < t \leqq 1$, $g(0) = 0$, satisfies lim $\|\Delta; z\| = 0$, without being of bounded variation.

**5. Further questions.** We have tried to present our result in an elementary form, keeping at a level which might be attained in a first course in complex variable. We have not given the name integral to the limit of the approximating sums, since doing so would perhaps involve the obligation of developing the properties of this limit.

In conclusion I mention that D. W. Hall once asked me a related question about the integral along Jordan arcs. Moreover, I suggest the following additional questions: What are the properties of the paths with lim $\|\Delta; z\| = 0$? To what extent is the stronger form of Cauchy's theorem preserved? Must new conditions be imposed if $f(z)$ satisfies merely a Lipschitz condition? For what paths do decompositions exist with arbitrarily small $\|\Delta; z\|$? What about the limit of the approximating sums as $\|\Delta; z\| \to 0$?

# A NECESSARY AND SUFFICIENT CONDITION OF WIENER

G. C. EVANS, University of California*

**1. Introduction.** A necessary and sufficient condition, that an object belong to a certain defined class, is a tautology in the sense that it is equivalent, from the point of view of logical analysis, to the original proposition. A sufficient condition, which is not necessary, offers apparently an immediate advantage in that its purpose is to point out a recognizable subclass of objects for which the given definition is valid. Similarly, a necessary condition, which is not sufficient, excludes a recognizable subclass which does not contain the defined objects. These one-sided conditions therefore serve to delimit workable situations. But unless we assume a mystical attitude toward mathematics, so that we remain in a continual state of surprise, the discovery of an essential application of a necessary and sufficient condition should strike us with more force than the usual discovery of mathematical fact—or else, we should realize that the concreteness of mathematical fact is as important as logical structure.

* Presented to the Northern California Section of the Mathematical Association of America at Berkeley, California, January 26, 1946.

**2. Regular points and Wiener's condition.** An interesting case of this kind is an application of Wiener's necessary and sufficient condition that a point be a *regular boundary point* with respect to a given domain and Laplace's equation.* It has been remarked that Wiener's condition is "beautiful," presumably because it analyzes the neighborhood of the boundary point in an appropriate way.

For domains with sufficiently smooth boundaries we know that there is one and only one function which is a solution of Laplace's equation within the domain and takes on the values of a given continuous function for arbitrary approach to the boundary. An arbitrary domain can be approximated arbitrarily closely by a domain with such a smooth boundary; indeed, an arbitrary domain $T$ is the limit as $n\rightarrow\infty$ of a sequence of such smooth boundary domains $T_n$, each domain of the sequence strictly containing the previous one.

Given a function $f$, which is continuous on the boundary, it can be extended so as to be continuous within a certain distance of the boundary. There will then correspond to each approximating domain a harmonic function $u_n$ which belongs to the given continuous function, taking on its values on the boundary of $T_n$. As $n\rightarrow\infty$, the function $u_n$ tends to a function $u$ which is harmonic in the domain $T$; the function $u$ depends merely on $f$ and not on its extension. The question remains as to how this sequence solution $u$ behaves on approach to a point $p$ of the boundary of $T$. The point $p$ is a *regular boundary point* for $T$ if, given arbitrarily the continuous boundary function $f$, we have

$$\lim_{P\rightarrow p} u(P) = f(p), \qquad \text{for } P \text{ in } T.$$

Lebesgue showed that if $p$ is the vertex of a sufficiently sharp spine reaching into $T$, $p$ is not regular. Lebesgue also gave a necessary and sufficient condition that $p$ be a regular boundary point in terms of the existence of a special function called a *barrier*, which is a harmonic function (or a harmonic function plus a potential of positive mass) with certain defined properties. To be exact, the barrier has to have a positive lower bound in $T$, no matter how small a neighborhood of $p$ is excluded, and has to tend to 0 at $p$. One use of this condition is the concreteness of its application as a sufficient condition; for example, it can be shown by this means or by Wiener's condition that if $p$ is the vertex of a small triangle which lies outside of $T$, then $p$ is regular.

Wiener's condition depends on the notion of the *capacity* of a bounded closed set in space; for example, the set might be a piece of conducting surface or a solid conductor. The capacity is defined as the upper bound (least upper bound) of the total positive mass which can be distributed on the set so that the potential of the distribution nowhere exceeds the value 1. The capacity of a sphere is equal to its radius while that of a circular disc is $(2/\pi)$ times its radius. Capacity is of dimension one in length. If we have two sets $F'$ and $F$, such that to each point $q'$ of $F'$ there corresponds a point $q$ of $F$, to distinct points $q'$, $r'$

* N. Wiener, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 3, 1924, pp. 24–51 and pp. 127–146.

of $F'$ there correspond distinct points $q$, $r$ of $F$, and the respective distances satisfy $|q'r'| \leqq |qr|$, then the capacity of $F'$ is not greater than the capacity of $F$. In particular, this condition is satisfied if $F'$ is contained in $F$. If the capacity of $F$ is positive there exists mass, equal to the capacity of $F$ (whose distribution we may call a conductor distribution), of which the potential (the conductor potential) takes on the value 1, but does not exceed that value. The conductor potential takes on the value 1 at all regular boundary points of the infinite domain whose boundary is the exterior frontier of $F$.

Now let $p$ be a boundary point of a given domain $T$, and $\lambda$ a positive number less than unity. Let $\gamma_k$ be the capacity of the closed set $e_k$ of points $q$ which do not belong to $T$ and which satisfy the condition

$$\lambda^k \leqq \text{distance } pq \leqq \lambda^{k-1}.$$

Consider the series

$$\sigma = \sum_1^\infty \left(\frac{\gamma_k}{\lambda^k}\right).$$

Then $p$ is regular if $\sigma$ diverges, but is irregular if $\sigma$ converges. This is Wiener's condition. One can show that Lebesgue's condition is equivalent to it.*

**3. Surfaces of minimum capacity.** Let us now consider the application. We know that in three dimensions there exist harmonic functions bounded over all space provided that we allow them to be multiple-valued. The multiplicity may be any finite value; in particular the function may be double-valued. It is this latter case which we are to consider. Corresponding to branch points in the plane we have branch curves in space. We wish to investigate the continuity of the double-valued function on these branch curves.

For example, the conductor potential of a circular disc is given by the formula

$$V(P) = 1 - \frac{2}{\pi} \arctan \frac{\sqrt{\mu}}{a},$$

where $\mu$ is the ellipsoidal parameter of the point $P$ in the system of spheroidal coordinates for which the level surface corresponding to $\mu$ degenerates into the disc when $\mu = 0$. This potential $V(P)$ can be continued harmonically across the disc in either direction by replacing $\sqrt{\mu}$ by $-\sqrt{\mu}$, and thus yields a double-valued non-negative function, harmonic in all space except on the circumference $s$ of the disc, equal to 0 and 2 respectively at $\infty$, and the sum of its two values for any $P$ remaining equal to 2. It will be noticed that in terms of this explicit expression, $V(P)$ is continuous on $s$; in fact, the behavior of $V(P)$ can be examined in detail in the neighborhood of $s$ by introducing coördinates $\alpha$, $\beta$ with

$$x^2 + y^2 + z^2 = a^2 + \alpha^2 + 2a\alpha \cos \beta, \qquad z = \alpha \sin \beta,$$

* O. D. Kellogg, Foundations of Potential Theory, Berlin, 1929, pp. 326–334.

so that, writing

$$\frac{x^2 + y^2}{a^2 + \mu} + \frac{z^2}{\mu} = 1,$$

$\mu$ will be given by the expression

$$\mu = \left(\frac{\alpha}{2}\right)\left\{2a \cos \beta + \alpha + [4a^2 + \alpha^2 + 4a\alpha \cos \beta]^{1/2}\right\}.$$

The potential $V(P)$ has the value 1 on the disc and on $s$. The disc plays the role of a cut, $s$ being the branch curve.

This analogy to Riemann surfaces was used by Sommerfeld* for the solution of a number of problems and has been noticed and applied by Courant, the present author, and others. Let us suppose that we have a closed curve $s$ in space, itself of zero capacity, such that the points of the interior of some sphere which contains it may be put into one to one correspondence with the points of a second sphere in such a way that $s$ corresponds to a circle inside the second sphere. We are able to show that there exists a particular surface cap $S$, bounded by $s$, such that, among all such caps, $S$ has minimum capacity. The cap $S$ is in fact the level surface $V=1$ of a uniquely determined double-valued function which is bounded in all space, harmonic except on $s$, and equal to 0 and 2, respectively, at $\infty$. The existence and properties of $S$ are obtained in terms of $V$. The surface $S$ separates no points from $\infty$. If $S$ is made a cut surface, the branch $V_1$ of $V$ which vanishes at $\infty$ is the conductor potential of $S$. At any point the sum of the two branch values remains equal to 2.†

Every point of $s$ is a limiting point of $S$, and of the infinite domain complementary to $S$, and every small closed curve which loops $s$ cuts $S$. It is not true, however, that $V$ is necessarily continuous on $s$, taking on the value 1 for arbitrary approach to a point of $s$. This statement is easily verified in the special case of the plane curve $s$ obtained by adding a thin flat spine to the disc which we have discussed above.

THEOREM: *It is true that $V$ is continuous at any point $p$ of $s$ where $s$ has a tangent line, or where $s$ has a forward and a backward tangent which do not make a zero angle.*

In order to prove this theorem let $t_1$ be the forward tangent at $p$, $t_2$ the backward tangent and $\pi$ a sector of a half-plane bounded by $t_1$ and $t_2$. Let $F$ be the set of points, in a closed spherical neighborhood $\Gamma$ of $p$, which belong to $S+s$. Draw two thin single-sheeted cones $C_1$ and $C_2$, with axes $t_1$ and $t_2$, respectively, and common vertex $p$, cutting the boundary of $\Gamma$ in small circles which are exterior to each other and separated; the radius of $\Gamma$ may be taken small enough

* A. Sommerfeld, Proceedings of the London Mathematical Society, vol. 28, 1897, pp. 395–429. H. Bateman, Partial Differential Equations of Mathematical Physics, New York, 1944, p. 466.

† G. C. Evans, Bulletin of the American Mathematical Society, vol. 47, 1941, pp. 717–733.

so that within $\Gamma$ the curve $s$ lies entirely inside the cones.

We now project certain of the points of $F$ circularly to the half plane $\pi$, in order to form a set $F'$. Consider the set of circles $C$ in $\Gamma$ with centers on $t_1+p$, in planes orthogonal to $t_1$, whose discs do not contain any points interior to $C_2$ but contain all the points of their plane sections with $C_1$. The circumferences $C$ loop $s$, and therefore each $C$ contains a point of $F$. By rotation along $C$ the point is projected into a point of the half plane $\pi$, and these projected points fill a certain sector with a non-zero angle at $p$. This plane sector plus the point $p$ constitute the set $F'$.

To each point of $F'$ corresponds, by the projection, one or more of the points of $F$, of which one may be chosen. To distinct points $q_1'$, $q_2'$ of $F'$ correspond distinct points $q_1$, $q_2$ of $F$, since $q_1'$, $q_2'$ lie on distinct circumferences $C$. Moreover, with regard to the respective distances, $|q_1'q_2'| \leqq |q_1q_2|$. In fact, if we take cylindrical coordinates $(r, \theta, z)$, $t_1$ being the axis of $z$, we find that

$$|q_1'q_2'|^2 = (r_2 - r_1)^2 + (z_2 - z_1)^2, \quad r_1^2 = x_1^2 + y_1^2, \quad r_2^2 = x_2^2 + y_2^2$$
$$|q_1q_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$
$$|q_1q_2|^2 - |q_1'q_2'|^2 = 2(r_1r_2 - x_1x_2 - y_1y_2) = 2r_1r_2[1 - \cos(\theta_2 - \theta_1)].$$

Hence the capacity of $F'$ cannot exceed the capacity of $F$.

Similarly, if we consider subsets $e_k$ of $F$ which are specified in Wiener's criterion and the subsets $e_k'$ of $F'$ which lie in the corresponding annular plane regions, each $e_k'$ is a circular projection of $e_k$. Hence the capacity $\gamma_k'$ of $e_k'$ cannot exceed the capacity $\gamma_k$ of $e_k$. Accordingly, the series $\sigma$ will diverge if the series

$$\sigma' = \sum_1^{\infty} \frac{\gamma_k'}{\lambda^k}$$

diverges.

But $F'$ contains a plane triangle with vertex at $p$. And as was mentioned above, $p$ is a regular point for the infinite domain whose sole boundary is this plane triangle. It follows, therefore, using the necessary part of the Wiener condition, that the Wiener series for this triangle diverges. But each term of this series is dominated by the corresponding term in $\sigma'$, since each annular portion of the triangle is contained in the corresponding $e_k'$ Hence $\sigma'$ diverges and $\sigma$ diverges. Accordingly, using the sufficiency part of the Wiener condition, $p$ is a regular boundary point for the infinite domain $T$ of which $S$ is the sole boundary, and

$$\lim_{P \to p} V_1(P) = 1, \qquad P \text{ in } T.$$

Also, if $V_2(P)$ is the other branch of the function $V$, since $V_1(P)+V_2(P) \equiv 2$, we have $\lim_{P\to p} V_2(P) = 1$. In other words, $V(P)$ is continuous on $s$.

## MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Send material for this department directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

### ON THE NEED FOR CARE IN USING A CERTAIN INTEGRAL FORMULA

A. E. TAYLOR, University of California, Los Angeles

The formula

$$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{\sqrt{a^2 - b^2} \tan \frac{1}{2}x}{a + b}, \qquad a^2 > b^2, \tag{1}$$

is given in many standard integral tables. It must be used with care, however, as the following paradox shows.

Let us take $a = 5$, $b = -3$, and calculate the value of the definite integral

$$\int_0^{3\pi/2} \frac{dx}{5 - 3 \cos x}. \tag{2}$$

The integrand in (2) is positive and continuous for all values of $x$, so the definite integral is certainly positive. However, if we use (1), we obtain a negative result, namely

$$\int_0^{3\pi/2} \frac{dx}{5 - 3 \cos x} = \frac{1}{2} \tan^{-1} (2 \tan \tfrac{1}{2}x) \Big|_0^{3\pi/2} = -\frac{1}{2} \tan^{-1} 2.$$

We have here an apparent failure of the fundamental theorem of calculus. If the function on the right in (1) be denoted by $F(x)$, it is true that

$$F'(x) = \frac{1}{a + b \cos x} \tag{3}$$

*for all values of $x$ for which $F(x)$ is defined.* But $F(x)$ is not defined when $x$ is an odd multiple of $\pi$. Furthermore, if $x_n = (2n+1)\pi$, $n = 0, \pm 1, \cdots$, the limits of $F(x)$ at $x = x_n$ from the right and left, respectively, are

$$F(x_n + 0) = \frac{-\pi}{\sqrt{a^2 - b^2}}, \qquad F(x_n - 0) = \frac{\pi}{\sqrt{a^2 - b^2}} \tag{4}$$

if $a+b>0$. If $a+b<0$, the right members of the two equations (4) must be exchanged. Hence there is no way of defining $F(x)$ at $x = x_n$ so as to make the function differentiable there.* Therefore, in applying (1) to evaluate a definite integral, we must take care that the interval of integration does not contain one of the points $x = x_n$ *in its interior.* If a point $x = x_n$ is at an end of the interval we may still use (1), but we must think of $F(x)$ as being defined at the end of the

* Another example of much the same kind, but less likely to occur in practice, was pointed out recently by C. I. Lubin (this MONTHLY, vol. 53, 1946, p. 586).

interval by its limiting value from within the interval. The correct evaluation of (2), for example, is as follows:

$$\int_0^{\pi} \frac{dx}{5 - 3\cos x} = F(\pi - 0) - F(0) = \frac{\pi}{4} - 0;$$

$$\int_{\pi}^{3\pi/2} \frac{dx}{5 - 3\cos x} = F\left(\frac{3\pi}{2}\right) - F(\pi + 0) = -\tfrac{1}{2}\tan^{-1} 2 + \frac{\pi}{4};$$

$$\int_0^{3\pi/2} \frac{dx}{5 - 3\cos x} = \frac{\pi}{2} - \tfrac{1}{2}\tan^{-1} 2 = 1.02.$$

An example of the occurrence of the foregoing situation in practice is furnished by the following problem. Consider an infinitely long cylindrical surface $x^2+y^2=a^2$. Let the two halves of this surface for which $y>0$ and $y<0$, respectively, be conductors which are insulated from each other and given potentials 1 and $-1$, respectively. The potential along the line with cylindrical coördinates $(r, \theta)$, $0 \leqq r < a$, is given by Poisson's integral formula

$$V(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi) \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\phi - \theta)}\, d\phi, \tag{5}$$

where $f(\phi)=1$ when $0<\phi<\pi$, $f(\phi)=-1$ when $\pi<\phi<2\pi$. We make the change of variable $x=\phi-\theta$, and use (1) with care. It is necessary to distinguish three cases. The results are

$$V(r, \theta) = 0 \quad \text{if} \quad \theta = 0 \quad \text{or} \quad \theta = \pi;$$

$$V(r, \theta) = \frac{2}{\pi}\tan^{-1}\left[\frac{a + r}{a - r}\operatorname{ctn}\frac{\theta}{2}\right] + \frac{2}{\pi}\tan^{-1}\left[\frac{a + r}{a - r}\tan\frac{\theta}{2}\right] \mp 1; \tag{6}$$

in the last formula we choose $-1$ if $0<\theta<\pi$, and $+1$ if $\pi<\theta<2\pi$.

The solution (6) is clumsy in form, and the work of deriving it is tedious. There is a better procedure for the evaluation of the integral (5). We let $x=\phi-\theta$ when $0\leqq\phi\leqq\pi$, and $y=\phi-\theta$ when $\pi\leqq\phi\leqq 2\pi$. Then

$$V(r, \theta) = \frac{1}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{(a^2 - r^2)dx}{a^2 + r^2 - 2ar\cos x} - \frac{1}{2\pi}\int_{\pi-\theta}^{2\pi-\theta} \frac{(a^2 - r^2)dy}{a^2 + r^2 - 2ar\cos y}.$$

In the second integral let $y=\pi+x$, and combine the result with the first integral. In this way we obtain

$$V(r, \theta) = \frac{1}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{4ar(a^2 - r^2)\cos x}{(a^2 + r^2)^2 - 4a^2r^2\cos^2 x}\, dx. \tag{7}$$

The integral (7) may be evaluated easily by the substitution $u=\sin x$. The final result obtained in this way is much neater than the form (6); it is

$$V(r, \theta) = \frac{2}{\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{a^2 - r^2}\right). \tag{8}$$

## POLYGONS HAVING A COMMON MEAN

V. O. McBRIEN, Holy Cross College

1. The following theorem on cyclic $n$-gons was recently discussed by R. Goormaghtigh, this MONTHLY, vol. 53, 1946, p. 525:

When two $n$-sided polygons $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ are inscribed in a circle $\Gamma$, the $n$ orthopoles, with respect to $A_1A_2 \cdots A_n$, of all the polygons obtained by omitting one of the vertices of $B_1B_2 \cdots B_n$ and the $n$ orthopoles, with respect to $B_1B_2 \cdots B_n$, of all the polygons obtained by omitting one of the vertices of $A_1A_2 \cdots A_n$ are $2n$ points on a circle having as center the midpoint of the segment joining the orthocenters of the polygons. If $\phi$ is the angle formed by the mean polygons of the given polygons, the radius of the circle equals that of $\Gamma$ multiplied by $\cos \frac{1}{2}n\phi$ when $n$ is even and by $\sin \frac{1}{2}n\phi$ when $n$ is odd.

2. It is of interest to discuss the condition for polygons which have a *common mean*, for then the circle reduces to the null circle. The properties of triangles having a common mean have been considered by O. J. Ramler, this MONTHLY, vol. 47, 1940, p. 140.

Using circular coördinates, let the cyclic $n$-gon $A_i$ have vertices $t_i$ and the cyclic $n$-gon $B_i$ have vertices $\tau_i$, where $|t_i| = |\tau_i| = 1$. Designate the symmetric functions of $t_i$ and $\tau_i$ by $s_{n,i}$ and $\sigma_{n,i}$, respectively. The orthopole of a variable vertex $t_i$ is given by

$$z = \tfrac{1}{2}(s_{n,1} + \sigma_{n,1}) - \tfrac{1}{2}\left[1 + (-1)^n \frac{\sigma_{n,n}}{s_{n,n}}\right] t_i. \tag{2.1}$$

This is the map equation of the circle of $2n$ orthopoles. The radius of the circle is $\frac{1}{2}\left|\left[1+(-1)^n\sigma_{n,n}/s_{n,n}\right]\right|$. The condition for a null circle is therefore,

$$(-1)^{n-1}\sigma_{n,n} = s_{n,n}. \tag{2.2}$$

For $n=3$, we have $\sigma_{3,3}=s_{3,3}$ which is the condition that a pair of cyclic triangles have a common mean triangle. In general we may say that two cyclic $n$-gons have a common mean if (2.2) is true. We then have as an important case of the theorem above, the COROLLARY:

*If two cyclic $n$-gons have a common mean, the $2n$ orthopoles with respect to the $n$-gons are coincident.*

## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.*

### THE TRIAL INTEGRAL METHOD

M. F. SMILEY, Northwestern University

The purpose of this note is to provide a workable and direct explanation* of a method of solution of the non-homogeneous linear differential equation

$$F(D_x)y = f(x), \tag{1}$$

in which $F(u)$ is assumed to be a polynomial in $u$ with real coefficients independent of $x$. We shall assume that $f(x)$ satisfies a *homogeneous* linear differential equation

$$G(D_x)f(x) = 0, \tag{2}$$

where $G(u)$ is a polynomial in $u$ with real coefficients independent of $x$.† We insist on real coefficients only because this has been the custom in engineering applications.

We shall suppose that a method of solution of *homogeneous* equations $H(D_x)y=0$, with $H(u)$ a polynomial in $u$ with real coefficients independent of $x$, has been explained. We feel that this discussion should culminate in a table of the following sort.

| Root of $H(u)=0$ | Multiplicity | Corresponding terms in the general solution |
|---|---|---|
| $u=\alpha$, real | Simple | $Ce^{\alpha x}$ |
| $u=\alpha$, real | $\mu$ | $e^{\alpha x}(C_1+C_2x+\cdots+C_\mu x^{\mu-1})$ |
| $u=\alpha\pm\beta i$ | Simple | $e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$ |
| $u=\alpha\pm\beta i$ | $\mu$ | $e^{\alpha x}[\cos\beta x(C_1+C_2x+\cdots+C_\mu x^{\mu-1})$ $+\sin\beta x(C_{\mu+1}+C_{\mu+2}x+\cdots+C_{2\mu}x^{\mu-1})]$ |

It should be emphasized that the functions $f(x)$ which satisfy our basic assumption are linear combinations of those appearing in this table.

If the polynomial $G(u)$ which occurs in (2) were available, we could reduce (1) to the *homogeneous* equation

* Phillip Franklin's Methods of Advanced Calculus treats this case in a manner quite similar to ours. We claim no originality for the method given here, but we feel that its advantages over the usual method deserve consideration by a teacher of elementary differential equations.

† This is the case in which the well known method of the trial integral applies. See A. Cohen, Differential Equations.

$$G(D_x)F(D_x)y = 0 \tag{3}$$

by applying the operator $G(D_x)$ to (1). The general solution of (3) could then be substituted in (1) and the additional constants introduced by the operator $G(D_x)$ could be evaluated. In fact, a knowledge of the *roots* of $G(u)=0$ and of $F(u)=0$ will suffice to determine the general solution of (3).

To find the roots of $G(u)=0$ which correspond to a given function $f(x)$ we suggest that the table displayed be used *in reverse*. Thus to $x^2e^{3x}$ (or to $(x^2+4)e^{3x}$) there corresponds the roots* (3, 3, 3); while to $xe^x \sin 5x$ (or to $xe^x \cos 5x$) there corresponds the roots $(1\pm 5i,\ 1\pm 5i)$. Having done this, we may then combine the roots of $G(u)=0$ with those of $F(u)=0$ to obtain the general solution of (3) from the displayed table.

Once the trial form of $y$ has been determined, we may proceed to substitute in (1) and evaluate the non-arbitrary constants. Of course, the general solution of (3) will contain the complementary function of (1) and this may be ignored if we desire only a particular integral of (1).

We conclude with a simple example.

$$(D_x^2 + 1)y = x^2 \sin x. \tag{4}$$

Here $F(u)=u^2+1$ and the roots of $F(u)=0$ are $\pm i$. The roots corresponding to $x^2 \sin x$ are $(\pm i,\ \pm i,\ \pm i)$. Hence the roots of $G(u)F(u)=0$ are $(\pm i,\ \pm i,\ \pm i,\ \pm i)$ and

$$y = (A + Bx + Cx^2 + Dx^3) \cos x + (E + Fx + Gx^2 + Hx^3) \sin x.$$

Substitution in (4) yields $D=-1/6$, $B=G=1/2$; $C=F=H=0$; $A$, $E$ arbitrary. With these values, $y$ is the general solution of (4).

## ON THE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

A. B. FARNELL, University of Colorado

Several special equations which are sufficient to illustrate the points in question will be considered here.

For example, consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{ax} \tag{1}$$

and the solution

$$y(x) = c_1e^x + c_2xe^x + \frac{e^{ax}}{(a-1)^2} \tag{2}$$

which satisfies the differential equation formally. The standard approach presented in most elementary texts to this solution is the method of complementary

* We adopt the usual notation in indicating the multiplicity of a root by repetition.

and particular solutions. For $a$ different from 1, the solution is usually left in this form.

Is (2) a solution of (1) for $a$ equal to 1? Most elementary texts infer that in this case (2) is not a solution and has no connection with the solution of (1), and some new approach is needed to find a particular solution; this new approach is usually pulled out of thin air. No cognizance is taken of the fact that $c_1$ and $c_2$ are functions of $a$. If (2) is written more precisely, the need for special considerations for various cases is avoided.

Let $y(0)=y_0$, $y'(0)=y_0'$. Imposing these initial conditions, (2) can be written in the form

$$y(x) = y_0e^x + (y_0' - y_0)xe^x + \frac{e^{ax} - e^x + (1-a)xe^x}{(a-1)^2}. \tag{3}$$

This solution is valid for $a$ different from 1, and as $a$ approaches 1, (3) becomes in the limit

$$y(x) = y_0e^x + (y_0' - y_0)xe^x + \tfrac{1}{2}x^2e^x \tag{4}$$

the correct solution for $a$ equal to 1.

For the differential equation

$$\frac{d^2y}{dx^2} + y = \sin ax$$

the solution can be written in the form

$$y(x) = y_0' \sin x + y_0 \cos x + \frac{\sin ax - a \sin x}{1-a^2}. \tag{5}$$

For $a$ different from (1), (5) gives the correct solution; and for $a$ equal to 1, the proper solution is

$$y(x) = y_0' \sin x + y_0 \cos x + \tfrac{1}{2}(\sin x - x \cos x), \tag{6}$$

which is obtained from (5) by letting $a$ approach 1.

Although the special methods of approach for various cases mentioned above have the undoubted advantage of reducing the labor involved in solving special equations, examples, such as the ones given above, should help in showing why and when these special methods work.

Finally, consider the differential equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = x.$$

Most students will readily agree that in this case the particular solution $y_p$ might contain $x$, $x^2$, and $x^3$. A little further argument will convince them that $y_p$ should be as general as its derivative and hence should contain a constant term. We thus take $y_p$ in the form

$$y_p = Ax^3 + Bx^2 + Cx + D,$$

which gives the particular solution in every case regardless of the values of the constants in the left-hand side of the differential equation; in one case $D$ might not be determined which means simply that it is arbitrary.

In the same way, for the differential equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = x \sin x$$

we have in all cases,

$$y_p = (Ax^2 + Bx + C) \sin x + (Dx^2 + Ex + F) \cos x.$$

After the student is convinced he can form the particular solution in all cases without any special knowledge of the complementary solution, it is then easier to justify the special methods adopted for various cases, and, the reduction in labor by using them will be obvious.

## A FUNCTION WITH A FINITE DISCONTINUITY

J. A. WARD, University of Georgia

An illustration of a function with a finite discontinuity that I have found useful in teaching sophomore calculus is the following:

$$y = \text{the degree in } u \text{ of: } (x - 1)u^3 + 5u^2 - 3u + 2.$$

Then $y$ is a discontinuous function of $x$; for at $x=1$, then $y=2$; but for all other values of $x$ we have that $y=3$. This function seems a little less artificial to sophomores than one of the type $y=(x^2-4)/(x-2)$ which has to be re-defined at $x=2$.

Other examples may readily be constructed along these lines, such as:

$$y = \text{the degree in } u \text{ of } (x - 1)u^{x^2+4} + 5u^2 - 3u + 2.$$

This gives in the $xy$-plane a parabola with the single discontinuity at $x=1$. Another variation is a continuous function whose derivative has two discontinuities:

$$y = \text{the degree in } u \text{ of } u^{x^2} + 2u^2 - 5u + 3.$$

In the $xy$-plane this is a parabola with the lower part of the arc discarded and replaced by a chord.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

*Correction.* Problem E 754 [1947, 39] was improperly credited to C. A. Richmond, Tyngsboro, Massachusetts. The correct proposer is S. T. Thompson, Tacoma, Washington.

E 761. *Proposed by C. R. Perisho, McCook Junior College, Nebraska*

An object with a smooth lower plane surface, and center of gravity $h$ units above this surface, is balanced on a sphere of radius $R$. Find the relation between $h$ and $R$ which insures stability of the object under small displacements.

E 762. *Proposed by J. R. Van Andel, Naval Air Experimental Station, Philadelphia, Pa.*

Let $A_1$ and $A_2$ be two circles with radii $a_1$ and $a_2$ and centers $(a_1, 0)$ and $(a_2, 0)$, respectively, with $a_2 > a_1 > 0$. Let $C$ be any circle in the crescent shaped area $M$ between $A_1$ and $A_2$, and tangent to both $A_1$ and $A_2$.

(a) The locus of the center of $C$ as it sweeps out $M$ is an ellipse with semi-axes $(a_1+a_2)/2$ and $\sqrt{a_1a_2}$.

(b) If $C_t$ is a circle of radius $r_t$ and center $P_t(x_t, y_t)$, where

$$\begin{aligned} r_t &= a_1a_2(a_2 - a_1)\phi_t, \\ x_t &= a_1a_2(a_2 + a_1)\phi_t, \\ y_t &= 2tr_t \\ \overset{-1}{\phi_t} &= a_1a_2 + t^2(a_2 - a_1)^2, \end{aligned}$$

then, for any real value of $t$, $C_t$ lies in $M$ and is tangent to $A_1$, $A_2$, and $C_{t-1}$.

E 763. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenters of the faces of a tetrahedron to the reflections in these faces of the points of intersection of the corresponding altitudes with the circumsphere, are concurrent at the Monge point of the tetrahedron.

E 764. *Proposed by R. J. Walker, Cornell University*

If $a_{ij}$ is the number in the $i$th row and $j$th column of a diabolic (pandiagonal magic) square of order five, show that

$$\sum_{i=1}^{5}\left(\prod_{j=1}^{5} a_{ij}\right) = \sum_{j=1}^{5}\left(\prod_{i=1}^{5} a_{ij}\right).$$

This relationship also holds for magic squares of order three. Does it, or any similar relation, hold for magic, or diabolic, squares of any other order—in particular, for diabolic squares of order seven?

E 765. *Proposed by Harold Becker, Omaha, Nebraska*

(a) What is the joint resistance of the general Wheatstone bridge as a function of the five component resistors? (b) What are forms for the components so that they and the resultant will all be integers?

## SOLUTIONS

### Power Points

E 705 [1946, 36 and 395]. *Proposed by Joseph Rosenbaum, The Milford School, Conn.*

A point in the interior of a simple closed curve, which is nowhere concave, will be called a *power point* if the product of the segments of a variable chord through the point is constant. The existence of how many distinct power points insures that the curve is a circle?

*Note by the Proposer.* R. A. Rosenbaum points out that this problem is not new, it having appeared as an article entitled *A characteristic property of the circle and sphere; second note,* by K. Yanagihara, in the Tôhoku Mathematical Journal, vol. 11 (1917), p. 55. In the article Yanagihara discusses both this problem and the analogous one for a simply closed surface. His proof for the sufficiency of two power points for the curve is essentially the same as that given by Kelly in the MONTHLY. However, as regards the surface, Yanagihara arrives at the false conclusion that there, too, the existence of two power points is a sufficient condition for the surface to be a sphere. That this is not true is shown by the following example.

Take a circle with a chord $AB$, and two points $M$ and $N$ on the circle on opposite sides of $AB$. Next let $c$ be an arc of any plane curve *other than a circle,* passing through $M$ and $N$ and not in the plane of the circle. Now the closed surface generated by the variable circle $ABP$ as $P$ moves from $M$ to $N$ along the curve $c$ is not a sphere, because $c$ is not an arc of a circle. The surface, however, possesses two power points; in fact every point on $AB$ is a power point of the surface, since, by the very manner in which this surface was generated, every section of it by a plane through $AB$ is a circle having $AB$ as a chord.

It is easily shown, however, that the surface above will fail to possess a tangent plane at either $A$ or $B$, and that the sufficiency of two power points under the additional condition that the surface possesses a unique tangent plane at every point is correct. Since this additional condition is not included in Yanagihara's theorem, his theorem, as stated, is false.

Inasmuch as the existence of three non-collinear power points was mentioned in the editor's note to this problem as being a sufficient condition for a simply closed surface to be a sphere, it may be well to supply the proof for it here.

LEMMA 1. *If a simply closed surface $F$ possesses two power points $A$ and $B$*

*within it, then every point on $AB$ is a power point of $F$.*

This follows immediately from the fact that a simply closed curve possessing two power points is a circle.

LEMMA 2. *If a simply closed surface $F$ possesses three non-collinear power points $A$, $B$, $C$ within it, then every point $P$ in the plane $ABC$ is a power point of $F$.*

Draw any line through $P$ cutting $AB$ and $BC$ in $I$ and $J$ respectively. By Lemma 1, $I$ and $J$ are both power points of $F$, and since $P$ lies on $IJ$ then, again by Lemma 1, $P$ is a power point of $F$.

THEOREM. *If a simply closed convex surface $F$ possesses three non-collinear power points $A$, $B$, $C$ within it, then $F$ is a sphere.*

The plane $ABC$ cuts $F$ in a circle $k$. By Lemma 2, every point in this circle is a power point of $F$. Now consider the sphere $S$ determined by the circle $k$ and a point $Q$ on $F$ outside the circle $k$. If $X$ is any point on $F$ on the opposite side of the plane of $k$ from $Q$, then the line $QX$ cuts the plane of $k$ in a point whose power with respect to $F$ is the same as that with respect to $S$ (both being equal to the power of that point with respect to circle $k$). Hence $X$ must lie on the sphere $S$. This proves that all points on $F$ which are on the opposite side of the plane of $k$ from $Q$ lie on $S$. This in turn, by the same argument, leads to the conclusion that every point on $F$ on the opposite side of the plane of $k$ from $X$ lies on the sphere $S$. Since these two sets of points constitute the entire surface $F$, the proof is completed.

### Consecutive Odd Integers

E 726 [1946, 333]. *Proposed by W. Nicholson, Chicago, Illinois*

If $p$ and $N$ are positive integers, $p>1$, show that $N^p$ is the sum of $N$ consecutive odd integers.

*Solution by Murray Barbour, Michigan State College.* The formula for the sum of an arithmetic progression is $s=\frac{1}{2}n[2a+(n-1)d]$. Here $s=N^p$, $n=N$, and $d=2$. Substituting these values and solving for $a$ we find $a=N^{p-1}-N+1$. This value for $a$ is necessarily odd, since, for $p>1$, $N(N^{p-2}-1)$ is always even.

Also solved by D. W. Alling, M. Aissen, Joshua Barlaz, Barney Bissinger, W. G. Brady, Paul Brock, W. E. Byrne, M. I. Chernofsky, H. J. Cohen, M. L. Constable, R. E. Crane, Monte Dernham, Benjamin Epstein, J. W. Gaddum, Sydney Glusman, Bernard Greenspan, Stanley Hughart, Meyer Karlin, N. D. Lane, Elmer Latshaw, B. R. Leeds, W. J. LeVeque, C. D. Olds, Margaret Olmsted, K. B. Patterson, C. R. Perisho, C. L. Perry, P. A. Piza, E. D. Schell, E. P. Starke, W. R. Talbot, R. H. Urbano, Alan Wayne, Michael Wilensky, Maud Willey, R. K. Zeigler, and the proposer.

Barlaz proposed the similar but more difficult problem: Let $p$, $N$ be positive integers, $p>1$. Find conditions on $k$, $a$, $b$ so that $N^p$ may be written as the sum of $k$ consecutive terms of the arithmetic progression $ax+b$, $(a, b)=1$, $ax+b>0$.

Dernham pointed out that the problem is a special case of the more general theorem: If $M$ and $N$ are positive integers, $M \geqq N$, then $MN$ is the sum of $N$ consecutive odd or $N$ consecutive even integers, depending upon whether $M$ and $N$ have the same or different parity; namely

$$(M - N + 1) + (M - N + 3) + \cdots \quad \text{to } N \text{ terms.}$$

**Probability of Adjacent King and Jack**

E 727 [1946, 333]. *Proposed by Frederick Mosteller, Princeton, N. J.*

What is the probability that a King and a Jack will appear side by side in a shuffled pack?

*Solution by N. D. Lane, St. Andrew's College, Ontario.* Let us consider the number of ways a King and a Jack will *not* appear together. We may have four, three, two, or none of the Kings side by side.

If no two Kings are side by side, remove the Kings and Jacks from the pack and arrange the other 44 cards in one of 44! different ways. Now the Kings may be inserted in the 45 available places in $P(45, 4)$ ways. The Jacks may now be inserted, one at a time, so that no Jack is next to a King, in $41 \cdot 42 \cdot 43 \cdot 44$ ways. Hence the number of ways no Jack and King will appear side by side when the the Kings are separated is $44!P(45, 4)P(44, 4) = a$, say.

Similarly, if we have two Kings together and the other two separated the number of ways in which a Jack and King will not be together is $44!P(4, 2)$ $P(45, 3)P(45, 4) = b$.

If we have the four Kings arranged in pairs the number is $44!P(4, 2)P(45, 2)$ $P(46, 4) = c$.

If we have three Kings side by side, the number of ways is $44!P(4, 3)$ $P(45, 2)P(46, 4) = d$.

If all four Kings are side by side, the number of ways is $44!P(4, 4)P(45, 1)$ $P(47, 4) = \text{e}$.

The probability that a King and a Jack will appear together is then $1 - (a+b+c+d+e)/52! = 0.48628$, or almost one-half.

Also solved by H. D. Grossman and John Riordan.

Grossman pointed out that, except for the substitution of 4-spots and 7-spots for Kings and Jacks, this is problem No. 578 of the *National Mathematics Magazine*, April 1945, pp. 367, 368. His solution is the one there presented, and is similar to the above.

Riordan employed the symbolic method (*cf.* Kaplansky, *Symbolic Solution of Certain Problems in Permutations*, Bull. A.M.S., 1944, pp. 906–914). He also showed that the probability of exactly one occurrence of King and Jack adjacent is 0.37193.

The proposer remarked that the problem occurs (incorrectly solved) in the volume *The World's Best Book of Magic* by Walter B. Gibson. The author assures the amateur prestidigitator that the required probability is about 2/3, on the reasoning that there are usually eight cards adjacent to the Kings and that each of these has one chance in twelve of being a Jack; $8(1/12) = 2/3$.

### A Property of the Astroid

E 731 [1946, 394]. *Proposed by Howard Grossman, New York City*

At any point $P$ on the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, the tangent to the curve is the shortest line through $P$ lying between the axes.

*Solution by C. F. Pinzka, North Plainfield, N. J.* The pencil of lines passing through $P(x_1, y_1)$ is given by $y-y_1=m(x-x_1)$, the intercepts being $x_0=x_1-y_1/m$ and $y_0=y_1-mx_1$. Then the length of any line through $P$ is $(x_0^2+y_0^2)^{1/2}$. Applying the usual criteria we find that for the minimum line we have $m=-(y_1/x_1)^{1/3}$. But the slope of the tangent at $P$ to the hypocycloid is found, upon differentiation, also to be $-(y_1/x_1)^{1/3}$. Hence these two lines are coincident.

Also solved by Michael Aissen, D. W. Alling, Murray Barbour, P. T. Bateman, Paul Brock, H. E. Fettis, I. M. Gardoff, E. A. Jacobs, H. L. Lee, S. T. Parker, C. R. Perisho, D. W. Matlock, W. R. McEwen, Emanuel Mehr, Herbert Reisman, E. D. Schell, W. R. Talbot, and P. D. Thomas.

### A Three Digit Number

E 732 [1946, 394]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find, in the system of base 9, a number of three digits which, when transformed to the system of base 13, is composed of the same three digits.

*Solution by Walter Penney, Washington, D. C.* Let $abc$ be the required number expressed in base 9. Since the leading digit cannot be the same in the two representations we have the following four cases:

$$81a + 9b + c = 169b + 13a + c,$$
$$81a + 9b + c = 169b + 13c + a,$$
$$81a + 9b + c = 169c + 13a + b,$$
$$81a + 9b + c = 169c + 13b + a.$$

The first equation, which reduces to $40b=17a$, obviously has no solution in integers less than 9. The second equation, which reduces to $40b=20a-3c$, yields the four solutions $abc=210$, 420, 630, 840. The third equation, which reduces to $42c=17a+2b$, yields the two solutions $abc=241$, 482. The last equation is found to yield no solution.

There are therefore the six solutions 210, 420, 630, 840, 241, and 482, expressed in base 9, corresponding to 102, 204, 306, 408, 124, and 248, respectively, in base 13.

Also solved by Michael Aissen, D. W. Alling, Murray Barbour, R. G. Blake, W. E. Bunyan, Monte Dernham, H. L. Lee, B. R. Leeds, Margaret Olmsted, S. T. Parker, Clay Perry, C. F. Pinzka, F. W. Saunders, E. D. Schell, W. R. Talbot, and the proposer. Many of these solutions gave only partial results.

# ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.*

## PROBLEMS FOR SOLUTION

4239. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

$AXBZ$ is a jointed rhombus connected with a fixed point $O$ by two equal rods $OA$, $OB$. $OCZD$ is a jointed rhombus and $YC$, $YD$ are equal rods. (Two Peaucellier cells, as it were "cross joined.") Prove that, as $Y$ describes a circle, $X$ describes a conic.

4240. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relations which must connect $N$, $B$, $B'$ in order that the number $N$ may be written with the same three digits in the system of numeration of base $B$ as in the system of base $B'$. Having given $B$, find $B'$ and $N$. Apply the results when $B=10$.

4241. *Proposed by R. Goormaghtigh, Bruges, Belgium*

Consider a parabola having its vertex at a variable point $M$ on a given plane curve, and its focus at $F$, the point dividing the radius of curvature $MC$ in a constant ratio; the parabola touches its complete envelope at $M$ and also at two other finite points. The corresponding chord of contact is perpendicular to the line joining $M$ to the midpoint of the radius of curvature at $C$ of the evolute of the given curve. If this chord of contact intersects $MF$ at $D$, then $DC=2MF$.

4242. *Proposed by W. O. Pennell, Exeter, New Hampshire*

Determine the sums of the following infinite series:

$$(1)\qquad \frac{n+2}{n+1}-\frac{2n+2}{(n+1)(2n+1)}+\frac{3n+2}{(n+1)(2n+1)(3n+1)}-\cdots,$$

$$(2)\qquad \frac{3n+7}{n+1}-\frac{8n+9}{(n+1)(2n+1)}+\frac{15n+11}{(n+1)(2n+1)(3n+1)} - \frac{24n+13}{(n+1)(2n+1)(3n+1)(4n+1)}+\cdots,$$

where $n$ is any real number except $0$, $-1$, $-1/2$, $-1/3$, etc.

4243. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The point $M$, situated in the interior of a tetrahedron $ABCD$, such that the volume of the tetrahedron having for vertices the points of intersection of the

lines $AM$, $BM$, $CM$, $DM$ with the opposite faces of $ABCD$ be a maximum, coincides with the centroid of $ABCD$.

## SOLUTIONS

### Convex Polyhedra

4176 [1945, 522]. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove the following two theorems in affine geometry of three dimensions:

(a) If all the faces of a convex polyhedron are parallelograms, their number is the product of two consecutive integers.

(b) If each face of a convex polyhedron has a center of symmetry, the whole polyhedron has a center of symmetry.

*Solution by Leo Moser, University of Toronto.* We prove (b) first. Any polygon having central symmetry is necessarily a parallel-sided $2m$-gon, since the point of central symmetry leads from any side to an equal and parallel side. We dissect each face into parallelograms. This can be done since every parallel-sided $2m$-gon can be dissected into a parallel sided $2(m-1)$-gon and a ribbon of $m-1$ parallelograms. Each edge of the polyhedron then determines a set of parallel sides and so a set of parallelograms called a zone. Now every parallelogram determines two zones which, by topological considerations, must intersect in another parallelogram which we will call the mate of the first parallelogram.

A parallelogram and its mate are congruent and parallel since the sides of the parallelograms are equal and parallel in pairs. Further a parallelogram cannot be parallel to more than one parallelogram, for then the polyhedron would be concave.

Clearly, two congruent parallelograms having sides parallel in pairs have a center of symmetry so that it only remains to show that the center of symmetry determined by one parallelogram and its mate is the same as that determined by any other parallelogram and its mate. We show this first for any two adjoining parallelograms and their mates. The center of symmetry for any parallelogram and its mate is determined by any point on one of the parallelograms. Take the point to be a point on a side of the parallelogram. Since this point belongs also to the adjoining parallelogram these two parallelograms and their mates must have the same center of symmetry. But we can pass from any parallelogram to any other parallelogram by crossing over a number of faces and edges. Hence all pairs of parallelograms have the same center of symmetry and the theorem is proved.

The proof of (a) follows easily. Every two zones (see b) must intersect. For, consider a fixed zone and a parallelogram not in it. Then the mate of this parallelogram is on the opposite side of the zone. Otherwise the polyhedron would be concave. Hence every zone intersects the zones determined by any parallelogram not in it, and so intersects every other zone. Further, as in (b), once two zones intersect they must intersect twice. Let the number of zones be $n$. Then the number of faces must equal the number of intersections of zones which is $2\binom{n}{2}=n(n-1)$ as required.

**Probability of Two Coincident Birthdays**

4177 [1945, 522]. *Proposed by P. R. Halmos, Syracuse University*

What is the smallest number of people sufficient to ensure that the probability that there be at least two with the same birthday is at least 1/2? It is to be assumed that any two days of the year are equally likely to be the birthday of a given individual and that there are no leap years.

*Solution by Z. I. Mosesson, Fort Monroe, Va.*

The probability that no two of $n$ people have the same birthday is

$$\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\cdots\left(\frac{366-n}{365}\right)=\frac{1}{(365)^n}\,\frac{365!}{(365-n)!}.$$

Hence the probability that at least two of $n$ people have the same birthday is

$$1-\frac{365!}{(365)^n(365-n)!}.$$

We wish to determine the least value of $n$ for which

$$1-\frac{365!}{(365)^n(365-n)!}\geqq\frac{1}{2}\quad\text{or}\quad\frac{(365)^n(365-n)!}{365!}\geqq 2$$

or

$$n\log_{10}365+\log_{10}(365-n)!\geqq\log_{10}2+\log_{10}365!.$$

Using a table of $\log_{10}n!$ such as is given in T. C. Fry's *Probability and its Engineering Uses*, it is easy to show by trial and error that $n=23$.

Solved also by David Alling, P. T. Bateman, D. H. Browne, H. S. M. Coxeter, Monte Dernham, Howard Eves, Bart Park and the proposer.

*Editorial Note.* Browne, Coxeter, and Eves stated that the problem and its solution are given in Ball-Coxeter's *Mathematical Recreations and Essays*, p. 45.

**A Maximum Product**

4180 [1945, 523]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The product of $k$ positive integers whose sum is $N$, where $N=kp+h$, is a maximum when $h$ of the factors are equal to $p+1$ and the $k-h$ others to $p$.

Dedicated to E. P. Starke.

*Solution by P. T. Bateman, Bryn Mawr College.* It is evidently intended that $0\leqq h<k$. If the difference between the largest and the smallest of a set of $k$ integers is unity, then the set clearly consists of $h$ elements $p+1$ and $k-h$ elements $p$. If the difference between the largest and the smallest integer of the set is greater than unity, we can construct a new set having the same number of integers, and the same sum, but a greater product. This is done by replacing one of the maximal elements $M$ by $M-1$ and one of the minimal elements $m$ by

$m+1$. Since $M-m>1$,

$$(M-1)(m+1) = Mm + M - m - 1 > Mm$$

and thus the product is increased. Since the number of possibilities is finite, the stated result follows.

Solved also by Murray Barbour, B. P. Gill, H. S. Grant, P. C. Hammer, and P. A. Piza.

*Editorial Note.* Similar elementary reasoning shows that, $k$ being unspecified, the maximum product which can be formed from any partition of $N$ $(N>1)$ into positive integers is given by

$$3^{N/3}, \qquad 2^2 \cdot 3^{(N-4)/3}, \qquad 2 \cdot 3^{(N-2)/3},$$

according as $N$ is congruent to 0, 1, 2, respectively, modulo 3. For any positive integer $q$ we can write

$$(1) \qquad q \leqq 2^{q/2}, \quad \text{or} \quad q \leqq 3 \cdot 2^{(q-3)/2},$$

according as $q$ is even or odd. By the original result, the maximum product for any $k$ is in the form $p^r(p+1)^s$. Without decreasing this product and without altering the sum of the factors we can replace it, according to (1), by a product of 2's and 3's. Finally, whenever the exponent of 2 exceeds 2, we can increase the product further by replacing $2^3$ by $3^2$ as often as possible. No further increase is now possible and the result appears as stated. The reader might care to go further and determine the maximum product of factors whose sum is $N$, when neither the factors nor the sum need be integers.

### Generalized Simson Lines

4181 [1945, 582]. *Proposed by P. D. Thomas, Lumberton, Miss.*

Lines are drawn from a point $P$ on the circumcircle of an equilateral triangle parallel to the three sides, thus determining six points, two on each side respectively. (1) Prove that the six points thus determined lie by threes on two straight lines. (2) If $Q$ is the point of intersection of these two lines, find the locus of $Q$ as $P$ moves on the circumcircle.

*Solution by R. Goormaghtigh, Bruges, Belgium.* Consider a point $P$ on the circumcircle $\Gamma$ of any triangle $A_1A_2A_3$. In a system of complex coördinates having $\Gamma$ as base circle (radius unity, center at origin), let $t_1$, $t_2$, $t_3$, and $\tau$ be the coördinates of $A_1$, $A_2$, $A_3$, and $P$, respectively, and let

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

If $\bar{a}$ is the conjugate of $a$, then $t_i\bar{t}_i = \tau\bar{\tau} = 1$ because $\Gamma$ is the base circle. If $\lambda = e^{2i\theta}$, it is easily verified that the side $A_2A_3$ and the line drawn from $P$ forming with $A_2A_3$ the angle $\theta$, will have the respective equations:

$$z + t_2t_3\bar{z} = t_2 + t_3, \qquad z + \lambda t_2t_3\bar{z} = \tau + \lambda t_2t_3\bar{\tau}.$$

Their intersection is given by

$$z_1 = \frac{\lambda t_2 + \lambda t_3 - \lambda t_2 t_3 \bar{\tau} - \tau}{\lambda - 1}.$$

The line

$$\tau z + \lambda s_3 \bar{z} = \frac{\tau^2 - \lambda s_1 \tau + \lambda s_2 - \lambda^2 s_3 \bar{\tau}}{1 - \lambda} \tag{1}$$

is easily seen to be satisfied by $z_1$, and also by the analogous points, $z_2$ and $z_3$, on the other sides of the triangle. This line is called the Simson line, for the angle $\theta$, of $P$ as to the given triangle. Since the two straight lines considered in the problem are obviously the Simson lines, for the angles $\pm\pi/3$, of $P$ as to the given equilateral triangle, the proof of the first part is complete.

The Simson line, for a second angle $\theta'$, of $P$ is given by (1) with $\lambda$ replaced by $\lambda' = e^{2i\theta'}$. The intersection $Q(x)$ of these two lines is given by

$$x = \frac{\lambda\lambda'}{(1-\lambda)(1-\lambda')}\left[s_3\bar{\tau}^2 - s_2\bar{\tau} + s_1 - \frac{(\lambda + \lambda' - 1)\tau}{\lambda\lambda'}\right]. \tag{2}$$

Hence the locus of $Q$ is in general a quartic. When the triangle $A_1A_2A_3$ is equilateral, so that $s_1 = s_2 = 0$, (2) reduces to a trinodal hypotrochoid. If also, as in the present problem, $\theta = -\theta' = \pi/3$, (2) becomes the required locus

$$x = \frac{s_3\bar{\tau}^2 + 2\tau}{3}, \tag{3}$$

which is the deltoid (three-cusped hypocycloid) having its cusps at the vertices of the given triangle.

Further results of interest, related to an equilateral triangle, may be obtained from (2). The locus of intersections of Simson lines, for the angles $\pm\pi/6$, is the circumcircle $\Gamma$. For the angles $\pm\pi/4$, (2) gives the regular trifolium. If we hold $P$ fixed while the triangle turns about its center, the locus of the intersection of the Simson lines for the angles $\pm\theta$ is a circle passing through $P$.

Solved also by H. E. Fettis, Ou Li, Irma Moses, and O. J. Ramler, using analysis similar to the above (and with references to Morley and Morley, *Inversive Geometry*); by J. H. Butchart, Howard Eves, and Richard Meyer, using synthetic geometry; by Claire F. Adler, W. E. Cox, D. H. Erkiletian, Jr., W. A. Rees, W. T. Short, A. Sisk, G. A. Williams, and R. H. Wilson, Jr., using standard analytic geometry; and by W. J. Robinson, using both the latter methods.

**Envelope of Simson Lines**

4115 [1944, 233]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

From a point $P$ on the circumcircle of a triangle lines are drawn inclined at angles $\theta$ to the sides of the triangle and meeting them in three collinear points.

Prove that as $P$ varies the line on the three points envelopes a three-cusped hypocycloid. Prove that this hypocycloid is the locus of a point on a circle of radius $R/2 \sin \theta$ which rolls inside another circle three times the radius, whose center $X$ is equidistant from the circumcenter $C$ and the orthocenter $O$ and is such that angle $OXC=2\theta$, $R$ being the radius of the circumcircle.

*Solution by the Proposer.* Take the circumcircle as the unit circle with center at the origin, denote the vertices by the complex constants, $t_1$, $t_2$, $t_3$, and set

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

If $\tau$ denotes the point $P$, the equation of this generalized Simson line is

$$\bar{z}(1-\lambda)s_3\tau - z(1-\bar{\lambda})\tau^2 = \tau^3\bar{\lambda} - s_1\tau^2 + s_2\tau - s_3\lambda, \tag{1}$$

where $\lambda = e^{2i\theta}$. Consider also the circle

$$z = \frac{s_1}{1-\bar{\lambda}} + \frac{u}{2\sin\theta} \tag{2}$$

where $u\bar{u}=1$. (1) and (2) intersect where

$$u = i\tau e^{-i\theta}, \qquad \frac{s_3e^{3i\theta}}{i\tau^2}, \tag{3}$$

that is, $u$ and $-s_3e^{i\theta}/iu^2$.

If $\theta=\pi/2$, the line (1) is the ordinary Simson line, the circle (2) is the nine-point circle, and the points (3) are $u$, $-s_3/u^2$. Therefore, the envelope when $\theta\neq\pi/2$ may be derived from the familiar result when $\theta=\pi/2$ by merely replacing the Simson line by the line (1), the nine-point circle by the circle (2), and $s_3$ by $s_3e^{i\theta}/i$. This gives the required results—and the additional information that the tangents at the cusps when $\theta\neq\pi/2$ are inclined to those in the case $\theta=\pi/2$ at angles $(\theta-\pi/2+2n\pi)/3$.

*Editorial Note.* In connection with his solution of 4181, O. J. Ramler points out that the envelope of the Simson line, for the angle $\theta$, of $P$ (see (1) in the solution of 4181 above) is obtained by differentiating partially with respect to $\tau$. This gives

$$z + \frac{\lambda^2 s_1}{1-\lambda} = s_3\bar{\tau}^2 + 2\tau,$$

which is a deltoid similar to (3) in the solution of 4181, but three times as large. In the case of an equilateral triangle $A_1A_2A_3$, $s_1=0$ and the two deltoids are concentric.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*College Algebra.* By A. A. Albert. New York and London, McGraw-Hill Book Co., 1946. 12+278 pages. $2.75.

In his introduction the author justifies his new textbook in college algebra and one can do no better than to let him present his own case: "College algebra has been a most abused subject. The time allotted to it is frequently inadequate for a genuinely good treatment, and indeed the entire course is sometimes omitted. This is due partly to a desire to bring students to a study of the calculus as early as possible. It is also due partly to the presentation of college algebra, in all texts thus far published, as a collection of seemingly unrelated topics. The desire to teach the calculus as early as possible tends to defeat its own ends. The building of a course in the calculus on what must be a weak foundation cannot result in a good student understanding of the subject. There is also no reason why the material of college algebra cannot be cohesively organized."

There is considerable justification for these views. Many college courses and texts have in the main persisted as collections of trick questions with too little emphasis on their systematic background. An excellent example of this type of presentation could until quite recently be found in the preparations for the first actuarial examinations. In algebra as in other fields of college instruction in mathematics there also exists a considerable amount of inertia which tends to make the choice of content in many courses almost dogmatically fixed. There are, of course, certain basic facts which will be with us forever, and which always must be included in the elementary books. But on the whole the newer trends in research and in the applications seem to be slow in exerting their influence. In mathematics there is no revolution as by quantum mechanics in physics, no atom bombs whose explosion immediately makes their impact felt down to the introductory texts.

The need for a revision of the instruction in elementary algebra, both in regard to content and presentation, fortunately has been recognized in a few of the recent texts and the present book is a significant step in the same direction.

The first few chapters give some of the traditional investigations of numbers, the laws of operation, the main properties of integers and the extension of the number system to the fields of real and complex numbers. Here in the introduction of the real numbers the reviewer finds one of the few instances where it seems that the presentation could be improved. The discussion is very brief and it may not give the students sufficient time and space to dwell on certain of the essential points.

In the subsequent chapters one finds the basic properties of polynomials up to unique factorization, the binomial theorem, and the method of indeterminate

coefficients. Two long chapters are devoted to the study of the algebraic equations. One finds, to begin with, a discussion of the decomposition into linear factors and of the relations between roots and coefficients. Imaginary roots and multiple roots are touched upon and the determination of rational roots is treated at some length. The determination of the real roots is discussed rather exhaustively, including bounds for the roots, methods for isolating roots, Descartes' rule of signs, Sturm's theorem, Horner's and Newton's methods for the computation of the roots. A useful feature is the inclusion of a chapter on vectors with applications to complex numbers, de Moivre's theorem, rotation of axes and conic sections.

The two last chapters represent an innovation for textbooks of this kind, containing a general discussion of determinants and matrix theory. This seems a most desirable inclusion. During the last decennials these topics have acquired an importance which is rapidly growing, not only in pure mathematics, but also in modern physical theories as well as in some of the now most common statistical methods. It is appropriate therefore that they should at present be given more generally and at an earlier level than previously. The author discusses determinants, their expansions and basic properties, and of course, their use in solving systems of linear equations. The matrix theory uses vectors in $n$-dimensional space, it includes multiplication of matrices, computation of inverses, elementary transformations and rank together with some discussion of similarity, symmetric and orthogonal matrices and applications to quadratic forms. Although it is clear that these themes cannot be given in their most general and complete form, there is enough to give the students some familiarity with the concepts.

To sum up, Albert's new textbook should prove a very valuable introduction to college algebra, both through its systematic outlook and through its inclusion of modern topics. The numerous problems add to its usefulness.

OYSTEIN ORE

*Curves*. By Lt. Col. R. C. Yates, AUS. West Point, N. Y., Department of Mathematics, United States Military Academy, 1946. 8+230 pages.

The detailed study of special plane curves is one of the most beautiful and at the same time one of the most neglected portions of the undergraduate curriculum. Its study is made more difficult by the fact that information concerning these curves is scattered widely throughout the mathematical literature, frequently in sources not easily obtainable in this country. And so it is a major operation to determine what are the known facts about many of the less common curves.

Colonel Yates has gone a long way toward remedying this difficulty by assembling the material contained in this book. Since there is no general theory of curves, the material is arranged in self-contained sections, one for each type of curve arranged alphabetically by the name of the curve. In each of these sections the reader will find a sketch of the history of the curve, a general description of

the curve including a graph, and its equations in a number of forms such as (1) cartesian, (2) polar, (3) parametric, (4) pedal, (5) Whewell, and (6) Cesaro forms. Further there is a collection of metric formulae for arc length, area, curvature, and the like, followed by a discussion of its relations to other curves and other general properties; and finally there is a list of references to the literature. There are also sections describing general methods of generating curves such as evolutes, envelopes, parallel curves, and pursuit curves, and there is a full section of methods of curve sketching. A table of contents and a detailed index provide ready access to this material.

The book is printed by offset from typescript and has a spiral binding. According to the title page it is "Prepared for use in the Department of Mathematics U. S. Military Academy," and it gives the impression of being a set of notes on curves rather than a textbook or a comprehensive treatise. In particular its shows a need for careful editing for it contains certain items which would not bother a well-trained reader, but which would confuse undergraduates. For example the first page contains a list of notation, presumably standard for the entire book. In this list $p$ is defined as the "Distance from Origin to Tangent," but $p$ appears in many other places with different meanings. Again on page 41 there is the statement: "for the conic, a tangent may be defined as a line meeting the curve in but one point" (!). The references suffer from not being in standard bibliographical form, and contain abbreviations and omissions of relevant data which complicate the task of locating the given material. It would be an injustice to the author, however, to continue such a list; for he has performed a very useful task in collecting such a wealth of facts on curves and has produced a book to which many mathematicians will turn with pleasure and profit.

C. B. ALLENDOERFER

*Calculus*. Second Edition. By F. H. Miller. New York, John Wiley and Sons, 1946, 16+416 pages. $3.50.

Those familiar with the author's first edition of *Calculus*, 1939, will welcome this new edition. Many changes have been made. The treatment of the limit concept has been amplified. An article on graphical differentiation and one on approximation integration will contribute to the usefulness of the book. The exercises have been revised and some of the more complicated problems in the first edition have been replaced.

Throughout Chapters II and III the derivative of $y$ with respect to $x$ is denoted by $D_xy$, and the ordinary notation for the derivative is not used until after the introduction of differentials. The natural logarithm of $x$ is denoted by $\ln x$, and the common logarithm by $\log x$. These changes seem to be in keeping with current trends. In dealing with indeterminate forms the symbols $0/0$, $\infty/\infty$, and so on, are avoided.

The author has written a detailed and carefully planned textbook which is directed toward the needs and interests of not only the future major in mathematics, but also the engineer. Geometrical and physical interpretations of both

differentiation and integration are introduced before the general processes of these subjects are discussed. Thus the student is given an immediate idea of the usefulness of the calculus before he is exposed to long lists of formulas.

Many readers will enjoy the manner in which Duhamel's theorem is avoided in the formulation of geometrical and physical problems. In fact, infinitesimals of higher order are not mentioned, nor does the need for this concept arise in the book.

Throughout the book definitions and theorems are carefully worded and well displayed on the page. The chapter on limits is very thorough and the fundamental theorems on limits are proved in an interesting and fairly rigorous manner. A figure or two would have helped to illustrate the general definition of a limit of a sequence. Moreover, the transition from the limit of a sequence to the limit of a function is a little abrupt. There is a good discussion of continuity, and throughout the book the author is careful to state what assumptions as to continuity are assumed and used. The average student, however, will find this chapter hard reading and much of the material will have to be broken down by the instructor.

Chapter IX on partial differentiation comes before the chapters on integration. Here the addition of figures illustrating functions of two variables and showing the geometrical meaning of the partial derivatives would have improved the discussion. There is no mention of directional derivatives.

In Chapter XI the definite integral is first presented as a limit of a certain sum and is shown to be a function of its upper limit. Then it is shown analytically to possess a derivative and hence the concept of an integral as an antiderivative arises. In this way the student is taught at the outset that the definite integral need not necessarily mean an area. An example actually showing the calculation of a limit of a sum would have been a welcome addition to the chapter.

There is a chapter on infinite series which includes Maclaurin's Integral Test and Taylor's Series with a remainder. The discussion of operations with power series is brief but to the point and the student is given ample warning about differentiating and integrating series.

In the chapter on multiple integrals the author is careful to distinguish between double integrals and iterated integrals. The applications mentioned here include most of the standard ones. There is a good discussion of the area of a surface.

The book ends with a chapter on differential equations. Here the discussion of separation of variables could be more rigorous. Moreover the solution of differential equations in series expansions is not included.

A few matters of detail might be mentioned. Exercises 14 and 15, p. 32, would be better placed after motion in the plane curve on p. 125. There are no problems to go with the discussion of derivatives of parametric equations on p. 42.

Some teachers will object to the introduction of the term infinitesimal on

page 7. Many modern texts are avoiding the term altogether. Also there will be disagreement with the discussion of differentials as presented, especially over the use of the equation $dy=f'(x)\Delta x$, with $f(x)=x$, to justify the taking of $dx$ to be always equal to $\Delta x$. It is a mistake to give students the impression that $dx$ is always equal to $\Delta x$.

On page 66 hyperbolic functions are introduced as exercises and there is no discussion of these functions in the text. In Exercise 43 on that page the student is asked to show that $\sinh^{-1}y=\ln(y+\sqrt{1+y^2})$. It is the reviewer's experience that students need a text discussion as well as help from the instructor in order to work such problems with confidence.

The discussion of maxima and minima does not include examples in which the critical point has an infinite slope. Nor are there problems in which the absolute maximum in a certain range occurs at a point for which $f'(x)\neq 0$.

The reviewer found few misprints and none of a serious nature. In places the text material looks a bit crowded and some of the discussions are too brief but considering the amount of material covered the author has written a clear and useful text and one in which applications are certainly not lacking. Answers are given to the odd-numbered exercises and those to the even-numbered ones are available. An appendix of algebra, geometry, trigonometry, and analytic geometry formulas is included. There are tables of the functions $e^x$, $e^{-x}$, $\ln x$, $\log x$, as well as a fairly complete table of integrals. A table of differentiation formulas is needed, for nowhere are such formulas listed for easy reference. All the excellent features of the original edition have been retained. The changes have added desirable features and should increase the effectiveness of the text as a teaching instrument.

C. D. OLDS

## NEW BOOKS RECEIVED

*Analytic Geometry and Calculus.* By J. F. Randolph and Mark Kac. New York, Macmillan Co., 1946. 9+642 pages. $4.75.

*Basic Mathematics for Technical Courses.* By C. E. Tuites. New York, Prentice-Hall Co., 1946. 14+344+132 pages. $5.00.

*Mathematical Aids for Engineers.* By R. W. Dull. New York and London, McGraw-Hill Book Co., 1946. 12+346 pages. $4.50.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

Of probable interest to members of mathematical clubs is the series of articles appearing in *The Pentagon*, official publication of *Kappa Mu Epsilon*, presenting bibliographies on subjects which are suitable for club programs. The bibliographies thus far presented together with the issue in which they appear are:

*Women as mathematicians*, vol. V, no. 2, Spring 1946
*The cattle problem of Archimedes*, vol. V, no. 2, Spring 1946
*Paper folding*, vol. V, no. 2, Spring 1946
*Mathematical prodigies*, vol. VI, no. 1, Fall 1946
*Calculating machines*, vol. VI, no. 1, Fall 1946
*The bee as a mathematician*, vol. VI, no. 1, Fall 1946

Some of the mathematics clubs have sent to the editor of this department their programs announcing coming events. Many of these programs have been very clever and stimulating. The editor is of the opinion that unusual effort in the advertising of events to students is well worth while in creating interest. Miss Marion Stark, Faculty Advisor of the Wellesley College *Mathematics Club*, contributes the following, which was mimeographed on a suitable folder:

"The Wellesley College Mathematics Club, a member of the Intercollegiate Mathematics Clubs of Greater Boston, cordially welcomes you to join in this year's activities.

"Our purpose is to take *Mathematics* off the bookshelves, dust it off, and have fun with it. *Mathematics* has been scaring us long enough; we feel it's time we put it in its place—and even play a few tricks on it. (Of course the joke may be on us, but that's all part of the fun.) The members of the club include everybody taking second and third grade *Mathematics*, and all juniors and seniors taking a first grade course.

"We'll keep you posted as to when and where the meetings will take place. We hope you'll come and help us make merry with deltas and epsilons. Remember—no dues, no duties, no drudgery; just fun, food, and entertainment, and (who knows?) you might even meet a handsome Tech student!"

### CLUB REPORTS, 1945-46

#### Delta-Y Club, D'Youville College

As an increment to the courses in mathematics, the *Delta-Y Club* took as a general topic for study, *History of Mathematics*. At regular monthly meetings there were prepared papers and informal discussions on:

*Primitive mathematics* by Marjorie Benzinger
*Mathematics takes wings* by Jane Deckop
*René Descartes* by Alice Staebell

*Apollonius of Perga* by Evelyn Kruse

*The problem of Apollonius* by Mary Loretta Hoar

*Leibniz* by Helen J. Hand

The activities for the year ended in May with a social meeting and spread. Guest speaker was Mrs. Louis D. Copley.

Officers for 1945–46: President, Mary Loretta Hoar; Secretary-Treasurer, Helen J. Rand.

**Kappa Mu Epsilon, Hofstra College**

The following papers were presented at regular meetings during the year:

*Mathematical problems of antiquity*, by Ruth Mayer

*Atomic energy*, by Wanda Scala

*Mathematics in chemistry*, by Professor J. George Lutz

*Problems of mechanics*, by Professor A. D. Capuro

*Mathematics and physics of the nerves*, by Professor Otto Schmidt, formerly of the University of Minnesota, with demonstration of the lie detector in action.

In April, a joint supper meeting was held at Adelphi College with the *Adelphi Mathematics Club*. At this meeting the following talks were given:

*Conic sections*, illustrated with light projections, lantern slides, and string models, by Professor H. von Baravalle of Adelphi.

*Kappa Mu Epsilon*, by Professor L. F. Ollmann

*The mathematical theory of spring fever*, by Professor E. R. Stabler.

At the annual initiation banquet the main address of the evening was:

*A comparison of American and English colleges*, by Mr. E. Trudeau Thomas, Hofstra director of admissions.

Other activities of the year included a Christmas square dance party, and a picnic in June at Belmont Lake State Park.

Officers for the year 1946–47 are: President, Edward Ryder; Vice-President and Secretary, Leo Malone; Treasurer, Professor J. George Lutz; Corresponding Secretary, Professor Albert Capuro; Faculty Sponsor, Professor L. F. Ollmann.

**Mathematics Club, Wellesley College**

The club assisted the Department of Mathematics in the presentation of two exhibitions. The first was a model exhibition for which the club had a room of mathematical toys, games and puzzles. The second, an exhibition of rare books in the College Library, at which the club took charge of an international table, containing books by mathematicians from as many countries as possible.

Students participated in two club programs by presenting talks on *Mathematics used in summer jobs* which they had held, and *Famous mathematicians of other countries*, as part of a United Nations Program at the College. At other meetings the members sang songs composed by students and faculty.

*Early days in Wellesley's mathematics* was the subject of a talk by Miss Helen Merrill, Emeritus Professor of Mathematics. Professor W. R. Ransom gave a lecture to the club at another meeting. Appreciation of the service rendered by Miss Jennie Copeland, retiring chairman of the mathematics depart-

ment was expressed in the form of a supper meeting at the close of the year at which time a gift was presented to her.

The officers for 1945–46 were: President, Ida Harrison; Vice-President, Eileen McGuire; Treasurer, Jean Marshall; Secretary, Patricia Peare; Junior Executive, Lois Wood; Faculty Advisor, Miss Marion Stark.

**Kappa Mu Epsilon, Texas Technological College**

Seven meetings were held during the year, including two formal initiation programs and a Christmas party. Eleven new members were initiated on November 23, 1945 and fourteen on May 3, 1946.

Among the several interesting papers presented during the year was one on: *A mathematical theory of aeronautics* by Assistant Professor Lida B. May.

The officers for the year 1946–47 are: President, James Fergerson; Vice-President, Helen Robin; Secretary, Betty Jones; Treasurer, Joel Simmons; Faculty Sponsor, Miss Virginia Bowman; Secretary Descartes, Miss Lida B. May.

**Kappa Mu Epsilon, University of New Mexico**

Monthly meetings were held by the *New Mexico Alpha* Chapter of *Kappa Mu Epsilon* with an initiation each term. The main event of the year was the visit of Dr. Fraenkel of the University of Jerusalem to the campus. At an open meeting sponsored by the chapter, Dr. Fraenkel spoke on *The recent controversies about the foundation of mathematics.*

Programs for the year included the following talks:

*Tesselations,* by Mrs. Eupha Buck Morris
*Photogrammetry,* by Mrs. Ruth Kendrick
*Our calendar,* by Dr. Martin Fleck
*The Central Valley project,* by Mr. Marvin May
*Functions of the polar planimeter,* by Dr. Arthur Rosenthal
*How to win on the horses,* by Dr. H. D. Larsen
*Magic squares,* by Mr. Frank Lane.

The officers for 1945–46 were: President, Bob Fox; Vice-President, William E. Dickerson; Secretary, Ted Hawley; Treasurer, Merle Mitchell; Faculty Sponsor, Miss E. Marie Hove.

The following officers were elected for 1946–47: President, Darrell Baker; Vice-President, Paul Barnhart; Secretary, Dorothy Lodter; Faculty Sponsor, Mrs. Eupha Buck Morris.

## NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.*

### THE SEVENTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The seventh annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, May 24, 1947. This Competition, made possible by the Trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The examination consists of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor George Mackey, Hunt Hall 12, Harvard University, Cambridge, Massachusetts, by a postcard request. All applications must be filed with Professor Mackey not later than May 1, 1947. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination, the books are to be sent by registered mail or by express to Professor Mackey, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are \$400, \$300, \$200 and \$100 in the order of their rank. In addition, there will be prizes of \$40, \$30, \$20 and \$10 awarded to the members of these teams according to the rank of the team; a prize of \$50 to each of the five highest contestants, and a prize of \$20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to the fifteen individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (at Radcliffe College, in the case of a woman) an annual \$1000 William Lowell Putnam Prize Scholarship to one of the first five contest-

ants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the six previous competitions and examination questions will be found in the MONTHLY for May 1938, 1939, 1940, 1941 and 1942, and for October, 1946.

### BROWN UNIVERSITY

The establishment of a graduate department of the History of Mathematics at Brown University has been announced. This department will be under the leadership of Professor Otto Neugebauer. Associated with him will be Dr. A. J. Sachs, who has been promoted to an assistant professorship. A course in Oriental History, by Professor Sachs, is planned for the academic year 1947–1948 and it is expected that a seminar on selected topics in ancient astronomy and mathematics will be offered in cooperation with the departments of mathematics, astronomy and classics.

### PRINCETON UNIVERSITY

A conference on the "Problems of Mathematics," one of a series of conferences in celebration of the Bicentennial of the founding of Princeton University, was held at Princeton on December 17, 18, and 19, 1946. The participants included, besides the local mathematicians, 76 from elsewhere, among whom 12 came from outside the United States. The conference was organized in the form of nine round tables on various subjects, with the discussion oriented as far as possible towards the formulation of problems for future work. The sessions were as follows: *Algebra*, chairman E. Artin, reporter G. P. Hochschild, discussion leaders G. Birkhoff, R. Brauer, N. Jacobson; *Algebraic Geometry*, chairman S. Lefschetz, reporter I. S. Cohen, discussion leaders W. V. D. Hodge, O. Zariski; *Differential Geometry*, chairman O. Veblen, reporter C. B. Allendoerfer, discussion leaders V. Hlavatý, T. Y. Thomas; *Mathematical Logic*, chairman A. Church, reporter J. C. C. McKinsey, discussion leader A. Tarski; *Topology*, chairman A. W. Tucker, reporter S. Eilenberg, discussion leaders H. Hopf, D. Montgomery, N. E. Steenrod, J. H. C. Whitehead; *New Fields*, chairman J. von Neumann, reporter V. Bargmann, discussion leaders G. C. Evans, F. D. Murnaghan, J. L. Synge, N. Wiener; *Mathematical Probability*, chairman S. S. Wilks, reporter J. W. Tukey, discussion leaders H. Cramér, J. L. Doob, W. Feller; *Analysis*, chairman S. Bochner, reporter R. P. Boas, discussion leaders L. V. Ahlfors, E. Hille, M. Riesz, A. Zygmund; *Analysis in the Large*, chairman M. Morse, reporter M. Shiffman, discussion leaders R. Courant, H. Hopf.

It is planned to issue shortly a descriptive pamphlet, and later a more complete monograph, covering the work of the conference. The monograph will contain an extensive list of the problems proposed.

### THE UNIVERSITY OF NORTH CAROLINA

The University of North Carolina completed in July 1946 the organization

of the all-University Institute of Statistics. Gertrude M. Cox is Director of the Institute, with W. G. Cochran and Harold Hotelling as Associate Directors in charge respectively of the Department of Experimental Statistics at Raleigh and the Department of Mathematical Statistics at Chapel Hill.

The Department of Mathematical Statistics is concerned primarily with research in the theory of statistics and with graduate training for the Ph.D. degree of students who expect to teach or develop statistical theory. Courses offered by regular members of the Department during the present year, 1946–47, include Statistical Inference, Mathematical Economics, and Least Squares and Time Series by Professor Hotelling; Correlation and Multivariate Analysis by Associate Professor P. L. Hsu; Introductory and Advanced Probability by Associate Professor Herbert Robbins; and Sequential Analysis and Rank Order Statistics by Instructor Edward Paulson. M. S. Bartlett of Cambridge University, Visiting Professor for the first half of the academic year 1946–47, offered courses on Stochastic Processes and on Estimation and Testing Hypotheses. Professor Harald Cramér of the University of Stockholm gave a series of three lectures in December 1946 on the Theory and Application of Stochastic Processes. Associate Professor W. G. Madow, now in Brazil, will join the Department at the end of 1947.

Students entering the Department of Mathematical Statistics must know mathematics through advanced calculus. Additional mathematics, including theory of functions and matrix algebra is required before the completion of their training. It is a fundamental principle of the Institute that students must acquire both a thorough knowledge of the mathematics of statistics and experience in a specific field of application. The latter is provided at Raleigh where the large staff deals with many branches of applied statistics, including sampling survey techniques, analysis of economic data, industrial statistics, design of experiments, and crop forecasting. Various research and consultation projects are in progress at Raleigh in which advanced students have an opportunity to work under critical supervision.

## PERSONAL ITEMS

Professor J. A. H. Duffie of Assumption College, University of Western Ontario has been appointed professor of chemistry at the University of Ottawa.

Professor J. P. Everett of Western Michigan College has retired with the title of professor emeritus.

J. E. Freund has been appointed to an assistant professorship at Alfred University, Alfred, New York.

Assistant Professor F. S. Harper of the University of Nebraska has been appointed to a professorship and head of the department of actuarial science at Drake University.

M. A. Hyman is employed as a mathematician at the Naval Ordnance Lab-

oratory in Washington, D. C. He is not at Rutgers University as was erroneously reported in the December issue of the MONTHLY.

Professor Oswald Veblen of the Institute for Advanced Study has been elected to membership in the Royal Danish Academy of Sciences and Letters and the Polish Academy of Sciences and Letters.

Professor Hermann Weyl of the Institute for Advanced Study has been elected a corresponding member of the Zürcher Naturforschende Gesellschaft and a foreign member of the Royal Swedish Academy of Sciences.

The following appointments to instructorships are announced:
Harvard University: Dr. R. C. James
Hofstra College: Frank Hawthorne
Queens College (Tutors in mathematics): Louise H. Ercolano, Florence E. Gerhardt, Aida Kalish, Mrs. Charlotte L. R. Knag, Bernice J. Lehrman
Northwestern University: Dr. A. O. Lindstrum
The University of Maine: J. A. Harmon, Leo Lapidus, Mrs. A. D. Mawhinney, Sutton Monro, Letitia Watson
The University of Illinois: Dr. Joseph Landin
Rider College: Emanuel Levine

D. E. Marrs of Pittsburg, California died May 28, 1946.

Dean Emeritus C. S. Slichter of the University of Wisconsin died October 4, 1946.

Associate Professor I. D. Stewart of Whitman College, Walla Walla, Washington, died July 21, 1946.

Professor G. B. Sweazey of Westminster College, Fulton, Missouri died August 10, 1946.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### THE THIRTIETH ANNUAL MEETING OF THE ASSOCIATION

The thirtieth annual meeting of the Mathematical Association of America was held at Swarthmore College, Swarthmore, Pennsylvania, on Thursday and Friday, December 26–27, 1946, in conjunction with meetings of the American Mathematical Society. About four hundred and ninety-eight persons attended the meetings, including the following two hundred and eighty members of the Association:

C. R. ADAMS, Brown University
LOUISE ADAMS, High Point College
V. W. ADKISSON, University of Arkansas
R. P. AGNEW, Cornell University
C. B. ALLENDOERFER, Haverford College
H. E. ARNOLD, Wesleyan University
EMIL ARTIN, Princeton University
SILVIO AURORA, Columbia University

FRANK AYRES, JR., Dickinson College
H. C. AYRES, U. S. Naval Academy
W. L. AYRES, Purdue University

R. P. BAILEY, U. S. Naval Academy
N. H. BALL, U. S. Naval Academy
D. H. BALLOU, Middlebury College
JOSHUA BARLAZ, Rutgers University
P. T. BATEMAN, Yale University
HELEN P. BEARD, Newcomb College
H. M. BEATTY, Ohio State University
A. A. BENNETT, Brown University
THEODORE BENNETT, Marietta College
A. L. BILLIG, High School, Allentown, Pa.
M. T. BIRD, Allegheny College
W. E. BLEICK, U. S. Naval Academy
R. P. BOAS, Brown University
T. A. BOTTS, University of Delaware
S. G. BOURNE, Johns Hopkins University
J. G. BOWKER, Middlebury College
H. W. BRINKMANN, Swarthmore College
FOSTER BROOKS, Kent State University
R. H. BRUCK, University of Wisconsin
C. T. BUMER, Bureau of Ordnance
R. S. BURINGTON, Bureau of Ordnance
HERBERT BUSEMANN, Smith College
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College

S. S. CAIRNS, Syracuse University
ELEANOR CALKINS, College of William and Mary
E. J. CAMP, Macalester College
H. H. CAMPAIGNE, U. S. Navy
P. A. CARIS, University of Pennsylvania
W. B. CARVER, Cornell University
REV. J. E. CASE, St. Louis University
F. L. CELAURO, Lehigh University
J. O. CHELLEVOLD, Lehigh University
W. F. CHENEY, Jr., University of Connecticut
RANDOLPH CHURCH, U. S. Naval Academy
R. V. CHURCHILL, University of Michigan
J. A. CLARKSON, University of Pennsylvania
J. W. CLAWSON, Ursinus College
A. B. COBLE, University of Illinois
NATHANIEL COBURN, University of Michigan
NANCY COLE, Connecticut College
J. A. COOLEY, University of Tennessee
T. F. COPE, Queens College
RICHARD COURANT, New York University
W. H. H. COWLES, Pratt Institute
G. F. CRAMER, U. S. Navy
H. B. CURRY, Pennsylvania State College
J. H. CURTISS, National Bureau of Standards
E. H. CUTLER, Lehigh University

JAMES ELMER DAVIS, Drexel Institute of Technology
L. J. DECK, Muhlenberg College
F. F. DECKER, Syracuse University
CARL DENBOW, U. S. Naval Academy
F. L. DENNIS, Ursinus College
A. H. DIAMOND, Oklahoma A. and M. College
L. L. DINES, Smith College
H. L. DORWART, Washington and Jefferson College
ARNOLD DRESDEN, Swarthmore College
D. M. DRIBIN, War Department
ROY DUBISCH, Syracuse University
NELSON DUNFORD, Yale University
JANET C. DURAND, Vassar College

J. E. EATON, Naval Research Laboratory
E. D. EAVES, University of Tennessee
NAT EDMONSON, Johns Hopkins University
SAMUEL EILENBERG, Indiana University

W. H. FAGERSTROM, College of the City of New York
WILLIAM FELLER, Cornell University
F. A. FICKEN, University of Tennessee
N. J. FINE, Washington, D. C.
DANIEL FINKEL, Amherst College
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, University of Georgia
R. M. FOSTER, Polytechnic Institute of Brooklyn
J. S. FRAME, Michigan State College

H. M. GEHMAN, University of Buffalo
B. H. GERE, U. S. Naval Academy
REV. F. J. GERST, Loyola University
J. H. GIESE, Ballistic Research Laboratories
B. P. GILL, College of the City of New York
J. W. GIVENS, Illinois Institute of Technology
A. M. GLEASON, Harvard University
MARY GOINS, Marshall College
MICHAEL GOLDBERG, Bureau of Ordnance
H. H. GOLDSTINE, Institute for Advanced Study
R. A. GOOD, University of Maryland
A. W. GOODMAN, Columbia University
S. H. GOULD, University of Toronto
A. A. GRAU, University of Kentucky
GEORGE GROSSMAN, DeWitt Clinton High School, New York
C. C. GROVE, University of Pennsylvania

V. H. HAAG, Hershey Junior College
THEODORE HAILPERIN, Lehigh University
R. W. HAMMING, Bell Telephone Laboratories
E. S. HAMMOND, Bowdoin College
C. E. HEILMAN, Syracuse University
M. H. HEINS, Brown University
G. C. HELME, Pratt Institute
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
R. P. HOBBS, Rinehart and Company, Inc.
D. L. HOLL, Iowa State College
T. R. HOLLCROFT, Wells College
D. B. HOUGHTON, Franklin Institute
E. MARIE HOVE, Hofstra College
M. GWENETH HUMPHREYS, Newcomb College
W. A. HURWITZ, Cornell University
H. D. HUSKEY, University of Oklahoma
W. R. HUTCHERSON, Berea College

J. E. IKENBERRY, Madison College
M. H. INGRAHAM, University of Wisconsin

E. D. JENKINS, Eastern Kentucky State Teachers College
FRITZ JOHN, New York University
ROBERTA F. JOHNSON, Wilson College
R. P. JOHNSON, Carnegie Institute of Technology
W. L. JOHNSON, Mississippi Southern College
B. W. JONES, Cornell University

MARK KAC, Cornell University
IRVING KAPLANSKY, University of Chicago
J. R. KLINE, University of Pennsylvania
H. L. KRALL, Pennsylvania State College
R. R. KUEBLER, JR., Dickinson College
H. W. KUHN, Ohio State University

MARY E. LADUE, Barnard College
GILLIE A. LAREW, Randolph-Macon Woman's College
J. A. LARRIVEE, University of Vermont
C. G. LATIMER, University of Kentucky
V. V. LATSHAW, Lehigh University
SOLOMON LEFSCHETZ, Princeton University
JOSEPH LEHNER, Hydrocarbon Research, New York
MARGUERITE LEHR, Bryn Mawr College
WALTER LEIGHTON, Washington University
CAROLINE A. LESTER, New York State College for Teachers
HARRY LEVY, University of Illinois
ANNE L. LEWIS, Woman's College, University of North Carolina
S. B. LITTAUER, Newark College of Engineering
A. T. LONSETH, Northwestern University

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MAC LANE, Harvard University
H. F. MAC NEISH, Brooklyn College
INGO MADDAUS, JR., Naval Research Laboratory
J. D. MANCILL, University of Alabama
F. L. MANNING, Ursinus College
R. H. MARQUIS, Ohio University
M. H. MARTIN, University of Maryland
W. T. MARTIN, Massachusetts Institute of Technology
E. D. MCCARTHY, University of Detroit
SOPHIA L. MCDONALD, University of California
D. L. MCDONOUGH, High School, Philadelphia
S. S. MCNEARY, Drexel Institute
MARY E. MEADE, University of Maryland
A. E. MEDER, JR., Rutgers University
EMANUEL MEHR, Johns Hopkins University
A. S. MERRILL, Montana State University
H. L. MEYER, JR., University of Chicago
A. N. MILGRAM, Institute for Advanced Study
D. D. MILLER, University of Tennessee
SOLOMON MITCHELL, Grinnell College
E. E. MOISE, University of Texas
MARTIN MOLIVER, Martin College
DEANE MONTGOMERY, Yale University
A. H. MOORE, Pratt Institute
LILLIAN MOORE, Far Rockaway High School, New York
RICHARD MORRIS, Rutgers University
IRMA R. MOSES, Temple University
E. D. MOUZON, JR., Southern Methodist University
C. W. MUNSHOWER, Colgate University
W. R. MURRAY, Franklin and Marshall College

C. A. NELSON, New Jersey College for Women
A. B. NEALE, Watson Laboratories
J. J. NEWMAN, Harvard University
C. V. NEWSOM, Oberlin College
P. B. NORMAN, Pratt Institute

C. O. OAKLEY, Haverford College
L. F. OLLMANN, Hofstra College
ISAAC OPATOWSKI, University of Michigan
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, Pennsylvania State College
J. C. OXTOBY, Bryn Mawr College

S. T. PARKER, University of Cincinnati
G. W. PATTERSON, University of Pennsylvania
E. K. PAXTON, Washington and Lee University
A. M. PEISER, Rutgers University
C. R. PERISHO, McCook Junior College
B. J. PETTIS, Yale University
H. R. PHALEN, College of William and Mary
C. R. PHELPS, Rutgers University
A. E. PITCHER, Lehigh University
G. B. PRICE, University of Kansas
A. L. PUTNAM, University of Chicago

J. F. RANDOLPH, Oberlin College
S. E. RASOR, Ohio State University
G. E. RAYNOR, Lehigh University
C. N. REYNOLDS, West Virginia University
R. G. D. RICHARDSON, Brown University
P. R. RIDER, Washington University
R. F. RINEHART, Case School of Applied Science
R. M. ROBINSON, Princeton University
R. E. ROOT, U. S. Naval Academy
J. B. ROSENBACH, Carnegie Institute of Technology
LOUISE J. ROSENBAUM, Reed College
R. A. ROSENBAUM, Reed College
P. C. ROSENBLOOM, Syracuse University
J. B. ROSSER, Cornell University
L. J. ROUSE, University of Michigan

RAPHAEL SALEM, Massachusetts Institute of Technology
HANS SAMELSON, University of Michigan
R. G. SANGER, Kansas State College
ARTHUR SARD, Queens College
I. J. SCHOENBERG, University of Pennsylvania
WLADIMIR SEIDEL, University of Rochester
W. E. SEWELL, War Department, Army Education Branch
W. P. SHARP, JR., Newark College of Engineering
L. W. SHERIDAN, College of St. Thomas
D. T. SIGLEY, Johns Hopkins University
L. L. SMAIL, Lehigh University
C. V. L. SMITH, Raytheon Manufacturing Co.
W. M. SMITH, Lafayette College
ERNST SNAPPER, University of Southern California
ANDREW SOBCZYK, Watson Laboratories
F. W. SOHON, Georgetown University
P. I. SPEICHER, Albright College
E. R. STABLER, Hofstra College
R. C. STALEY, University of North Dakota
E. P. STARKE, Rutgers University
F. H. STEEN, Allegheny College
H. W. STEINHAUS, Equitable Life Assurance Society, New York
R. C. STEPHENS, Knox College
ELLEN C. STOKES, New York State College for Teachers
M. H. STONE, University of Chicago
HELEN F. STORY, Pennsylvania State College
IRVING SUSSMAN, University of Dayton
J. L. SYNGE, Carnegie Institute of Technology
OTTO SZÁSZ, University of Cincinnati
GABOR SZEGÖ, Stanford University

FEODOR THEILHEIMER, Trinity College
R. M. THRALL, University of Michigan
LEONARD TORNHEIM, University of Michigan
A. W. TUCKER, Princeton University
J. W. TUKEY, Princeton University

G. L. WALKER, Temple University
R. J. WALKER, Cornell University
A. D. WALLACE, University of Pennsylvania
J. L. WALSH, Harvard University
JEAN B. WALTON, University of Pennsylvania
W. R. WASOW, Swarthmore College
C. W. WATKEYS, University of Rochester
G. C. WEBBER, University of Delaware
M. S. WEBSTER, Purdue University
J. V. WEHAUSEN, David Taylor Model Basin
MARIE J. WEISS, Newcomb College
E. T. WELMERS, Bell Aircraft Corporation
ANNA PELL WHEELER, Bryn Mawr College
J. H. WHITE, U. S. Naval Academy
A. L. WHITEMAN, Navy Department
E. A. WHITMAN, Carnegie Institute of Technology
P. M. WHITMAN, Tufts College
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, Texas Technological College
S. S. WILKS, Princeton University
K. P. WILLIAMS, Indiana University
MARY E. WILLIAMS, Skidmore College
W. L. WILLIAMS, University of South Carolina
R. H. WILSON, JR., Temple University
CLEMENT WINSTON, Department of Commerce
H. A. WOOD, Chance Vought Aircraft
FRANCES M. WRIGHT, Triple Cities College

BERTRAM YOOD, Yale University
J. W. T. YOUNGS, Indiana University

Rooms for members of the organizations and their families were provided in the college dormitories, and meals were served in the college dining room in Parrish Hall. Tea was served on Thursday and Friday afternoons in the parlor of Bond Memorial Hall by the ladies of the Swarthmore College Department of Mathematics. An excellent program of chamber music was presented Friday evening in the auditorium of the Clothier Memorial Building by Phyllis Groff, Leona Gold, Helen Arens, Betty Benthin, Robert N. Hilkert, Ralph H. Fox, and Arnold Dresden.

A dinner for the two organizations was held at 6:30 Friday evening in the college dining room. Professor C. O. Oakley acted as toast-master and introduced President John W. Nason of Swarthmore College who welcomed the visiting organizations in a few happily chosen words. He was followed by Professor W. T. Martin who spoke on the general subject of the post-war graduate student in mathematics. Resolutions were presented by Professor C. G. Latimer expressing the thanks of the visiting organizations to President Nason and the Board of Managers of Swarthmore College, to the members of the Swarthmore Mathematics Department and their wives, and to the various members of the College staff who helped with the friendly and efficient arrangements which made the meeting so enjoyable. The resolutions were adopted by a rising vote.

The American Mathematical Society held its sessions between noon Friday and noon Sunday. The Josiah Willard Gibbs lecture was delivered by Professor Subrahmanyan Chandrasekhar of the University of Chicago on Thursday evening, his subject being "The transfer of radiation in stellar atmospheres"; and on Saturday at 2:00 P.M. Professor A. P. Morse of the University of California gave an address, by invitation, on "Derivatives and their integrals."

The Mathematical Association held its sessions on Thursday afternoon and Friday morning, the program having been arranged by a committee consisting of I. J. Schoenberg, chairman, D. W. Hall, and Ivan Niven. The following papers were presented:

### First Session of the Association

"Parallelism, solid angle and curvature," by Professor C. B. Allendoerfer, Haverford College.

"Mathematics in the Army Education Program," by Colonel W. E. Sewell, Army Education Branch.

"The theory of braids," by Professor Emil Artin, Princeton University.

### Second Session of the Association

"On the multiplication of series," by Professor Antoni Zygmund, University of Pennsylvania.

"Random walk and the theory of Brownian motion," by Professor Mark Kac, Cornell University.

Professor M. S. Knebelman of the State College of Washington was to have presented a paper on "The teaching of college mathematics today" at the Friday morning session, but he was unfortunately unable to attend the meeting.

## Meeting of the Board of Governors

The Board met on Thursday at 7:30 P.M. in the News Bureau room in Parrish Hall, seventeen members of the Board being present.

Among the more important items of business transacted were the following.

Ninety-two persons as listed below, were elected to membership on applications duly certified:

SISTER M. ANITA, A.M. (Seton Hall Coll.) Instr., Caldwell Coll., Caldwell, N. J.

D. L. ARENSON. Student, Illinois Inst. of Tech., Chicago, Ill.

R. P. BAILEY, Ph.D. (Pennsylvania) Asso. Prof., U. S. Naval Acad., Annapolis, Md.

DOROTHEA M. BAUMANN, B.S. (Marquette) Teacher, Rufus King High School, Milwaukee, Wis.

MAX BEBERMAN, B.S. (C.C.N.Y.) Teacher, Math. and Sci., Nome Territorial High School, Nome, Alaska

PAUL BELGODÈRE, Agrégé de Math. (Paris, E.N.S.) Secrétaire Général, Intermédiaire des Recherches Mathematiques, 55 rue de Varenne, Paris 7, France

J. H. BELL, Ph.D. (Wisconsin) Asst. Prof., Michigan State Coll., East Lansing, Mich.

W. D. BERG, Ph.D. (Iowa) Visiting Asst. Prof., Kenyon Coll., Gambier, Ohio

ARTHUR BERNHART, Ph.D. (Michigan) Asst. Prof., Univ. of Oklahoma, Norman, Okla.

R. P. BRADY, M.S. (Chicago) Hunter Coll., New York, N. Y.

W. H. BUNCH, A.M. (Oregon) Head of Dept., High School, Adrian, Ore.

J. J. BURCKHARDT. Prof., Univ. of Zurich, Zurich, Switzerland

G. P. BURNS, M.S. (W. Va. Univ.) Asso. Prof., Marshall Coll., Huntington, W. Va.

GRACE M. CANADA, A.M. (Columbia) Asst. Prof., East Central State Coll., Ada, Okla.

H. N. CARTER, B.S. (Northeastern Okla. St. C.) Asst. Prof., Univ. of Tulsa, Tulsa 4, Okla.

C. L. CLARK, Ph.D. (Virginia) Asso. Prof., Oregon State Coll., Corvallis, Ore.

W. W. S. CLAYTOR, Ph.D. (Pennsylvania) Prof., Hampton Inst., Hampton, Va.

NATHANIEL COBURN, Ph.D. (Mass. Inst. of Tech.) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

BROTHER CYPRIAN LUKE (Roney), M.S. (Catholic Univ.) Head of Dept., Sacred Heart Coll., Las Vegas, N. M.

CONSTANCE H. DAVIS (Mrs. Leslie). 238 Audley St., South Orange, N. J.

R. B. DERFLINGER. Student; Asst., Geneva Coll., Beaver Falls, Pa.

ALICE B. DICKINSON, A.M. (Columbia) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich.

MARY P. DOLCIANA, A.M. (Cornell) Fellow, Cornell Univ., Ithaca, N. Y.

ROY DUBISCH, Ph.D. (Chicago) Asst. Prof., Syracuse Univ., Triple Cities Coll., Endicott, N. Y.

R. W. ENGEL. Hotel Kahler, Rochester, Minn.

D. M. FRIEDLEN. Student, Illinois Inst. of Tech., Chicago, Ill.

C. M. FULTON, Ph.D. (Tech. Hochschule, Munich) Lecturer, Univ. of California at Los Angeles, Los Angeles, Calif.

S. H. GOULD, Ph.D. (Yale) Asso. Prof., Victoria Coll., Toronto, Ont., Canada

E. C. GRAS, A.M. (Harvard) Instr., U. S. Naval Acad., Annapolis, Md.

W. T. GUY, JR., B.S. (A. and M. Coll. of Texas) Instr., Univ. of Texas, Austin, Tex.

W. R. HANSON, M.S. (Chicago) Instr., San Francisco Jr. Coll., San Francisco, Calif.

G. C. HELME, M.S. (Washington Univ.) Instr., Pratt Inst., Brooklyn, N. Y.

R. B. HERRERA, A.M. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif.

V. A. HOERSCH, Ph.D. (Iowa) Asst. Prof., Univ. of Illinois, Urbana, Ill.

CARL HOLTOM, Ph.D. (Chicago) Asst. Prof., Purdue Univ., Lafayette, Ind.

LEROY HOLUBAR, B.S.E.E. (Colorado) Instr., Engg. Math., Univ. of Colorado, Boulder, Colo.

R. E. HORTON, A.B. (U.C.L.A.) Asso., Univ. of California at Los Angeles, Los Angeles, Calif.

H. B. HOYLE, JR., A.M. (North Carolina) Prof., Head of Dept., Queens Coll., Charlotte, N. C.

P. L. HSU, D.Sc. (London) Asso. Prof., Univ. of North Carolina, Chapel Hill, N.C.

ELAINE HUNDERTMARK, A.M. (Illinois) Instr., Univ. of Arkansas, Fayetteville, Ark.
H. V. HUNEKE, A.M. (Oklahoma) Instr., Northwestern State Coll., Alva, Okla.
MRS. FAY H. JOHNSON, A.B. (Howard Payne Coll.) Instr., Howard Payne Coll., Brownwood, Tex.
H. F. S. JONAH, Ph.D. (Purdue) Asso. Prof., Purdue Univ., West Lafayette, Ind.
H. S. KIEVAL, Ph.D. (Cincinnati) Instr., Brooklyn Coll., Brooklyn, N. Y.
W. J. KLIMCZAK, A.M. (Yale) Instr., Yale Univ., New Haven, Conn.
J. T. KRATTIGER, A.M. (Southern Methodist) Instr., Univ. of Oklahoma, Norman, Okla.
R. R. KUEBLER, JR., A.B. (Dickinson Coll.) Instr., Dickinson Coll., Carlisle, Pa.
J. P. LASALLE, Ph.D. (Calif. Inst. of Tech.) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.
CHARLES LOEWNER, Ph.D. (Prague) Prof., Syracuse Univ., Syracuse, N. Y.
G. P. LOWEKE, Ph.D. (Berlin) Asst. Prof., Engg. Mech., Wayne Univ., Detroit, Mich.
R. V. LYNCH, B.S. (Harvard) Instr., Phillips Exeter Acad., Exeter, N. H.
R. A. LYTLE, A.M. (Virginia) Adj. Prof., Univ. of South Carolina, Columbia, S. C.
A. L. MAYERSON, B.S. (Michigan) Grad. Student; Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich.
ELNA B. MCBRIDE (Mrs. J. S.), M.S. (Tennessee) Asst. Prof., Memphis State Coll., Memphis, Tenn.
A. W. MCGAUGHEY, Ph.D. (Cincinnati) Prof., Chm. of Dept., Westminster Coll., New Wilmington, Pa.
W. K. MCNABB, A.M. (Michigan) Instr., The Hockaday School, Dallas 6, Tex.
P. E. MEADOWS, A.B. (Carleton Coll.) Asst. Prof., Washington and Lee Univ., Lexington, Va.
SERGE MINOIS, Agrégation des Sci. Math. (Paris) Asst. Prof., Lyceé Lakanol e Sceaux. 7 rue Candelot, Bourg-la-Reine (Seine), France
A. H. MOORE, B.M.E. (Pratt Inst.) Instr., Pratt Inst., Brooklyn, N. Y.
DOROTHY J. MORROW, M.S. (Washington) Asst. Prof., Statistics, George Washington Univ., Washington, D. C.
D. J. MYATT, B.M.E. (Univ. of Louisville) Asst. Engr., James Clark, Jr. Electric Co., Louisville, Ky.
(MISS) ANDREWA R. NOBLE, Ph.D. (California) Instr., Montana State Univ., Missoula, Mont.
A. M. OSTROWSKI, Ph.D. (Göttingen) Prof., Univ. of Basle, Basle, Switzerland
A. O. QUALLEY, A.M. (Iowa) Instr., Lehigh Univ., Bethlehem, Pa.
MARY K. RAPP, A.M. (Illinois) Instr., Illinois Inst. of Tech., Chicago, Ill.
R. W. RECTOR, A.M. (Stanford) Instr., U. S. Naval Acad., Annapolis, Md.
LOUIS ROSS, A.M. in Educ. (Univ. of Akron) Instr., Univ. of Akron, Akron, Ohio
C. E. RUSCH, A.M. (Wisconsin) Prof., Mission House Coll., R.R. 3, Plymouth, Wis.
R. F. SCHWETYE. Chem. Engr., Mexican Zinc Co., Rosita, Coahuila, Mexico
W. H. SIMONS, M.A. (Univ. of B.C.) Lecturer, Univ. of British Columbia, Vancouver, B. C., Canada
DOROTHY E. SMITH, B.S. in Educ. (N. Ill. St. T.C.) Teacher, Community High School, Erie, Ill.
MALCOLM SMITH. Student, Illinois Inst. of Tech., Chicago, Ill.
L. C. SNIVELY, M.S.E.E. (Colorado) Asst. Prof., Engg. Math., Univ. of Colorado, Boulder, Colo.
O. S. SPEARS, M.S. (Ala. Poly. Inst.) Instr., Univ. of Oklahoma, Norman, Okla.
J. C. STEWART, Ph.D. (Illinois) Asst. Prof., Lawrence Coll., Appleton, Wis.
IRVING SUSSMAN, B.S. (Columbia) Asst. Prof., Univ. of Dayton, Dayton, Ohio
MARGARET O. TAYLOR, M.S. (Pittsburgh) Mathematician, Gulf Research and Development Co., Pittsburgh, Pa.
W. B. TEMPLE, A.M. (Louisiana State) Asst. Prof., A. and M. Coll. of Texas, College Station, Tex. *On leave. Instr., Univ. of Texas, Austin, Tex.*
FEODOR THEILHEIMER, Ph.D. (Berlin) Asst. Prof., Trinity Coll., Hartford, Conn.
LEONARD TORNHEIM, Ph.D. (Chicago) Instr., Univ. of Michigan, Ann Arbor, Mich.
G. R. VICK, A.M. (Sam Houston, St. T.C.) Asst. Prof., Sam Houston State Teachers Coll., Huntsville, Tex.
B. M. WALL, A.M. (Sam Houston, St. T.C.) Asso. Prof., Sam Houston State Teachers Coll., Huntsville, Tex.
MRS. LILLIE C. WALTERS, A.M. (Colorado) Instr., Engg. Math., Univ. of Colorado, Boulder, Colo.

W. R. WASOW, Ph.D. (New York Univ.) Asst. Prof., Swarthmore Coll., Swarthmore, Pa.

G. C. WEBBER, Ph.D. (Chicago) Asso. Prof., Univ. of Delaware, Newark, Del.

HELEN R. WHITE, A.B. (North Carolina) Statistician, Bureau of Census, Dept. of Commerce; Grad. Student, George Washington Univ., Washington, D. C.

A. L. WHITEMAN, Ph.D. (Pennsylvania) Mathematician, Office of Chief of Naval Operations, Washington, D. C.

MRS. BERYL W. WILLIAMS, A.M. (Maine) Instr., Agric. and Tech. Coll., Greensboro, N. C.

FLORENCE A. WIRSCHING, A.B. (N.J.St.T.C., Montclair) Instr., Purdue Univ., West Lafayette, Ind.

L. G. WORTHINGTON, A.M. (N. Tex. State T.C.) Instr., Univ. of Texas, Austin, Tex.

C. B. WRIGHT, Ph.D. (Pittsburgh) Prof., East Texas State Teachers Coll., Commerce, Tex.

H. M. ZERBE, M.S. (Pennsylvania State) Instr., Pennsylvania State Coll., Hazleton Undergraduate Center, Hazleton, Pa.

The Secretary reported the deaths of the following members of the Association:

JOSEPH ALLEN, Associate Professor of mathematics, Retired, College of the City of New York. (March 4, 1946)

DANIEL ARANY, Professor emeritus of mathematics, Royal School for Industry, Budapest.

HARRY BATEMAN, Professor of mathematics, California Institute of Technology. (January 21, 1946)

L. M. BERKELEY, Lawyer, New York, N. Y.

H. F. BLICHFELDT, Professor emeritus of mathematics, Stanford University. (November 16, 1945)

A. A. BLUMBERG, Assistant Professor of mathematics, A. and M. College of Texas. (October 21, 1945)

A. L. CANDY, Professor emeritus of mathematics, University of Nebraska. (July 18, 1945)

F. H. CLUTZ, Professor emeritus of civil engineering, Gettysburg College. (December 30, 1945)

A. R. CONGDON, Professor emeritus of secondary education, University of Nebraska. (November 11, 1945)

A. R. CRATHORNE, Professor emeritus of mathematics, University of Illinois. (March 7, 1946)

FLETCHER DURELL, Head of department emeritus, Lawrenceville School, New Jersey. (March 25, 1946)

C. W. EMMONS, Professor of mathematics, Simpson College. (December 29, 1945)

FEDERIGO ENRIQUES, Professor of mathematics, Retired, University of Rome. (June 14, 1946)

A. M. FREEMAN, Director of mathematical laboratory, Boston Fiduciary and Research Associates. (May 20, 1946)

HANS FRIED, Lecturer in mathematics, Swarthmore College. (December 23, 1945)

C. F. GUMMER, Professor of mathematics, Queen's University, Kingston, Canada. (January 21, 1946)

LAURENCE HADLEY, Professor of mathematics, Purdue University. (March 21, 1946)

DUNHAM JACKSON, Professor emeritus of mathematics, University of Minnesota. (November 6, 1946)

D. E. MARRS, Teacher, Senior High School, Pittsburg, California. (May 28, 1946)

A. S. MCMASTER, Assistant Professor of engineering mathematics, University of Colorado. (May 24, 1946)

T. A. PIERCE, Professor of mathematics, University of Nebraska. (August 18, 1945)

R. G. M. SABEL, Teacher, Bristol (Connecticut) High School. (February 24, 1946)

I. D. STEWART, Associate Professor of mathematics, Whitman College. (July 21, 1946)

G. B. SWEAZEY, Dean, Westminster College, Fulton, Missouri. (August 10, 1946)

ALTHEOD TREMBLAY, Professor of mathematics, Laval University, Quebec, Canada. (March 1, 1946)

E. E. WHITFORD, Professor emeritus of mathematics, College of the City of New York. (May 3, 1946)

The Board voted to accept the invitation from the University of Wisconsin to hold the Summer Meeting of 1948 at Madison, Wisconsin, and the invitation from the University of Colorado to hold the Summer Meeting of 1949 at Boulder, Colorado. Also it was voted to hold the Annual Meeting of 1948 in New York City in conjunction with the meetings of the American Association for the Advancement of Science.

On nominations by the Executive Committee the Board elected C. B. Allendoerfer as Second Vice-President for the two years 1947–48; also G. T. Whyburn as Representative on the National Research Council for the term July 1947 to July 1950, and W. T. Martin as Representative on the Council of the American Association for the Advancement of Science for the two years 1947–48.

The Board approved of the appointment by President MacDuffee of a Nominating Committee for 1947 consisting of W. F. Cheney, Jr., W. M. Whyburn, and L. L. Dines, chairman.

The Board voted to resume the making of exchange arrangements between the MONTHLY and other mathematical periodicals, a practice which had been discontinued by action of the Board on September 12, 1943. Also the Board approved of the plan to place H. M. Gehman in charge of all such exchange arrangements, including the matter of the disposal of all periodicals received by exchange.

The Board appointed C. V. Newsom, R. M. Foster, J. F. Randolph, and W. B. Carver as a committee with power to make a new contract for the printing of the MONTHLY.

On the nomination of the Editor-in-Chief, C. V. Newsom, the Board elected the following Associate Editors of the MONTHLY for the year 1947:

| | | |
|---|---|---|
| C. B. ALLENDOERFER | H. P. EVANS | W. T. MARTIN |
| E. F. BECKENBACH | HOWARD EVES | L. F. OLLMANN |
| L. M. BLUMENTHAL | B. F. FINKEL | R. F. RINEHART |
| N. B. CONKWRIGHT | B. W. JONES | E. P. STARKE |
| H. S. M. COXETER | N. H. MCCOY | E. P. VANCE |

## ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday at 9:30 A.M., President MacDuffee presiding.

The following results of the balloting for officers was announced:

L. R. Ford, Illinois Institute of Technology, was elected President for the term 1947–48.

D. H. Lehmer, University of California, and W. T. Martin, Massachusetts Institute of Technology, were elected Governors at Large for the term 1947–49.

The Secretary also announced the election by the Board of C. B. Allendoerfer as Second Vice-President for the term 1947–48.

Upon recommendation by the Board of Governors the following changes in the By-laws of the Association were adopted:

ARTICLE III, Section 8 (a). The words "by this membership in constituencies (hereinafter called 'Regions') established by the Board" shall be replaced by the words "by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist."

ARTICLE III, Section 8 (c). In the first sentence the word "Region" is to be replaced by the word "Section," the word "biennially" by the word "triennially," and the word "two" by the word "three." The second sentence is to be replaced by the sentence, "For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section."

ARTICLE III, Section 8 (g). At the end of the first sentence the phrase "by the Board" shall be replaced by the words "by the President with the approval of the Board." In the second sentence the words "two months" shall be changed to "six months." In the third sentence the word "Board" in both places where it occurs is to be replaced by the words "Nominating Committee."

ARTICLE IV, Section 2. The whole first sentence is to be replaced by the sentence, "The Board shall hold a meeting each year immediately preceding the annual meeting of the Association."

ARTICLE V, Section 2. At the end of the sentence there shall be added the phrase, "except as the Board may provide."

W. B. CARVER, *Secretary-Treasurer*

## CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS, Wichita, April 19, 1947
KENTUCKY
LOUISIANA-MISSISSIPPI, Hattiesburg, Mississippi, April 25–26, 1947
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, April 19, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST, Vancouver, British Columbia, April 10–11, 1947
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April 18–19, 1947
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

COMPOSED, PRINTED AND BOUND BY GEORGE BANTA PUBLISHING COMPANY

# THE AMERICAN
# MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 4


CONTENTS




APRIL 1947

# The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)



EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association and advertising in the Monthly, should be sent to the Secretary-Treasurer, W. B. Carver, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the first of the month in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# MATHEMATICS IN THE ARMY EDUCATION PROGRAM

COLONEL W. E. SEWELL, Washington, D. C.*

**1. Introduction.** The purpose of this paper is to present the facts with regard to mathematics in the Army Education Program and some of the implications which can be derived from these facts. In particular, I shall consider the question: How much demand for mathematics can be expected from the veteran?

Mathematics has been one of the most popular subjects in the curriculum of the Army Education Program. In order that the reader may understand the full significance of this popularity, some of the pertinent points of the Army Education Program should be explained. This program was begun prior to our entry into World War II, and had as its purpose the furnishing to the serviceman of educational opportunities to fit his needs or desires, or both. In particular, the subjects offered were selected (1) to enhance the value of the individual to the Army, (2) to prevent a complete interruption of the serviceman's education, (3) to satisfy intellectual interests, and (4) to prepare the serviceman for his return to civilian life. The method used at first was the furnishing of correspondence courses and lesson service to individuals upon their request. The program was gradually enlarged to include self-study courses, off-duty and on-duty classes, and university extension courses from some seventy-five colleges and universities. The curriculum begins with literacy training, and includes, in addition to the academic courses, a wide variety of vocational subjects. The United States Armed Forces Institute at Madison, Wisconsin, is the principal installation in the implementation of this program. The Institute is in reality a large correspondence school, a testing agency, and a supply depot, and is operated jointly by the War and Navy Departments.

There is one feature of this program which should be emphasized, namely, that participation has always been voluntary on the part of the soldier. It is this fact which makes the popularity which courses in mathematics enjoyed especially startling. When a man becomes a member of the student body of the United States Armed Forces Institute, he does so simply because he wants to learn something that he does not know, and for this reason alone. There is no social prestige, no football team, no paternal alma mater involved in his decision. Of course, men are encouraged to take advantage of the opportunities offered and are given counsel and guidance in the selection of subjects. About ten percent of the personnel participate in the program, and this is encouraging because the surroundings, environment, and facilities are not particularly conducive to study. There are no well-lighted comfortable classrooms, no warm inviting study halls with instructional help and ample library facilities. During hostilities the conditions were particularly difficult, and official duties left time for little else. During the period of demobilization there was much more time available, but the facilities for study were largely improvised and the environment far from ideal.

* Chief, Education Branch, Information and Education Division, War Department.

**2. Courses offered in the program.** As stated above, the curriculum runs the gamut of mathematics. It begins with literacy training in which the most elementary principles of arithmetic are taught. In the high school category such books as *Basic Mathematics* by Betz, *Algebra* by Wells and Hart, and *Trigonometry* by Curtiss and Moulton are the texts. Hart's *College Algebra*, three standard trigonometries, Love's *Analytic Geometry*, Granville, Smith and Longley's *Calculus*, and Murray's *Differential Equations* are examples of the college texts. These books show what type of subject matter the servicemen have been choosing; certainly they are texts designed for learning and not for amusement. The above courses are offered by the United States Armed Forces Institute itself.

Further courses in mathematics are available to the serviceman through the extension divisions of the universities and colleges under contract with the Institute. However, in this paper only Institute courses are considered.

It should be mentioned here that most of the books used in the Army Education Program were bound in paper for reasons of economy in purchase price and shipping space. The majority of the texts are identical in content with the high school or college edition. Some have been revised by the insertion of additional explanation, exercises, and reviews, so as to enable the student to understand and absorb the subject matter without the aid of an instructor. The purpose is a "self-teaching text." It was the Army's plan to provide self-teaching texts for all subjects amenable to such treatment; however, the project was curtailed after V-J Day. Consequently, of the total number of texts in the curriculum of the United States Armed Forces Institute, only a few are "self-teaching."

**3. Popularity of courses.** In considering the popularity of books and courses, the wide variety of offerings is important. The curriculum included more than 400 courses when these figures were compiled. There were courses in everything from "Learning to Read" to "Shakespeare," from "Pork Production" to "Art Through the Ages."

Of the twenty most popular books, seven are mathematics, all high school texts, and four others are bookkeeping and accounting. The most popular book of all is *Auto-Mechanics*; second is *Bookkeeping and Accounting;* third is *A First Course in Algebra, Part I;* and fourth is *A First Course in Algebra, Part II.* The only college level books among the twenty most popular are *Principles of Accounting* and *Modern Electric and Gas Refrigeration.* All of the books except the last two are "self-teaching."

Of the twenty most popular high school correspondence courses, five are mathematics. The most popular course is Beginning Algebra; second is Review Arithmetic; and third, Bookkeeping. Trigonometry is ninth on the list. Twenty percent of all those enrolled in high school correspondence courses are taking mathematics. The second most popular field is bookkeeping, which has fifteen percent of the enrollees. Of those studying mathematics, six percent are taking arithmatic, five percent algebra, and four percent plane geometry.

Of the twenty most popular college correspondence courses, four are mathematics. The most popular course is College Algebra and Trigonometry; second

Introduction to Accounting; and third, English Composition. Differential Calculus ranks tenth; Differential Equations is twenty-eighth; and Integral Calculus, thirty-sixth. Twenty-four percent of all those enrolled in college correspondence courses are taking mathematics. The second most popular field is accounting, which has twelve percent of the enrollees. The mathematics course with the highest percentage enrollment is College Algebra and Trigonometry; nine percent of all the enrollees are taking this subject. Differential Calculus enjoys a surprising demand; five percent of those enrolled are taking this course.

A poll taken in July of this year shows that eighteen percent of the men in the Army have signed up for USAFI correspondence courses. Over 75,000 men have enrolled for mathematics courses at the Institute. Of the courses currently being distributed, about nine percent are mathematics courses. Sixteen percent of the active enrollees, as of July 1, 1946, were taking mathematics. Of total completions, as of July 1, 1946, approximately eight percent were in mathematics courses.

**4. Reasons for the popularity of mathematics.** Now, what do all these figures mean? The first question is why such a horrible subject as mathematics should be so popular? Why should so many men of their own free will and accord elect to study algebra or the calculus? One reason is certainly that military operations require quantitative thinking. The old maxim of Nathan Bedford Forrest that the way to win a battle is "to get there fustes with the mostes men" is still pertinent with slight revision, say, "to get there fustes with the mostes power." Every part of this statement involves quantities. Power means men, ammunition, guns, equipment, vehicles, fuel, and so on. Time and distance and speed are involved. In fact, a military operation involves figures from start to finish. Every man in the operation works against these figures, and there are very few participants who are not acutely conscious of their part in a colossal mathematical problem. It is undoubtedly true that many men were so continuously bombarded with figures and calculations that they resolved to learn something about arithmetic and algebra in order to better understand the task before them. Others saw this lack of knowledge hampering them in the job they were doing or preventing them from being promoted. Certainly the military environment encouraged the study of mathematics, and accounts for some of the popularity which the subject enjoys.

But this is not the predominant factor. Many of the men who took these courses enrolled at a time when the fighting was nearing its end and when they were looking forward not to spending much more time in the service, but to getting out of the Army and going back to school or to work. It is true that their recent years of Army life had emphasized quantitative thinking and this had its effect on their choice of subjects. On the other hand, their post-war career was uppermost in their minds. Many men were planning to go back to school, and they realized that they did not have the mathematical background which they needed. They were more interested in what was valuable for their future than in what credits they needed to get a diploma.

**5. Decrease in emphasis on mathematics.** This brings up a point which is very important to every mathematician. It is the decreasing mathematical content of the high school and college curricula during the past twenty or thirty years. Twenty-five years ago, every student was required to take two years of good, sound algebra in order to graduate; in addition, two more years of mathematics were offered as electives and had many subscribers. Moreover, two years of mathematics were required for an A.B. degree in many colleges. The present situation is a far cry from that, even in the most conservative institutions. Mathematics has gradually been removed from the various curricula until there is very little left that is useful or even recognizable. Many of the courses which are called mathematics are a disgrace to the name. They are designed for amusement, and anything which might be thought-provoking is carefully avoided.

It is difficult to understand why such a purge could take place in the face of the trend of modern civilization. The old arguments for studying mathematics were that it developed logical processes which were useful in practically every walk of life and that it was indispensable for many trades and professions. These arguments are certainly just as valid today as they ever were. Furthermore, the present emphasis on science and technology makes a knowledge of some mathematics necessary if one is to understand, even superficially, the foundation of modern progress.

Notwithstanding, arithmetic and algebra have been placed in the high school museum, and college courses in mathematics have been retained for those few who want to study science or engineering. The only way to put the subject back in its proper place in college is to replace it in the high school curriculum. It was removed from the high schools because it was reputed to be dry, difficult, and impractical. As a matter of fact, this reputation was probably deserved in many cases, and no doubt the very instructors who were handling the subject admitted just that. Certainly mathematics is difficult for many students; it cannot be made easy and still be mathematics. On the other hand, there is no reason why it should be dry and impractical if it is presented properly. Proper presentation means suitable textbooks and capable teachers, and these are forces which can stem the present tide, a tide which will lead inevitably to the virtual extermination of mathematics as a universal subject.

In the years before the war, professors of mathematics in colleges, and especially in engineering schools, saw the effect of the diminishing mathematical knowledge of freshmen. It was obvious after two or three weeks that a high percentage of the entering students did not know enough arithmetic, much less algebra, to pursue the course in college algebra. There were only two solutions: (1) to eliminate those who could not make the grade, or (2) to go back and teach them arithmetic and high school algebra. Of course, the second solution was preferable for very obvious reasons: the number was too large to eliminate, such action would result in an indefensible waste of talent, and many boys who were capable but just poorly trained would be done a rank injustice. By establishing a course in arithmetic and high school algebra, many colleges were able to sal-

vage those who had the ability and determination to do the work; however, the students involved lost a semester which had to be made up at a later date. Moreover, much college faculty time was devoted to teaching high school mathematics. If such a situation is allowed to continue, one of two things must happen: either the mathematics once belonging to the secondary school category must be reclassified as college mathematics and the level lowered, or else the colleges must extend their courses beyond the present four years. In any event, college professors can expect more and more of their time to be devoted to teaching what they now consider high school mathematics.

**6. A correction of this situation.** This situation can be corrected, and the way mathematics has fared in the Army Education Program is an indication of why it should be corrected and it shows that now is the time. Many thousands of GI's who, as high school students, were allowed to skip over mathematics without learning it, registered for arithmetic, algebra, and geometry courses in the Army. They resent having been allowed to omit as superfluous something which later proved to be essential. They know now that it is inadvisable to allow a high school student to take only those subjects which he selects, because they are easy or because he likes them. They saw themselves faced with problems for which they had not been properly prepared, even though they could and should have been prepared in the normal course of events. As men, they realized that their judgments as boys were not sufficiently developed to rely upon in vital areas. These men believe in mathematics and the type of mathematics which has content. They are allies in putting the subject back in its place in the schools and colleges.

This means better teachers and better texts, and it is up to the mathematicians to develop them. It also means educating the public to the advantages of a knowledge of mathematics and to the disadvantages of an ignorance of the subject. The public is ready to listen to this explanation because it is now infiltrated with millions of men who saw the necessity for such knowledge during the war and with thousands who did something about it by studying on their own time and of their own volition.

But replacing arithmetic and algebra in the high school curriculum is only part of the job. Many of those men who became interested in mathematics while they were in the Army will never become full-time students again. They left school years ago, and circumstances do not permit them to return at this late date. However, they know what mathematics has to offer and they would welcome an opportunity to pursue the subject further. But the opportunity must be real; it should be an invitation. Such men are not likely to become engrossed in the contents of an algebra book written for the adolescent, geared to average student indifference, and dependent upon instructional explanation. They will respond to texts designed for self-teaching and directed to the adult in explanation, in illustrations, and in speed. This does not mean watered-down subject matter designed to be amusing. In fact, it means just the opposite; it means

real, down-to-earth mathematics designed to be interesting and practical. The text should be challenging and hence should be built around the average adult's experiences. Unless such books are made easily available and those who are interested encouraged to make use of them, many prospective customers will be lost.

The aim is to get mathematics into the adult education programs now gaining impetus in this country. This will require in addition to suitable books a concerted effort toward informing the public. The author will probably be criticized for calling for an advertising campaign, but mathematics is an article which is worth advertising and which can be sold. When one considers how many customers there were among the GI's without particular effort on anyone's part, one naturally wonders what could be done intentionally, knowing that startling results were obtained incidentally. All of the participants in the Army Education Program are adults and they were using books designed for high school and college students. How many more participants would there have been if there had been sufficient time to develop more suitable texts!

**7. Continuation of the Army Education Program.** Finally, it should be added that the Army Education Program will continue and that mathematics will be given as much opportunity in the future as in the past. The program will follow about the same principles as heretofore. Due to the decreased size of the Army, the numbers involved will be much smaller. Also, now that the draft has been suspended and emphasis is on voluntary enlistments, the educational level has dropped. The demand is centering on high school and vocational courses, and many of the college courses are no longer sufficiently popular to be retained in the curriculum. In fact, at the last meeting of the War-Navy Committee on the United States Armed Forces Institute it was decided to drop from the Institute offerings all courses above the freshman college level except those of particular value to the military. It is interesting to note that in this streamlining, the only mathematics course to be deleted was Differential Equations; even Differential and Integral Calculus were retained because they are still, in the reduced Army, very much in demand. Of course, every type of college course still remains available to Army personnel through college extension courses. As far as mathematics is concerned, the Army will continue to offer a wide choice of subjects directly from the Institute and in classes, and there is every reason to expect that the GI will continue to study everything from arithmetic through the calculus.

# ESTIMATING ELECTROSTATIC CAPACITY*

G. PÓLYA, Stanford University

**1. Introduction.** To find the electrostatic capacity of a given solid is a clear-cut, relatively simple problem of applied mathematics.

It is indeed a clear-cut problem. In order to solve it, we have to find first an integral of a differential equation under given boundary conditions (a function satisfying Laplace's equation outside the solid, that reduces to 1 on the boundary of the solid and to 0 at infinity). Then, we have to pick out a coefficient from the expansion of the integral we have found (the coefficient of $1/r$ in the expansion into spherical harmonics valid at sufficiently great distance from the origin). We can take the value of this coefficient as the definition of the electrostatic capacity of the solid if we are pure mathematicians.

To find the electrostatic capacity so defined, though difficult enough, is a problem that is simple in comparison with several others in applied mathematics. There are at least a few cases (ellipsoid, lens, spindle, anchor-ring) in which an exact solution can be found. That is, in these cases we can express the capacity as an integral or an infinite series whose numerical evaluation may still be quite difficult. In a very few cases, however, the expression of the capacity is simple, and it is quite simple in case of the sphere whose capacity is numerically equal to its radius. Yet for many familiar solids (as cube, right circular cylinder, and so on) we do not know the capacity, and have little hope of obtaining an exact expression for it.†

Yet "even when analysis fails to give a solution in the mathematical sense, we need not be altogether in the dark as to the magnitudes of the quantities with which we are dealing."‡ We may still obtain approximate formulas or definite bounds. Helped by a grant from the Office of Naval Research and a few collaborators, Mr. Szegö and the author undertook to explore systematically bounds for, and approximations to, the electrostatic capacity. We regard the evaluation of capacities as a sort of test case, hoping that some of the methods may have a wider scope and be used also for related problems of applied mathematics. Of the various results that we found some will be outlined in what follows. The methods of proof, however, cannot even be suggested in this short communication.§

**2. Volume-radius.** Capacity has an intuitive meaning for the physicist, but

* Presented at the meeting of the Northern California Section of the Mathematical Association of America, San Francisco, January 25, 1947.

† It was related by Jacobi and Kirchhoff that Dirichlet, shortly before his death, solved the problem of distribution of electricity on a rectangular parallelepiped. This solution appears to be lost.

‡ Lord Rayleigh, The Theory of Sound, 2nd ed., vol. 2, (1896) p. 180.

§ All new facts outlined in the following result from the joint work of Mr. Szegö and the present author, undertaken in September 1946, as a sequel to the work presented in the paper, G. Pólya and G. Szegö, Inequalities for the capacity of a condenser, American Journal of Mathematics, vol. 67, (1945) pp. 1–32.

another interpretation of the same quantity may appeal more to people less familiar with electrical experiments. We imagine a body embedded in a uniform infinite medium whose temperature at infinity is 0. If the surface of the solid is kept at a constant temperature (as the skin of a man or a cat approximately is) a steady flow of heat will pass from the body into the medium. Let 1 be the constant temperature of the boundary. Then the amount of outgoing heat per unit time is called the *thermal conductance* and this quantity is, except for a constant factor depending on the nature of the medium, equal to the electrostatic capacity of the embedded body.

Now we can all observe what a cat does when he prepares himself for sleeping through a cold night: he pulls in his legs, curls up, and, in short, makes his body as spherical as possible. He does so, obviously, in order to minimize the thermal conductance or, what is the same thing, his capacity. Thus, he seems to have a sort of knowledge of the following general theorem: *Of all solids with a given volume, the sphere has the minimum capacity.*

This theorem can actually be proved,* and since the capacity of a sphere is known, it can be restated as follows:

THEOREM. *The capacity of a solid is never less than the radius of a sphere with equal volume.*

Let $C$ denote the capacity and $V$ the volume of the solid. Then the fact just stated can be expressed by the inequality

$$C > \left(\frac{3V}{4\pi}\right)^{1/3}, \tag{1}$$

which is valid unless the solid is a sphere. The quantity on the right side of (1) may be called the *volume-radius* of the solid having the volume $V$. Thus, for any solid, the ratio of the capacity to the volume-radius is greater than 1. Examples show that this ratio can be arbitrarily large. The volume-radius yields a simple lower bound but no upper bound for the capacity.

**3. Surface-radius.** If the area of the surface of a solid is $S$, the radius of a sphere with the same surface area is $(S/4\pi)^{1/2}$, and we call this quantity the *surface-radius*. The ratio of capacity to surface-radius can attain any given positive value if the solid is appropriately chosen. In other words, this ratio varies between 0 and $\infty$. Thus, the surface-radius yields no usable upper or lower bound for the capacity.

This situation changes, however, if we restrict ourselves to the consideration of *convex* solids. Then the ratio in question has a positive lower bound, and we conjecture that *of all convex bodies, with a given surface area, the circular disk has*

* The theorem was discovered by H. Poincaré, Figures d'équilibre d'une masse fluide, Paris, 1903, pp. 17–22, yet without a complete proof. The first complete proof was given by G. Szegö, Über einige Extremalaufgaben der Potentialtheorie, Mathematische Zeitschrift, vol. 31 (1930) pp. 583–593.

*the minimum capacity.* This rather plausible conjecture is expressed by the inequality

$$C > \frac{8^{1/2}}{\pi}\left(\frac{S}{4\pi}\right)^{1/2}, \tag{2}$$

which we expect to be valid unless the convex body becomes a circular disk. In case of the disk, the two members of (2) become equal.

Our conjecture is not without a certain basis. We have verified it in various special cases and, although we did not prove (2), we did prove the weaker inequality

$$C > \frac{2}{\pi}\left(\frac{S}{4\pi}\right)^{1/2}. \tag{2'}$$

**4. Mean radius.** Here we restrict ourselves from the outset to the consideration of convex solids. The distance between two parallel tangent planes (or, more generally, "supporting" planes) is called the width of the solid in the direction perpendicular to those planes. Integrating over all directions (over the uniformily covered surface of the unit sphere), we find the *average width* of the solid, and we call one half of the average width the *mean radius.* As we can see immediately from a formula due to Minkowski, the mean radius of a "smooth" convex solid is also equal to $M/4\pi$, where $M$ denotes the surface integral of the mean curvature extended over the whole boundary of the solid. The mean radius can easily be computed for any convex polyhedron. For instance, it is $3a/4$ for a cube with edge $a$.

In general we have the following inequalities between the three radii here defined:

$$\left(\frac{3V}{4\pi}\right)^{1/3} < \left(\frac{S}{4\pi}\right)^{1/2} < \frac{M}{4\pi} \tag{3}$$

unless the convex solid is a sphere, in which case all three radii become equal. This is a restatement of classical results. The first of these inequalities expresses the "isoperimetric" property of the sphere; the second is due to Minkowski.[*]

We come now to the following result[†] which is not so easy to foresee intuitively as inequalities (1) or (2):

THEOREM. *The capacity of a convex solid is never superior to the capacity of a prolate ellipsoid whose major semi-axis is the mean radius of the solid and whose minor semi-axis is the surface-radius.*

---

[*] See H. Minkowski, Gesammelte Abhandlungen (1911) vol. 2, pp. 103–279, or W. Blaschke, Kreis und Kugel (1916), or T. Bonnesen and W. Fenchel, Theorie der konvexen Körper(1934), expecially p. 66 and p. 110 of the last quoted work.

[†] G. Szegö, Über einige neue Extremaleigenschaften der Kugel, Mathematische Zeitschrift, vol. 33 (1931) pp. 419–425. The formulation here given is new.

This states that, unless the solid is a sphere,

$$C < \frac{M}{4\pi} \frac{2\epsilon}{\log \frac{1+\epsilon}{1-\epsilon}}, \tag{4}$$

where the positive quantity (eccentricity) $\epsilon$ is defined by the equation

$$\epsilon^2 = 1 - \frac{4\pi S}{M^2}.$$

From (4), we can derive the weaker inequality

$$C < \frac{\frac{M}{4\pi} + 2\left(\frac{S}{4\pi}\right)^{1/2}}{3} \tag{4'}$$

and the still weaker relation

$$C < \frac{M}{4\pi}. \tag{4''}$$

Joining the inequalities (1) and (4″), we obtain

$$\left(\frac{3V}{4\pi}\right)^{1/3} < C < \frac{M}{4\pi} \tag{5}$$

unless the convex solid is a sphere. A comparison of (3) and (5) reveals a remarkable parallelism between the surface-radius and the capacity.*

**5. Conformal radius.** The following consideration is restricted to solids of revolution. A plane passing through the axis of revolution intersects the solid in a closed curve which we call the *meridian section.* Thus, the meridian section is a plane curve having an axis of symmetry.

We map the infinite part of the plane exterior to the meridian section onto the exterior of a circle of radius $\bar{r}$, so that the points at infinity correspond to each other and the linear magnification at infinity is 1. As is well known, the quantity $\bar{r}$ is uniquely determined. We call $\bar{r}$ the *conformal radius* of the solid of revolution.

The ratio of capacity to conformal radius can be arbitrarily small but it certainly cannot be arbitrarily large. We conjecture that

$$C < \frac{4}{\pi}\bar{r} \tag{6}$$

* For further remarks about this see the joint paper by G. Pólya and G. Szegö, quoted above, On the capacity of a condenser, Bulletin of the American Mathematical Society, vol. 51 (1945) pp. 325–350.

unless the solid of revolution becomes a circular disk and its meridian section a straight line-segment. We cannot prove this conjecture yet, but we did prove that

$$C < 1.2754\bar{r} \tag{6'}$$

which is not so bad numerically since

$$\frac{4}{\pi} = 1.2732 \cdots .$$

A simply closed plane curve which has two axes of symmetry perpendicular to each other can be rotated about each of these axes and thereby generates two solids of revolution which are, in general, different. Let $C$ and $C'$ denote the capacities of these two solids, respectively, and let $\bar{r}$ stand again for their common conformal radius. We conjecture that

$$C + C' < 2\bar{r} \tag{7}$$

unless the meridian section becomes a circle and the two solids of revolution become equal spheres, in which case $C = C' = \bar{r}$. Again we did not prove (7) but we did prove that

$$C + C' < 2.082\bar{r}. \tag{7'}$$

**6. Numerical applications.** The foregoing results are not only of great generality and simplicity, but also are readily applicable. Take any convex solid. We can compute $V$, $S$, and $M$ incomparably more easily than $C$. For solids of revolution we should also compute $\bar{r}$, a quantity which is of more intricate nature than $V$, $S$, or $M$, but still more accessible than $C$. Applying the bounds considered in the foregoing discussion, we often can find an acceptable estimate for $C$. Let us consider just two examples.

EXAMPLE 1. For a cube with edge $a$, we find

$$.62211a < C < .71055a.$$

The upper estimate is based on (4). The lower estimate is based on the hypothetical (2), but the well established (1) gives an estimate not much worse, namely, $.62033a$.

EXAMPLE 2. Consider the right circular cylinder obtained by rotating a square with side $a$ about the line joining the midpoints of two opposite sides. We find

$$.57236a < C < .59017a.$$

The lower estimate is based on (1), which in this case is better than (2). The upper estimate is based on the hypothetical (7). Observe that in the present case $C = C'$.

We see that in case of the cube we can derive an approximate value for the capacity differing by less than 7% from the unknown correct value. In case of

the cylinder just considered, the difference between obtainable approximation and unknown correct value is only about 1.5% of the latter.

I may venture the opinion that it would be desirable to treat other physical quantities in the same manner as the capacity was treated in the foregoing discussion. We should establish general bounds readily applicable to special cases and yielding estimates with errors of only a few percents. This is not easy, and the outlined theory needs considerable further work even in the relatively simple case of the electrostatic capacity here discussed.

## A TRIGONOMETRIC INFINITE PRODUCT

R. P. AGNEW AND R. J. WALKER, Cornell University

**1. Introduction.** Seventy years ago, Dobinski [1] gave a curious trigonometric formula which we shall discuss. The Dobinski formula is

$$(\tan x)(\tan 2x)^{1/2}(\tan 4x)^{1/4}(\tan 8x)^{1/8}\cdots = 4\sin^2 x??. \tag{1}$$

Assuming that the individual factors are unambiguously defined, the left side of (1) is an infinite product, and the meaning of (1) is the following: if $P_n(x)$ is the product of the first $n$ factors of (1), namely,

$$P_n(x) = (\tan x)(\tan 2x)^{1/2}(\tan 4x)^{1/4}\cdots(\tan 2^n x)^{1/2n}, \tag{2}$$

then

$$\lim_{n\to\infty} P_n(x) = 4\sin^2 x. \tag{3}$$

The two question marks in (1) are intended to question the manner in which the factors are defined and the validity of the formula.

Let $A$ denote the set of real numbers $x$ not of the form $h\pi/2^k$, where $h$ is an odd integer and $k$ is a positive integer. When $x$ is in $A$, no one of the arguments $x, 2x, 4x, 8x, \cdots$ is an odd multiple of $\pi/2$ and the tangents in (1) all exist. Dobinski gave two "proofs" that (1) holds for each $x$ in $A$. It appears that he gave two proofs because he considered each to be interesting, and not because he considered each to be defective and hence needed the support of the other. We shall present his first "proof," with appropriate remarks required to clarify the formula.

**2. Validity of the formula.** The "proof" is based on the identity

$$\tan\theta = \frac{2\sin^2\theta}{\sin 2\theta}\cdot$$

Taking $\theta$ to be successively $x, 2x, 4x, \cdots, 2^n x$, we obtain

$$\tan x = \frac{2(\sin x)^2}{(\sin 2x)}$$

$$(\tan 2x)^{1/2} = \frac{2^{1/2}(\sin 2x)}{(\sin 4x)^{1/2}}$$

$$(\tan 4x)^{1/4} = \frac{2^{1/4}(\sin 4x)^{1/2}}{(\sin 8x)^{1/4}}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$(\tan 2^n x)^{2^{-n}} = \frac{2^{2^{-n}}(\sin 2^n x)2^{-n+1}}{(\sin 2^{n+1}x)^{2^{-n}}}.$$

Multiplication and simplification give

$$P_n(x) = \frac{[2^{1+1/2+1/4+\cdots+2^{-n}} \sin^2 x]}{(\sin 2^{n+1}x)^{2^{-n}}}. \tag{4}$$

Since the quantity in brackets converges to $4 \sin^2 x$ as $n \to \infty$, (1) holds if and only if*

$$\lim_{n\to\infty} (\sin 2^{n+1}x)^{2^{-n}} = 1. \tag{5}$$

At this point we pause to look at (1) more critically. It is easy to show that for each $x$ in $A$ not of the form $m\pi$, $m$ an integer, some of the arguments $2x$, $4x$, $8x$, $\cdots$ lie in the second or fourth quadrants and hence have negative tangents. Thus (1) involves roots of negative numbers. As we shall see in Section 3, (1) will hold only in trivial cases if we define $q^{1/r}$, when $q<0$ and $r$ is an even positive integer, in the simplest and familiar unambiguous way by the formula

$$q^{1/r} = \exp [(1/r)(\log |q| + \pi i)]. \tag{6}$$

Hence we shall see what can be done by allowing the symbol $q^{1/p}$ to stand for any one of the $p$th roots of $q$ which we decide to select.

Choose a value for each of the factors and consider the sequence of complex numbers $P_n(x)$. For $P_n(x)$ to converge to $4 \sin^2 x$ it is necessary and sufficient that

$$\lim_{n\to\infty} \operatorname{amp} P_n(x) = 2m\pi, \qquad m \text{ an integer,} \tag{7}$$

$$\lim_{n\to\infty} |P_n(x)| = 4 \sin^2 x. \tag{8}$$

We first consider (7). Suppose $\operatorname{amp}(\tan kx)^{1/k}$, $k=2, 4, 8, \cdots$, is confined to the interval $-\pi<\theta\leqq\pi$, and let $\operatorname{amp} P_n(x)$ have the value,

$$\operatorname{amp} P_n(x) = \operatorname{amp} \tan x + \operatorname{amp} (\tan 2x)^{1/2} + \cdots + \operatorname{amp} (\tan 2^n x)^{2^{-n}}.$$

Then (7) is equivalent to

* Here Dobinski's argument breaks down, as his "proof" of (5) involves an unjustifiable interchange of two limit processes.

$$(9) \qquad \sum_{n=0}^{\infty} \text{amp } (\tan 2^n x)^{2^{-n}} = 2m\pi.$$

That the terms of this series can be chosen to make the series converge to the proper value is evident from the fact that the amplitudes of $(\tan 2^n x)^{2^{-n}}$ are equally spaced at intervals of $2\pi/2^n$. Thus in particular we can make successive choices so that we always have

$$0 \leqq \text{amp } P_n(x) \leqq \frac{2\pi}{2^n},$$

thereby satisfying (9) for $m=0$.

In discussing condition (8) we shall confine our attention to those values of $x$ in $A$ which lie in the interval $0<x<\pi/2$. The extension of our results to other values of $x$ in $A$ is a simple matter. Putting absolute values in the formulas leading to (5), we see that (8) is equivalent to

$$\lim_{n\to\infty} |\sin 2^{n+1}x|^{2^{-n}} = 1,$$

or

$$(10) \qquad \lim_{n\to\infty} 2^{-n} \log_2 |\sin 2^{n+1}x| = 0.$$

Any number $x$ in the interval $0<x<\pi/2$ can be expressed in the dyadic form

$$(11) \qquad x = \tfrac{1}{2}\pi(.d_1d_2 \cdots d_n \cdots),$$

each $d_i$ being 0 or 1. The values of $x$ for which there are only a finite number of 1's, or of 0's, are precisely those which were excluded from the set $A$, and so we may assume that the expression for $x$ contains an infinite number of both 0's and 1's.

We have

$$2^{n+1}x = \pi(d_1d_2 \cdots d_n . d_{n+1}d_{n+2} \cdots),$$

and

$$|\sin 2^{n+1}x| = \sin \pi(.d_{n+1}d_{n+2} \cdots).$$

CASE I. $d_{n+1}=0$. For this case,

$$0 < \pi(.d_{n+1}d_{n+2} \cdots) < \frac{\pi}{2}.$$

Now when $0<\theta<\pi/2$, it can easily be proved that

$$\frac{2}{\pi} < \frac{\sin \theta}{\theta} < 1.$$

Hence

$$|\sin 2^{n+1}x| = A_n(x)\pi(.0d_{n+2}d_{n+3} \cdots),$$

where $A_n(x)$ is a number between $2/\pi$ and 1. Taking the logarithm of each member gives

$$\log_2 |\sin 2^{n+1}x| = \log_2 A_n(x)\pi - z_n - \phi_n,$$

where $z_n$ is the number of 0's following $d_n$ before the first 1 makes its appearance, and $\phi_n$ is a number between 0 and 1. Thus

$$2^{-n} \log_2 |\sin 2^{n+1}x| = 2^{-n}(\log_2 A_n(x)\pi - \phi_n) - 2^{-n}z_n.$$

Now

$$\lim_{n\to\infty} 2^{-n}(\log_2 A_n(x)\pi - \phi_n) = 0,$$

and so (10) cannot hold unless

$$\lim_{n\to\infty} 2^{-n}z_n = 0. \tag{12}$$

CASE II. $d_{n+1}=1$. In this case,

$$\begin{aligned}\sin \pi(.1d_{n+2}d_{n+3}\cdots) &= \sin \pi(1 - .0e_{n+2}e_{n+3}\cdots)\\ &= \sin \pi(.0e_{n+2}e_{n+3}\cdots)\end{aligned}$$

where $e_k+d_k=1$, $k=n+2, n+3, \cdots$. The argument of Case I can now be applied to show that (10) cannot hold unless

$$\lim_{n\to\infty} 2^{-n}w_n = 0, \tag{13}$$

where $w_n$ is the number of 1's following $d_n$ before the first 0 makes its appearance.

Thus (12) and (13) are necessary conditions for (10). Taken together they are also a sufficient condition, for each value of $n$ comes under one of the two cases.

To sum up, we can say that Dobinski's formula (1) holds for a real number $x$ provided that

**A.** $x$ does not have the property that $2x/\pi$, expressed as a dyadic fraction, has such phenomenally long blocks of 0's or 1's that (12) or (13) fail to hold.

**B.** The values of $(\tan 2^n x)^{2^{-n}}$ are suitably chosen complex numbers.

**3. Failure of** (1). Each subinterval of the interval $0<x<\pi/2$ contains points of the form $h\pi/2^k$ where $h$ is an odd integer and $k$ is a positive integer; for such values of $x$, at least one of the tangents in (1) is undefined and the formula fails to hold. The analysis of the preceding section shows that each subinterval contains additional points (some of the points of the set $A$ defined above) for which (1) fails to hold. In fact, for each subinterval there is an integer $k$ such that the subinterval contains points

$$x = \tfrac{1}{2}\pi(.d_1d_2\cdots d_kd_{k+1}\cdots)$$

where the numbers $d_k, d_{k+1}, \cdots$ form a sequence consisting in order of $n_1=2^k$ zeros, $n_2=2^{k+n_1}$ ones, $n_3=2^{k+n_1+n_2}$ zeros, and so on. For each such $x$, (12) and (13) fail to hold, and hence also (8) and (1) fail to hold.

Suppose now that $x$ does not have the form $h\pi/2^k$ where $h$ and $k$ are integers,

so that the tangents in (1) all exist and are all different from zero. When the roots in (1) are determined unambiguously by use of (6), the series of amplitudes in the left member of (9) becomes

$$\pi\left(\alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2^2} + \frac{\alpha_4}{2^4} + \cdots\right) \tag{14}$$

where each $\alpha$ is 0 or 1 and the $\alpha$'s are neither all 0 nor all 1. The series (14) necessarily converges to an amplitude between 0 and $2\pi$. Accordingly (7) and Dobinski's formula (1) fail to hold, except in trivial cases, when the roots are unambiguously determined in the simplest and usual way.

**4. Application of Borel's theorem on dyadic expansions.** Borel [2] showed that for each $x$ in $0<x<1$, except for those in a set having Lebesgue measure zero, the dyadic expansion of $x$, namely,

$$x = .d_1d_2d_3\cdots$$

has the following property: if $p_n$ is the number of zeros in the first $n$ places, then

$$\lim_{n\to\infty} \frac{p_n}{n} = \frac{1}{2}\cdot \tag{15}$$

If (15) holds, then

$$\lim_{n\to\infty} \frac{z_n}{n} = 0, \qquad \lim_{n\to\infty} \frac{w_n}{n} = 0;$$

and *a fortiori*

$$\lim_{n\to\infty} \frac{z_n}{2^n} = 0, \qquad \lim_{n\to\infty} \frac{w_n}{2^n} = 0.$$

It follows that (8) holds for each $x$ in the interval $0<x<\pi/2$ except for a set having measure zero and that, under the proviso **B** above, Dobinski's formula (1) also holds.

**5. Complex values of $x$.** We finally consider the behavior of (1) when $x$ is complex. Let $\theta=a+ib$. Then

$$\begin{aligned}
|\sin\theta| &= |\sin a\cosh b + i\cos a\sinh b| \\
&= [\sin^2 a\cosh^2 b + \cos^2 a\sinh^2 b]^{1/2} \\
&= [\sin^2 a\cosh^2 b + \cos^2 a(\cosh^2 b - 1)]^{1/2} \\
&= [\cosh^2 b - \cos^2 a]^{1/2} \\
&= \cosh b\left[1 - \frac{\cos^2 a}{\cosh^2 b}\right]^{1/2} \\
&= \tfrac{1}{2}e^b(1+e^{-2b})\left[1 - \frac{\cos^2 a}{\cosh^2 b}\right]^{1/2}.
\end{aligned}$$

Putting $\theta = 2^{n+1}x$, where $x = p+iq$, $q \neq 0$, we get

$$| \sin 2^{n+1}x |^{2^{-n}} = \left[\frac{1}{2} e^{2^{n+1}q}(1 + e^{-2^{n+2}q}) \left(1 - \frac{\cos^2 2^{n+1}p}{\cosh^2 2^{n+1}q}\right)^{1/2}\right]^{2^{-n}}$$
$$= e^{2q}\left[\frac{1}{2} (1 + e^{-2^{n+2}q}) \left(1 - \frac{\cos^2 2^{n+1}p}{\cosh^2 2^{n+1}q}\right)^{1/2}\right]^{2^{-n}}.$$

Hence

$$\lim_{n\to\infty} | \sin 2^{n+1}x |^{2^{-n}} = e^{2q}.$$

Combining this with an obvious modification of the argument following equation (9), we obtain the following general theorem.

Let $x$ be any complex number, and $\theta$ any real number. Then values of $(\tan 2^n x)^{2^{-n}}$ for $n = 1, 2, \cdots$ can be chosen so that

$$\lim_{n\to\infty} \tan x(\tan 2x)^{1/2}(\tan 4x)^{1/4} \cdots (\tan 2^n x)^{2^{-n}} = 4e^{2I(x)+i\theta} \sin^2 x,$$

provided that $x$ does not belong to a certain set of measure zero on the real axis.

### References

1. Dobinski, G. 1876. Product einer unendlichen Factorenreihe. Archiv der Mathematik und Physik, vol. 59, pp. 98–100. Dobinski's note in volume 59 of the Archiv der Mathematik und Physik bears the name G. Dobiciecki, but a correction following the index in the volume says the name should be G. Dobinski.
2. Borel, E. 1909. Les probabilities denombrables et leurs applications arithmetiques. Rendiconti del Circolo Matematico di Palermo, vol. 27, pp. 247–271.

# ORTHOPOLAR AND ISOPOLAR LINES IN THE CYCLIC QUADRILATERAL

R. GOORMAGHTIGH, Bruges, Belgium

**1. Introduction.** According to a theorem given by Servais [1], the orthopoles of a straight line $l$ with respect to the four triangles formed by three out of four vertices of any quadrilateral are on a straight line $L$, called the orthopolar line of $l$ for the given quadrilateral. McBrien [2] has proved the same property and also mentioned several interesting envelopes of orthopolar lines in the case when the quadrilateral is cyclic.

It will be shown that these last results and some others may all be easily derived from two simple properties of the orthopolar line, which will also be extended to the isopolar line.

**2. Two principal directions.** Using complex coördinates, let $t_1$, $t_2$, $t_3$, $t_4$ be the turns corresponding to the vertices of a cyclic quadrilateral $A_1A_2A_3A_4$, inscribed in the base circle $\Gamma$ having the center $O$, while $\Omega$ is the unit-point, and denote by

$\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the elementary symmetric functions of the $t_i$; that is,

$$\sigma_1 = \sum t_1, \qquad \sigma_2 = \sum t_1 t_2, \qquad \sigma_3 = \sum t_1 t_2 t_3, \qquad \sigma_4 = \sum t_1 t_2 t_3 t_4.$$

Parallels drawn through $\Omega$ to $A_1A_2$ and $A_3A_4$ cut $\Gamma$ again at $t_1t_2$ and $t_3t_4$, so that the mid-points of the arcs on $\Gamma$ between these two points are $\pm\sigma_4^{1/2}$.

This shows that the bisectors of the angles between two opposite sides, or between the diagonals of the quadrilateral are parallel to two perpendicular directions $\Delta, \Delta'$, which is a well known property [3], and that, if $O\Omega$ is parallel to $\Delta, \Delta'$, $\sigma_4 = 1$.*

**3. Properties of the orthopolar lines.** The projection of $A_1$ on the straight line $l$†

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1$$

eing

$$\frac{a}{2} + \frac{t_1}{2} - \frac{a}{2\bar{a}t_1},$$

the orthopole of $l$ with respect to $A_1A_2A_3$ is

$$x = \frac{1}{2}\left(a + t_1 + t_2 + t_3 + \frac{t_1t_2t_3\bar{a}}{a}\right)$$

or

$$x = \frac{1}{2}\left(a + \sigma_1 - t_4 + \frac{\bar{a}\sigma_4}{at_4}\right)$$

and, as

$$\bar{x} = \frac{1}{2}\left(\bar{a} + \frac{\sigma_3}{\sigma_4} - \frac{1}{t_4} + \frac{at_4}{\bar{a}\sigma_4}\right),$$

the elimination of $t_4$ leads to the equation of the orthopolar line $L$, namely,

$$2ax + 2\bar{a}\sigma_4\bar{x} = a\sigma_1 + \bar{a}\sigma_3 + a^2 + \sigma_4\bar{a}^2.$$

Suppose now that $O\Omega$ is parallel to $\Delta$ or $\Delta'$; then the last equation becomes

$$2ax + 2\bar{a}\bar{x} = a(\sigma_1 + a) + \bar{a}(\bar{\sigma}_1 + \bar{a}).$$

Hence, we have two simple properties of the orthopolar line:

*A given line $l$ and its orthopolar line $L$ for the cyclic quadrilateral have symmetric directions to those of the bisectors of the pairs of opposite sides and diagonals.*

---

* These directions are those of the axes of the equilateral hyperbola circumscribed to the quadrilateral.

† $\bar{u}$ is the conjugate to $u$.

Further, $L$ passes through the point $\sigma_1/2+a/2$; but $\sigma_1/2$ is the center $\omega$ of the equilateral hyperbola circumscribed to the quadrilateral, and $a/2$ is the projection of $O$ on $l$.

*The orthopolar line $L$ contains the extremity of the segment equipollent to the distance from $O$ to $l$, having as origin the center $\omega$ of the equilateral hyperbola circumscribed to the quadrilateral.*

**4. Envelopes of orthopolar lines.** Suppose that $l$ envelopes a curve of which the pedal curve as to $O$ will be denoted by $C$. Let $\omega X$ and $\omega Y$ be two straight lines forming angles with $\Delta$ and $\Delta'$ equal to $\pi/4$. If, for the center $\omega$ and the ratio 2, $C'$ is homothetic to the curve derived from $C$ by a translation of vector $O\omega$, then the envelope of $L$ is also the envelope $C_0$ of the straight line joining the projections on $\omega X$ and $\omega Y$ of a point moving on $C'$.

Various particular cases of this last envelope are well known. Some of these are listed below.

*If a line $l$ passes through a fixed point, $C$ is a circle passing through $O$, and the orthopolar line $L$ of $l$ for the quadrilateral envelopes a deltoid.*

When $l$ passes through $O$, then $L$ passes through $\omega$.

It is also easy to prove that $C_0$ will reduce to a point when $C'$ and $C$ are equilateral hyperbolas passing respectively through $\omega$ and $O$ and having their asymptotes parallel to $\omega X$ and $\omega Y$.

*When $l$ envelopes the antipedal curve as to $O$ of an equilateral hyperbola passing through $O$ and having its axes parallel to $\Delta$ and $\Delta'$, $L$ passes through a fixed point.*

Finally, according to a well known definition of the astroid, we find a generalization of a property given by McBrien [2]; namely,

*When $l$ envelopes a circle concentric to the circumcircle of the quadrilateral, $L$ envelopes an astroid having $\omega X$ and $\omega Y$ for cusp-tangents; the constant segment cut off by these on $L$ is equal to the diameter of the circle, the envelope of $l$.*

**5. Extension to isopolar lines.** The two properties of the orthopolar line, given in Section 3, may be extended to the isopolar line.

The straight lines,

$$\frac{x}{\bar{x}} = p_1 \qquad \frac{x}{\bar{x}} = \lambda p,$$

form an angle $\theta$ such that $\lambda = e^{2i\theta}$; hence a straight line through $A_1$, forming an angle $\theta$ with $l$, cuts $l$ at the point

$$\frac{(-\lambda a + t_1 + a\lambda/\bar{a}t_1)}{(1-\lambda)}.$$

Further, the equation of the line through that point forming an angle $\pi - \theta\pi$ with $A_2A_3$ is

$$(1-\lambda)\left(x + \frac{t_2t_3\bar{x}}{\lambda}\right) = -a\lambda + t_1 + \frac{a\lambda}{\bar{a}t_1} + \frac{\bar{a}t_2t_3}{\lambda} - \frac{t_2t_3}{t_1} - \frac{\bar{a}t_1t_2t_3}{a\lambda}.$$

The isopole of $l$ for the angle $\theta$ as to $A_1A_2A_3$ has, therefore, as coördinate

$$x = \frac{(-a\lambda + t_1 + t_2 + t_3 - \bar{a}t_1t_2t_3/a\lambda)}{(1-\lambda)}$$

or

$$x = \frac{(-a\lambda + \sigma_1 - t_4 - \bar{a}\sigma_4/a\lambda t_4)}{(1-\lambda)};$$

and, since

$$\bar{x} = \frac{(\bar{a} - \lambda\bar{\sigma}_1 + \lambda/t_4 + a\lambda^2 t_4/\bar{a}\sigma_4)}{(1-\lambda)},$$

the elimination of $t_4$ shows that the isopoles of $l$ for the four triangles of the quadrilateral lie on the straight line

$$a\lambda^2 x + \bar{a}\sigma_4\bar{x} = \frac{(-a^2\lambda^3 + \bar{a}^2\sigma_4 + a\lambda^2\sigma_1 - \bar{a}\lambda\sigma_4\bar{\sigma}_1)}{(1-\lambda)};$$

when $O\Omega$ is parallel to $\Delta$ or $\Delta'$, the equation of the isopolar line becomes

$$\frac{ax + \bar{a}\bar{x}}{\lambda^2} = \frac{[a\sigma_1 - a^2\lambda + \bar{a}(\bar{a} - \lambda\bar{\sigma}_1)/\lambda^2]}{(1-\lambda)}.$$

*The isopolar line for the angle $\theta$ of a line $l$ forms an angle $2(\pi - \theta)$ with the symmetric of $l$ as to the principal directions $\Delta$ and $\Delta'$ of the quadrilateral.*

Further, as the isopolar line passes through the point

$$\frac{(\sigma_1 - a\lambda)}{(1-\lambda)},$$

we have the following property:

*The isopolar line passes through the image of $O$ in the point of the axis of the segment between $\omega$ and the projection of $O$ on $l$, from which that segment subtends an angle $2\theta$.*

**References**

1. Servais, Cl., Sur l'orthopôle, Mathesis, 1923. p. 11.
2. McBrien, V. O., On some systems of orthopolar and Kantor lines of lines referred to a quadrangle, The Catholic University of America Press, Washington, D. C., 1942.
3. -F. G.-M., Exercises de Géométrie, Tours, 7th ed., p. 288.

# MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

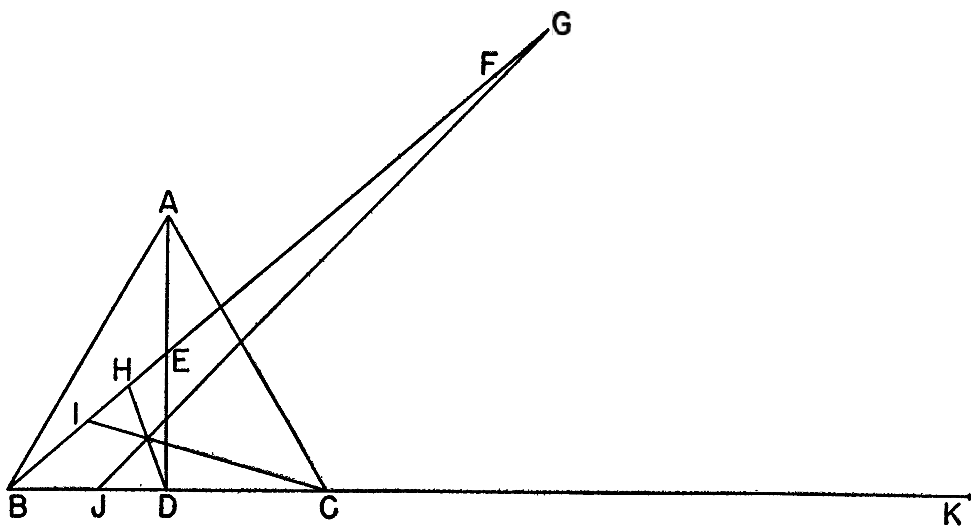
*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

## AN APPROXIMATE CONSTRUCTION FOR e

H. F. SANDHAM, Trinity College, Dublin, Ireland

In the figure the triangle $ABC$ is equilateral. The altitude $AD$ is bisected at $E$. The line $BE$ is produced to $F$ and then to $G$ so that $BF=2BC$, and $FG=\frac{1}{3}BE$. The points $H$ and $I$ are taken on $BE$ so that $BH=BD$, and $BI=\frac{1}{3}BC$. The point $J$ is taken on $BC$ so that $HD$, $IC$, and $JG$ are concurrent. The side $BC$ is produced to $K$ so that $CK=2BC$.

Then $JK/BC$ is approximately equal to $e$.




COMPUTATION.
Since

$$AD = \frac{\sqrt{3}}{2} BC, \qquad DE = \frac{\sqrt{3}}{4} BC,$$

we have

$$BE = \frac{\sqrt{7}}{4} BC.$$

From

$$BF = 2BC, \qquad FG = \frac{\sqrt{7}}{12} BC,$$

it follows that

$$BG = \left(2 + \frac{\sqrt{7}}{12}\right) BC.$$

Also

$$GI = \left(\frac{5}{3} + \frac{\sqrt{7}}{12}\right) BC.$$

Now

$$(BJ/JC):(BD/DC) = (BG/GI):(BH/HI).$$

Hence

$$\frac{BJ}{JC} = \frac{24 + \sqrt{7}}{3(20 + \sqrt{7})}.$$

Therefore

$$\frac{JK}{BC} = \frac{228 + 11\sqrt{7}}{84 + 4\sqrt{7}} = \frac{4711 + 3\sqrt{7}}{1736} = 2.718,281,828,(30).$$

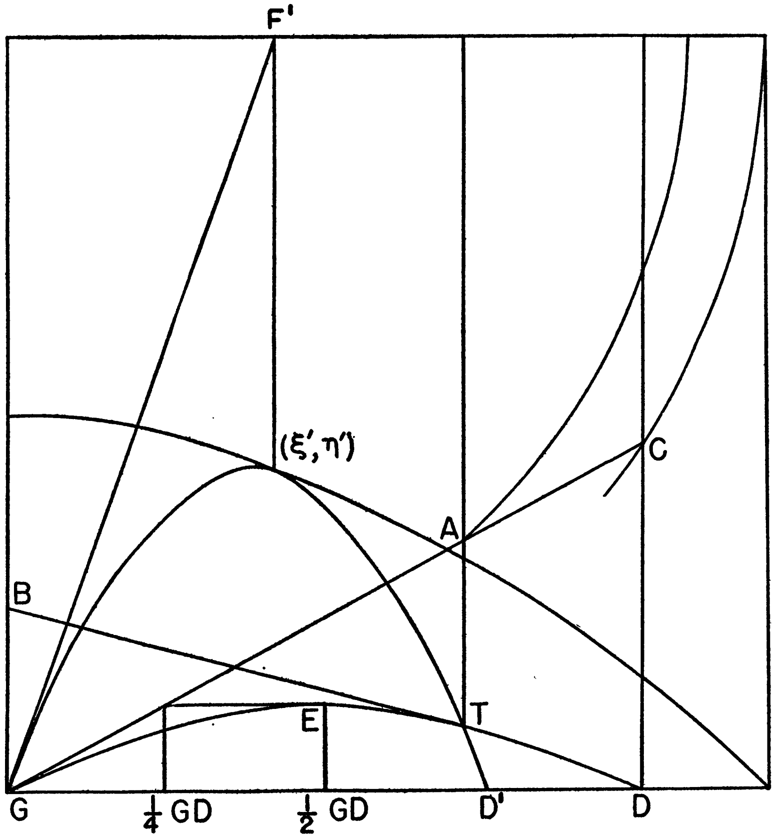
Thus since $e = 2.718,281,828,(46)$, the error is approximately 0.000,000,000,16.

## THE TRAJECTORY IN VACUO

J. M. Thomas, Duke University

R. J. Walker, *An artillery problem*, this Monthly, vol. 54, 1947, pp. 33–34, has shown how to construct geometrically the angle of departure, the angle of fall, and the time of flight for a projectile fired in vacuo, the muzzle velocity $v_o$, the acceleration of gravity $g$ and the positions of gun and target being known. His construction employs a fixed hyperbola. The present note gives for the above and related quantities ruler-and-compass constructions which the author used in lectures given during the late war. The discussion here is limited to a description of the constructions because elementary, simple proofs are easily made.

Let $R$ be the maximum range. Use an accent to distinguish quantities belonging to the upper trajectory from those belonging to the lower. Denote by $(a, b, r)$ the circle with center $(a, b)$ and radius $r$. In a system of rectangular cartesian coördinates (see figure) plot the gun at $G = (0, 0)$ and the target at $T = (x_o, y_o)$. The distance $GT$ is the slant range $r$. Let $A$, $A'$ with ordinates $a$, $a'$ be the intersections of $(x_o - r, R, R - y_o)$ with the vertical $x = x_o$. Let $B$, $B'$ be the points $(0, a - y_o)$, $(0, a' - y_o)$, which are most easily constructed by transferring the distances $TA$, $TA'$. Then $GA$, $GA'$ and $TB$, $TB'$ are the tangents to the trajectories at $G$ and $T$, respectively. The times of flight are the lengths $GA$, $GA'$, if the unit of length is made $v_o$ in reading them.




The intercepts of the trajectories on the $x$-axis are found by continuing $GA$, $GA'$ until they intersect the circle $(0, R, R)$ in $C$, $C'$ and then projecting $C$, $C'$ onto the $x$-axis at $D$, $D'$.

Through the intersection of $x=\frac{1}{4}GD$ and $GA$ draw a horizontal line. Its intersection with $x=\frac{1}{2}GD$ is the summit $E$ of the trajectory.

Once $D$ and $E$ have been found, the familiar projective construction employed in mechanical drawing can be used to find any number of points on the trajectory as the intersections of corresponding rays of a pencil of equally spaced vertical lines and a pencil of rays drawn from the summit to equally spaced points on the vertical through $G$ and that through $D$.

The envelope of the trajectories is the parabola which has vertex $(0, \frac{1}{2}R)$, has focus $(0, 0)$, and passes through $(R, 0)$. Points within or on the envelope are within range and can be hit. Points outside the envelope cannot be hit.

Let $(\xi, \eta)$ be the point of contact of the lower trajectory with the envelope. Let $GA$ meet $y=R$ in $F$. Let the line of departure complementary to $GA$ meet the vertical through $F$ at $H$. Then $\xi$ is the abscissa of $F$ and $\eta$ is $\frac{1}{2}HF$.

The circle $(0, y_0-a, GA)$ is a curve of constant fuse setting. As the point $(x_0, y_0)$ describes a trajectory, the center of this circle behaves like a particle

freely falling from rest at the gun. When the trajectories have been drawn, these circles can be used to parametrize them. Simultaneous positions of shells fired simultaneously on two trajectories can be compared by means of these circles.

There are other interesting loci connected with the problem. The envelope of the curves of equal fuse setting is the same as the envelope of the trajectories, the point of contact being imaginary if $v_0 t < R$. The locus of summits is the ellipse with vertices $(\pm\frac{1}{2}R, \frac{1}{4}R)$, $(0, 0)$, $(0, \frac{1}{2}R)$.

## A NOTE ON INTERPOLATION

P. M. Hummel, University of Alabama

In a recent article* the following theorem was proved:

*Let $f(x)$ be a function which together with its first three derivatives is continuous throughout the interval $a \leqq x \leqq b$. Moreover let $f''(x)$ and $f'''(x)$ be of constant sign throughout the interval. If $a$, $b$, $f(a)$, $f(b)$, and $x$ are given and $f(x)$ is determined by linear interpolation, the error† satisfies the inequalities*

$$\frac{(b-x)(x-a)}{(b-a)^2}[f(b)-f(a)-(b-a)f'(a)]$$
$$\leqq \textit{error} \leqq \frac{(b-x)(x-a)}{(b-a)^2}[(b-a)f'(b)-f(b)+f(a)],$$

*or these inequalities reversed, according as $f'''(x)$ is positive or negative.*

While the inequalities above give rather sharp limits for the error, they look somewhat formidable. The purpose of this note is to give alternate limitations for the error.

Using Taylor's finite expansion, one readily verifies that

$$f(b)-f(a)-(b-a)f'(a) = \tfrac{1}{2}(b-a)^2 f'(\theta_1), \qquad a < \theta_1 < b,$$
$$(b-a)f'(b)-f(b)+f(a) = \tfrac{1}{2}(b-a)^2 f'(\theta_2), \qquad a < \theta_2 < b.$$

When these results are substituted into the inequalities in the theorem, and obvious simplifications are made, we get

$$\tfrac{1}{2}(b-x)(x-a)f''(\theta_1) \leqq \textit{error} \leqq \tfrac{1}{2}(b-x)(x-a)f''(\theta_2), \quad a < \theta_1, \theta_2 < b. \tag{1}$$

Since $f''(x)$ is continuous, there exists a value $\theta$, between $\theta_1$ and $\theta_2$, such that

$$\textit{error} = \tfrac{1}{2}(b-x)(x-a)f''(\theta), \qquad a < \theta < b. \tag{2}$$

Equation (2) is an exact expression for the error, and, by use of the extreme values of $f''(x)$ in the interval $(a, b)$, gives limits for the error. This expression can be simplified further, though slightly weakened, by using the fact that the

* P. M. Hummel, The accuracy of linear interpolation, this Monthly, vol. 53, 1946, p. 364.

† The error due to interpolation is defined as *error = interpolated value − true value.*

quadratic factor $(b-x)(x-a)$ has a maximum value of $\frac{1}{4}(b-a)^2$. This fact enables us to write

$$|\, error\,| \leqq \frac{(b-a)^2}{8}\,|\, f''(\theta)\,|, \qquad a<\theta<b. \tag{3}$$

The relations (1), (2), and (3) are usually not as sharp as the inequalities given in the theorem, but in many cases are easier to use and are certainly easier to remember.

Finally it should be noted that all these results, including the inequalities in the theorem, do not include the error due to rounding off. Thus if $f(a)$ and $f(b)$ are not exact values but contain an error due to rounding off, then the interpolated value of $f(x)$ likewise contains an error due to rounding off. It is easy to show, however, that the rounding off error in $f(x)$ lies between the rounding off errors in $f(a)$ and $f(b)$.

### NOTE

It has been pointed out by Paul D. Thomas that the method of evaluating $\int_0^\infty e^{-x^2}dx$ presented by H. F. Sandham, this MONTHLY, vol. 53, 1946, p. 587, had been found previously, and appears, for instance, in W. E. Byerly, *Integral Calculus*, second edition, 1888, p. 99.—E. F. B.

## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

***All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.***

### DERIVATION OF THE EQUATIONS OF CONICS

F. HAWTHORNE, Hofstra College

In most analytic geometry texts the equations of conics are derived by use of the distance formula. Alternative derivations which do not involve radicals seem to provoke some student interest. The procedure can be illustrated by the following derivation of the equation of the ellipse.

Let the foci be the points $(\pm c, 0)$; and let $R$ be the distance from a point on the curve to the left-hand focus and $r$ be the corresponding distance to the right hand focus. Then by definition:

$$R^2 = y^2 + (x+c)^2, \quad \text{and} \tag{1}$$

$$r^2 = y^2 + (x-c)^2. \tag{2}$$

Adding and subtracting we have:

$$R^2 + r^2 = 2(y^2 + x^2 + c^2), \quad \text{and} \tag{3}$$

$$R^2 - r^2 = 4cx = (R + r)(R - r). \tag{4}$$

But by definition of the ellipse

$$R + r = 2a. \tag{5}$$

Hence from (4) and (5) we have that

$$R - r = 2cx/a. \tag{6}$$

Adding and subtracting (5) and (6) gives:

$$R = a + cx/a, \quad \text{and} \tag{7}$$

$$r = a - cx/a. \tag{8}$$

Putting the results of (7) and (8) into (3) gives:

$$a^2 + \frac{c^2x^2}{a^2} = c^2 + x^2 + y^2, \tag{9}$$

from which the standard equation is found by putting $b^2 = a^2 - c^2$. The expressions for the focal radii in terms of the eccentricity may be pointed out in equations (7) and (8).

## A SUBSTITUTION FOR SOLVING TRIGONOMETRIC EQUATIONS

R. W. WAGNER, Oberlin College

The substitution $t = \tan(\theta/2)$ which is used in the calculus to integrate some expressions containing both $\sin\theta$ and $\cos\theta$ can also be used to advantage in the solution of equations in which the unknown appears only as the argument of one or more trigonometric functions. The usefulness of this substitution arises from the fact that the trigonometric functions of $\theta$ are simple rational functions of $t$. From

$$\tan(\theta/2) = t,$$

one gets, by using the double angle formula,

$$\tan\theta = \frac{2t}{(1 - t^2)},$$

and hence that

$$\sin\theta = \frac{2t}{\sqrt{(2t)^2 + (1 - t^2)^2}} = \frac{2t}{1 + t^2}.$$

There is no ambiguous sign in this last equation. For $\sin\theta$ and $t$ have the same algebraic sign. Also, one gets

$$\cos\theta = \frac{1-t^2}{1+t^2}.$$

The neatness of this method of solution will be shown by applying it to an example:

EXAMPLE. *Find the angles between* $-\pi$ *and* $\pi$ *radians such that*

$$5\sin\theta - 21\cos\theta = 10.$$

On making the above substitutions one gets

$$\frac{10t}{1+t^2} - 21\frac{1-t^2}{1+t^2} = 10.$$

When this equation is cleared of fractions and terms collected the result is

$$11t^2 + 10t - 31 = 0.$$

Hence,

$$t = \frac{-10 \pm \sqrt{100+1364}}{22}$$
$$= 1.285 \quad \text{or} \quad -2.194.$$

From a table of trigonometric functions in radian measure one finds that

$$\theta/2 = 0.910 \quad \text{or} \quad -1.143,$$

and that

$$\theta = 1.820 \quad \text{or} \quad -2.286 \text{ radians}.$$

The main advantage of this method of solving the given equation lies in the fact that there is no difficulty in determining the quadrant in which an angle may lie. It is easy to bear in mind that negative values of $t$ correspond to negative angles and positive values of $t$ correspond to positive angles.

## EVALUATION OF A TRIGONOMETRIC INTEGRAL

J. P. HOYT, United States Naval Academy

The purpose of this note is to present a quick and easy method for evaluating $\int \sin^{2m} x \cos^{2n} x dx$, where $m$ and $n$ are positive integers. No table of formulas is required.

To illustrate the method consider the integration of $\int \sin^4 x \cos^6 x dx$. Since $\sin nx = (1/2i)(e^{inx} - e^{-inx})$ and $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$, we let $y = e^{ix}$, and find that $\sin nx = (1/2i)(y^n - 1/y^n)$ and $\cos nx = \frac{1}{2}(y^n + 1/y^n)$. Then

$$\sin^4 x = \frac{\left(y - \frac{1}{y}\right)^4}{2^4} = \frac{y^4 - 4y^2 + 6 - \frac{4}{y^2} + \frac{1}{y^4}}{2^4}.$$

If we notice that the powers of $y$ decrease by 2 from term to term, we may write this expression in the following symbolic form:

$$\sin^4 x = \frac{1 - 4 + 6 - 4 + 1}{2^4}.$$

Now instead of expanding $\cos^6 x$ in the same manner and then forming the product, it is easier and quicker to multiply the expansion of $\sin^4 x$ by the expression for $\cos x$ six times in succession. For the coefficients of each expansion are obtained from the coefficients of the preceding one in the same manner as the coefficients of $(a+b)^{n+1}$ are obtained from the coefficients of $(a+b)^n$; that is, in the form used in Pascal's Triangle.

The successive expansions appear as follows in the symbolic notation:

$$\begin{aligned}
\sin^4 x &= (1 - 4 + 6 - 4 + 1)/2^4 \\
\sin^4 x \cos x &= (1 - 3 + 2 + 2 - 3 + 1)/2^5 \\
\sin^4 x \cos^2 x &= (1 - 2 - 1 + 4 - 1 - 2 + 1)/2^6 \\
\sin^4 x \cos^3 x &= (1 - 1 - 3 + 3 + 3 - 3 - 1 + 1)/2^7 \\
\sin^4 x \cos^4 x &= (1 + 0 - 4 + 0 + 6 + 0 - 4 + 0 + 1)/2^8 \\
\sin^4 x \cos^5 x &= (1 + 1 - 4 - 4 + 6 + 6 - 4 - 4 + 1 + 1)/2^9 \\
\sin^4 x \cos^6 x &= (1 + 2 - 3 - 8 + 2 + 12 + 2 - 8 - 3 + 2 + 1)/2^{10}
\end{aligned}$$

or

$$\sin^4 x \cos^6 x = \left(y^{10} + 2y^8 - 3y^6 - 8y^4 + 2y^2 + 12 + \frac{2}{y^2} - \frac{8}{y^4} - \frac{3}{y^6} + \frac{2}{y^8} + \frac{1}{y^{10}}\right) \Big/ 2^{10}.$$

Regrouping the last expression we obtain:

$$\begin{aligned}
\sin^4 x \cos^6 x = \Bigg[\left(y^{10} + \frac{1}{y^{10}}\right) + 2\left(y^8 + \frac{1}{y^8}\right) - 3\left(y^6 + \frac{1}{y^6}\right) \\
- 8\left(y^4 + \frac{1}{y^4}\right) + 2\left(y^2 + \frac{1}{y^2}\right) + 12\Bigg] \Big/ 2^{10}
\end{aligned}$$

which can be rewritten:

$$\sin^4 x \cos^6 x = (\cos 10x + 2\cos 8x - 3\cos 6x - 8\cos 4x + 2\cos 2x + 6)/2^9.$$

So immediately we have the final result:

$$\int \sin^4 x \cos^6 x dx = \left[\frac{\sin 10x}{10} + \frac{\sin 8x}{4} - \frac{\sin 6x}{2} - 2\sin 4x + \sin 2x + 6x\right] \Big/ 2^9 + C.$$

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 759 [1947, 107]. *Corrected. Proposed by Theodore Running, Ann Arbor, Michigan*

Show that $x^n-(x-a)^n$ can be expressed as the difference of two squares in at least one way for all positive integral values of $n$, $x$, and $a$, $a$ less than $x$, not counting the obvious way when $n$ is even.

E 766. *Proposed by H. E. G. P.*

Professor Umbugio, who was introduced to our readers in our foregoing April number, invented a remarkable scheme for reviewing books. He divides the time he allows himself for reviewing into three fractions, $\alpha$, $\beta$, and $\gamma$. He devotes the fraction $\alpha$ of his time to a deep study of the title page and the jacket. He devotes the fraction $\beta$ to a spirited search for his name and for quotations from his works. Finally, he spends the fraction $\gamma$ of his alloted time in a proportionally penetrating perusal of the remaining text. Knowing his characteristic taste for simple and direct methods, we cannot fail to be duly impressed by the differential equations on which he bases his scheme:

$$(1) \qquad dx/dt = y - z, \qquad dy/dt = z - x, \qquad dz/dt = x - y.$$

He considers a system of solutions $x$, $y$, $z$ which is determined by initial conditions depending on a (small) parameter $\epsilon$, independent of $t$. Therefore $x$, $y$, and $z$ depend on both $t$ and $\epsilon$, and we appropriately use the notation:

$$(2) \qquad x = f(t, \epsilon), \qquad y = g(t, \epsilon), \qquad z = h(t, \epsilon).$$

The functions (2) satisfy the equations (1) and the initial conditions

$$(3) \qquad f(0, \epsilon) = 1/3 - \epsilon, \qquad g(0, \epsilon) = 1/3, \qquad h(0, \epsilon) = 1/3 + \epsilon.$$

Professor Umbugio defines his important fractions $\alpha$, $\beta$, and $\gamma$ by

$$\lim_{\epsilon\to 0} f(2, \epsilon) = \alpha, \qquad \lim_{\epsilon\to 0} g(5, \epsilon) = \beta, \qquad \lim_{\epsilon\to 0} h(279, \epsilon) = \gamma.$$

Deflate the Professor! Find $\alpha$, $\beta$, $\gamma$ without much numerical computation.

E 767. *Proposed by Milton Schwartz, Temple University*

From any point $P$ on $BC$ of parallelogram $ABCD$ line segments are drawn to $A$ and $D$. From any point $Q$ on $AD$ line segments are drawn to $B$ and $C$. Through the intersections of these four segments ($PA$, $PD$, $QB$, $QC$) a line is drawn meeting $AB$ in $R$ and $CD$ in $S$. Prove that $BR$ equals $DS$.

E 768. *Proposed by Irving Kaplansky, University of Chicago*

A number $n$ has the property that for any $p<q<n$,

$$S = p + (p+1) + \cdots + q$$

is never divisible by $n$. Show that this is true if and only if $n$ is a power of 2.

E 769. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a plane quadrangle $ABCD$, the perpendicular at $A$ to side $AB$ cuts the opposite side $CD$ in $M$, and the perpendicular at $A$ to side $AD$ cuts the opposite side $BC$ in $N$. Show that the radical axis of the circles described on $AM$ and $AN$ as diameters coincides with the tangent at $A$ to the equilateral hyperbola circumscribing the quadrangle.

E 770. *Proposed by Norman Miller, Queen's University, Canada*

A function $f(x)$ has the following definition, $n$ being a positive integer,

$$f(x) = x^n \sin \pi/x, \qquad x \neq 0,$$
$$f(0) = 0.$$

Show that, if $n=2m$ or $2m+1$ ($m$ a positive integer), then at $x=0$ the function possesses a derivative of the $m$th but no higher order, and that, at $x=0$, $f^{(m)}(x)$ is continuous or discontinuous according as $n=2m+1$ or $n=2m$.

## SOLUTIONS

### A Characteristic Root

E 733 [1946, 394]. *Proposed by E. D. Schell, Arlington, Va.*

Show that a square matrix of order $n$, whose elements form a magic square, has a characteristic root equal to $n(n^2+1)/2$.

*Solution by the Proposer.* Denote the vector consisting of a row of $n$ ones by $u$. Since the sums $s$ of the $n$ columns of the matrix $A$ are equal, $uA=us$. This may be written as $u(A-sI)=0$, where $I$ is the identity matrix. Hence $s$ is a characteristic root of $A$.

The sum of the first $n^2$ integers is $n^2(n^2+1)/2$, which is the sum of all $n$ columns of the matrix. Hence the sum of a single column is $s=n(n^2+1)/2$.

Also solved by D. W. Alling, Alfred Brauer, and Daniel Finkel.

Alfred Brauer points out that this problem is a special case of the following theorem proved in his paper, *Limits for the characteristic roots of a matrix* (Duke Mathematical Journal, vol. 13 (1946), pp. 387–395).

Let $A=(a_{\kappa\lambda})$ be a square matrix of order $n$, $R_\kappa=\sum_{\lambda=1}^{n}|a_{\kappa\lambda}|$, and $R=\max(R_1, R_2, \cdots, R_n)$. The absolute value of one of the characteristic roots of $A$ equals $R$ only if

$$(1) \qquad R_1 = R_2 = \cdots = R_n = R$$

and

$$\arg(a_{\kappa\lambda}) = \phi + \phi_\kappa - \phi_\lambda, \tag{2}$$

where $\phi$ and $\phi_1, \phi_2, \cdots, \phi_n$ are arbitrary real numbers. If (1) and (2) hold, then $Re^{i\phi}$ is a characteristic root.

**Non-Linear Spring Problem**

E 734 [1946, 394]. *Proposed by Henry Scheffé, University of California at Los Angeles*

A body in the form of a plate is supported by a smooth horizontal surface, which we take as the $x, y$-plane. A vertical peg is fastened to the body through its center of gravity. Two similar springs of natural length $l$ run from this peg to pegs fixed in the plane, one at $(0, l)$, and the other at $(0, -l)$, the springs being parallel to the plane. The body is released from rest in a position where its center of gravity is on the $x$-axis. Assuming the forces exerted by the springs proportional to their elongations, and the masses of the springs negligible, show that for small oscillations the frequency is proportional to the amplitude.

I. *Solution by P. T. Bateman, Yale University.* Let $k$ be the constant of each spring, $m$ the mass of the body, and $v$ the velocity of the body. If the body is at the point $(x, 0)$, the force of each spring is

$$k\{(x^2 + l^2)^{1/2} - l\}.$$

The components in the $y$-direction cancel, but the components in the $x$-direction add up to

$$2k\{(x^2 + l^2)^{1/2} - l\} \frac{-x}{(x^2 + l^2)^{1/2}} = \frac{-2kx^3}{\{(x^2 + l^2)^{1/2} + l\}(x^2 + l^2)^{1/2}} \cdot$$

Thus for small $x$ we have approximately

$$mv\frac{dv}{dx} = \frac{-2kx^3}{2l^2} = \frac{k}{l^2}x^3.$$

Integration gives $v^2 = k(a^4 - x^4)/(2ml^2)$, where $a$ is the amplitude. Thus the period corresponding to the amplitude $a$ is

$$T_a = 4\int_0^a \frac{dx}{\{k(a^4 - x^4)/(2ml^2)\}^{1/2}} = 4l\left(\frac{2m}{k}\right)^{1/2}\int_0^a \frac{dx}{(a^4 - x^4)^{1/2}} \cdot$$

Hence

$$T_{ta} = 4l\left(\frac{2m}{k}\right)^{1/2}\int_0^{ta} \frac{dx}{(t^4a^4 - x^4)^{1/2}} = 4l\left(\frac{2m}{k}\right)^{1/2}\int_0^a \frac{tdw}{(t^4a^4 - t^4w^4)^{1/2}} = \frac{T_a}{t} \cdot$$

II. *Solution by Elmer Latshaw, ACF-Brill Motors Co., Philadelphia, Pa.* Let $\theta$ be the angle between an extended spring and its initial or free position, and $R/2$ the scale of a spring. Then the resultant of the tensions in the stretched springs is along the $x$-axis and is equal to

$$A = \frac{\sqrt{2}\, c}{T\phi^{1/2}} = \frac{4\sqrt{2}\, cf}{\phi^{1/2}},$$

where $f$ is the frequency. Thus, for small oscillations, the frequency is proportional to the amplitude.

Also solved by W. J. Nemerever and T. H. Simester.

This problem furnishes a simple example on small vibrations in which the frequency is *not* independent of the amplitude.

**Isosceles $n$-Points**

E 735 [1946, 394]. *Proposed by Paul Erdös, Stanford University*

Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in space which cannot be so arranged?

*Solution by L. M. Kelly, University of Missouri.* We shall refer to a set of $n$ points, each of whose triples forms an isosceles triangle, as an *isosceles n-point.* We propose first to establish that some point of a plane isosceles 6-point must have the property that at least three of the segments joining it to the remaining five points are equal. That is, that at least three of the segments radiating from some vertex are equal.

Certainly some vertex, say 1, has at least two equal radiating segments. Let $12=13=a$, $14=x$, $15=y$, $16=z$, and let us suppose that no vertex has the desired property. Then $x\neq a$, $y\neq a$, $z\neq a$. Furthermore, from triangle 124, we conclude that either $24=a$ or $24=x$. Similarly, from triangle 134, $34=a$ or $34=x$. Both 34 and 24 cannot equal $x$, since then vertex 4 would satisfy the desired condition. Thus either 24 or 34 must equal $a$. No loss of generality is entailed if we assume $34=a$. Then, in triangle 135, 35 must equal $y$. Again, triangle 125 implies that $25=a$, and triangle 136 implies that $36=z$. Finally, triangle 126 presents us with a dilemma, since 26 must equal $a$ or $z$, and in either case we have a contradiction of the assumption. Thus at least one vertex must have the announced property. That is, an isosceles 6-point must possess an isosceles triangle and its circumcenter.

We now wish to prove that a plane isosceles 6-point cannot contain certain special configurations.

1. *Three linear points.* Suppose 1, 2, 3 linear, with $12=23=a$. Then 24 must equal $a$, otherwise angles 124 and 324 would both be base angles of isosceles triangles, which is impossible. This implies that angle 143 is a right angle and triangle 134 is right isosceles. Any other point 5, which together with 1, 2, 3, 4 forms an isosceles 5-point, must be the reflection of 4 in 13. We thus see that any plane isosceles 5-point containing a linear triple forms the vertices and center of a square. Clearly, then, no plane isosceles 6-point with a linear triple is possible.

2. *Equilateral triangle and circumcenter.* Suppose 4 the circumcenter of equi-

lateral triangle 123. Set $12=13=23=a$, $14=24=34=b$. Consider a fifth point 5 forming, if possible, with 1, 2, 3, 4 an isosceles 5-point. Now $15\neq 25$, since otherwise 3, 4, 5 would be linear. From triangle 125, either $15=a$ or $25=a$. We may assume $25=a$. Similar considerations show that either $35=a$ or $15=a$, and in either case there results a linear triple.

3. *Two equilateral triangles with common base.* Let $12=23=13=43=14=a$, $24=b$. Suppose, as before, a fifth point 5. Now $15=35$ would imply a linear triple. So, from triangle 135, we conclude either $15=a$ or $35=a$. Assume, without loss, $35=a$. Also $45\neq 25$ and, hence, either $45=b$ or $25=b$. We may again assume $45=b$. Triangle 145 then demands that $15=b$. Now triangle 125 provides a contradiction, since 25 must equal $a$ or $b$, which is impossible.

4. *Configuration with* $12=13=23=34=a$, $14=24=b$, $b<a$. Consider a fifth point 5. Now $15\neq 25$, since otherwise 3, 4, 5 would be linear. Therefore, from triangle 125, either $15=a$ or $25=a$. Assume, without loss, that $25=a$. Now, from triangle 245, either $45=a$ or $45=b$. If $45=a$, 5 would be the reflection of 3 in 42, and a simple computation shows that triangle 145 is not isosceles. Thus $45=b$. Triangle 345 compels $35=b$, since if $35=a$ configuration 3 would result. Finally, $15=b$. Now the 5-point is completely labeled and can easily be seen to be impossible of realization in the plane.

5. *Configuration with* $12=13=23=34=a$, $14=24=b$, $b>a$. Consider a fifth point 5. Now $15\neq 25$, since otherwise configuration 1 would result, and $15=a$ or $25=a$. Suppose, without loss, that $25=a$. Observe that $35\neq a$, since otherwise configuration 3 would result. Thus, since $15\neq a$, we have, from triangle 135, that $35=15$. From this it follows that $45\neq a$, since otherwise angle 135 would be obtuse. From triangle 245, 45 must equal $b$. Hence $35=15=b$. This causes 1, 3, 4, 5 to form configuration 4.

*Characterization of plane isosceles 6-point.* From our opening result we know that a plane isosceles 6-point must contain the circumcenter of one of the isosceles triangles. Let $12=13=14=a$, $24=34=b$, $23=c$. We know that $a\neq b$, since otherwise configuration 3 would result; $a\neq c$, since otherwise configuration 5 would result; $c\neq b$, since otherwise configuration 2 would result. Also, $25\neq 35$, since otherwise configuration 1 would result. Therefore, from triangle 235, we see that either $25=c$ or $35=c$. Suppose, without loss, that $35=c$. We now maintain that 15 must equal $a$. For suppose not. Then, from triangle 135, $15=c$, and from triangle 125, $25=a$. We now try to label 45. Triangle 245 implies that $45=b$ or $a$; triangle 345 implies that $45=b$ or $c$; triangle 145 implies that $45=a$ or $c$. These conditions are mutually impossible. Thus $15=a$. Now $45\neq b$, since otherwise 1, 4, 3 would be linear. Hence, from triangle 345, $45=c$. Finally, $25=b$. Examination of this configuration reveals that 2, 3, 4, 5 are four of the five vertices of a regular pentagon with 1 as its center. We can now conclude that the only possible sixth point must be the remaining vertex, and we have characterized plane isosceles 6-points as the vertices and center of a regular pentagon.

A necessary consequence of the above characterization of plane isosceles 6-points is that there can be no plane isosceles 7-points.

In three space the above kind of analysis would become much more difficult,

However, we can show that the answer to the question for three space cannot be less than nine. For consider a regular pentagon and its circumcenter. Take a point above the plane of the pentagon directly over the center and at a distance equal to the radius of the pentagon from it. Also consider the reflection of this point in the plane. These eight points form a space isosceles 8-point.

*Editorial Note.* Associated with the problem of characterizing the maximum isosceles $n$-points in euclidean spaces is the characterization of the maximum two distance sets in euclidean spaces, that is, the characterization of the maximum number of points in a euclidean space such that the various distances consist of no more than two distinct numbers. Kelly has shown that the maximum two distance set in the plane consists of five points which form the vertices of a regular pentagon. This set, along with its circumcenter, constitutes a plane isosceles 6-point, and, in the above solution, Kelly shows this set to be the maximum plane isosceles $n$-point. From this one might be led to conjecture that the maximum isosceles $n$-point in euclidean $r$-dimensional space would be the vertices, constituting a two distance set, of an $r$-dimensional polytope inscribed in a sphere, along with the center of the sphere. H. S. M. Coxeter asserts that the polytopes of this kind in 3, 4, 5 dimensions which he believes to be the best possible are those called "pure Archimedean" in his first published paper: *Proc. Camb. Phil. Soc.*, 24 (1928), pp. 1–9. (Coxeter remarks that these were described earlier by Thorold Gosset, *Messenger of Math.*, 29 (1900), pp. 43–48.) The one in 3 dimensions is the equilateral triangular prism with square side-faces. The one in 4 dimensions has 10 vertices, the midpoints of the 10 edges of the regular simplex. The one in 5 dimensions has 16 vertices, which are alternate vertices of the 5-dimensional hyper-cube. A polytope of this kind in 6 dimensions is the subject of a more recent paper (with some fine pictures) by Coxeter: *The polytope $2_{21}$ whose 27 vertices correspond to the lines on the general cubic surface*, Amer. J. of Math., 62 (1940), pp. 457–486.

Of course, the concluding paragraph of Kelly's solution shows that the above conjecture is not true. Nevertheless, Coxeter's examples for the higher spaces probably constitute the best results yet known for these dimensions.

### A Property of $k$ Positive Integers

E 736 [1946, 462]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots < a_k \leqq n$, where $k > [(n+1)/2]$, be $k$ positive integers. Then $a_i + a_j = a_r$ is solvable. (*Cf.* No. 3739 [1937, 120].)

*Solution by Leo Moser, University of Toronto.* The $k-1$ positive integers $a_2 - a_1$, $a_3 - a_1$, $\cdots$, $a_k - a_1$, are clearly all distinct. These, together with the $k$ given distinct $a$'s, give $2k-1 > n$ positive integers, each not greater than $n$. Hence at least one integer is common to both sets, so that at least once $a_r - a_1 = a_i$, or $a_i + a_1 = a_r$.

Also solved by P. T. Bateman, P. A. Clement, William Gustin, J. B. Kelly, and the proposer. The proposer pointed out that the sequence

$$[n/2]+1,\ [n/2]+2, \cdots, n$$

shows that for $k=[(n+1)/2]$ the result is false.

**Divergence of a Series**

E 737 [1946, 462]. *Proposed by V. L. Klee, Jr., University of Virginia*

Establish the divergence of the series $\sum_{n=1}^{\infty} n^{-a} \cos(b \log n)$ for all values of $a \leqq 1$, regardless of the value of $b$.

*Solution by Norman Miller, Queen's University, Canada.* If $b=0$, the series clearly diverges. Take $b \neq 0$. Then $\lim_{n\to\infty} |b| \log n = \infty$ and $\lim_{n\to\infty} [b \log(n+1) - b \log n] = 0$. Hence, given any positive integer $M$, we can find a positive integer $p$ so large that between $(p-1/3)\pi$ and $(p+1/3)\pi$ there are at least $M$ consecutive values of $|b| \log n$. For all of these values $\cos(b \log n)$ has the same sign and $|\cos(b \log n)| > 1/2$. Hence the sum of the corresponding $M$ consecutive terms of the given series is in absolute value greater than $\frac{1}{2}\sum_{n=q}^{q+m} n^{-a}$ for some integer $q$. Since this sum can be made arbitrarily large by a proper choice of $M$, the series fails to satisfy the Cauchy criterion for convergence.

Also solved by P. T. Bateman, Max Wyman, and the proposer. Wyman utilized the Cauchy integral test; Bateman and the proposer showed, somewhat as above, that the Cauchy criterion for convergence is not satisfied. Bateman pointed out that the divergence of the given series implies the well known divergence of $\sum_{n=1}^{\infty} n^{-z}$, $R(z) \leqq 1$. This is easily seen since, for real $b$, $\sum n^{-a} \cos(b \log n) = \sum R\{n^{-(a+ib)}\}$.

**A Property of the Monge Point**

E 738 [1946, 462]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the four spheres passing through the Monge point and the nine point circles of the faces of a tetrahedron are equal to each other.

*Solution by the Proposer.* Let $O$, $G$, $\Omega$ be the circumcenter, centroid, and Monge point of the tetrahedron $ABCD$, and let $M$, $N$, $P$, $Q$, $S$, $T$ be the midpoints of the edges $BC$, $CD$, $DB$, $AB$, $AC$, $AD$.

The symmetrics $A'$, $B'$, $C'$, $D'$, with respect to $G$, of the vertices $A$, $B$, $C$, $D$ are vertices of tetrahedra $A'MNP$, $B'NST$, $\cdots$, inversely homothetic to $ABCD$ in the ratio $-1/2$. The tetrahedron $A'MNP$ is symmetric, with respect to $G$, to the tetrahedron $ASTQ$, whence the circumsphere of the first passes through $\Omega$, the symmetric, with respect to $G$, of the circumcenter $O$, and the points $A'$ and $\Omega$ are diametrically opposite on this sphere. The theorem now follows.

*Otherwise,* let $A''$ be the point diametrically opposite to $A$ on the circumsphere. The $A'\Omega$ is the transform of $AA''$ under the homothety $(G_a, -1/2)$, where $G_a$ is the centroid of the face $BCD$. Again the theorem follows.

*Editorial Note.* The four equal spheres under consideration have radii equal to half that of the circumsphere, and are anologous to the circles $B'OC'$, $C'OA'$,

$A'OB'$ of a triangle $ABC$, where $O$ is the circumcenter and $A'$, $B'$, $C'$ are the midpoints of the sides $BC$, $CA$, $AB$.

### Inscribed Ellipse

E 739 [1946, 462]. *Proposed by L. M. Kelly, University of Missouri*

An ellipse inscribed in the triangle $ABC$ is tangent to $AB$ at $D$. Show that the midpoints of $CD$ and $AB$ are collinear with the center of the ellipse.

I. *Solution by H. E. Fettis, Dayton, Ohio.* Let $P$ and $Q$ be the midpoints of $AB$ and $CD$, and let $O$ be the center of the ellipse. Draw the diameter $DO$ of the ellipse and let the tangent at the other end, $G$, of this diameter cut $AC$ in $M$ and $BC$ in $N$.

$AN$ and $BM$ intersect in a point $S$ which lies on $DG$ and also on $CP$. Then, if $CG$ intersects $AB$ in $H$, we have, from similar triangles,

$$AD/GN = AS/SN = AB/MN,$$
$$BH/GN = BC/CN = AB/MN.$$

Therefore $AD=BH$, or $P$ is the midpoint of $DH$, so that $OP$ is parallel to $GH$ and bisects $CD$ at $Q$.

*Otherwise.* The result may be considered as a special case of the well known fact that the centers of all conics inscribed in a given quadrilateral are collinear with the midpoints of the diagonals of the quadrilateral. In the special case when two sides of the quadrilateral coincide, we have that the centers of all conics inscribed in a given triangle, and touching one side of the triangle at a fixed point, are collinear with the midpoint of that side and the midpoint of the line joining the opposite vertex with the fixed point of contact. This last statement is equivalent to the proposition stated in the problem.

II. *Solution by Joseph Rosenbaum, Milford School, Connecticut.* Project the ellipse orthogonally into a circle. We are then confronted with the corresponding theorem for a circle inscribed in a triangle. Preserving the lettering, consider the triangle $A'B'C$, where $A'$, $B'$ are the intersections with $CA$ and $CB$ of a line parallel to $AB$ and tangent to the incircle $(O)$ of $ABC$ at $E'$. Since $(O)$ is the excircle of triangle $A'B'C$ for the side $A'B'$, the intersection $E$ of $CE'$ with $AB$ is the point of contact of the excircle of triangle $ABC$ for side $AB$. Hence the midpoint $M$ of $AB$ is also the midpoint of $DE$. This, together with the fact that $O$ is the midpoint of $DE'$, implies the conclusion of the problem.

Also solved by E. F. Allen, J. C. Currie, J. M. Kingston, J. H. Simester, the proposer, and jointly by J. H. Butchart, Ressa Olson, Jean Rowe, and Jacquelin Wilbanks. The joint solution and the solutions by Currie and the proposer employed orthogonal projection as in solution II. The remaining solutions were analytic, Allen establishing the result for any central conic touching the three sides of a triangle.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4244. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider squares of $2n$ digits which can be formed by bringing together two consecutive $n$-digit numbers. Show that in every system of numeration there is at least one unending sequence of such squares. Show also that, if $n$ and $B$ are given and if $x^2$ is such a square, then $y^2$ is another where $x+y=B^n+1$ (provided squares with initial zeros are admitted). As a particular example let $B=12$, $n=3$.

4245. *Proposed by J. H. Butchart, Arizona State College*

The envelopes of two families of lines, $PQ$, $PQ'$, making angles of $\pm 30°$ respectively with the tangents to a deltoid at their points of contact $P$ are two deltoids, larger than the given one in the ratio $3^{1/2}:1$. Show also that $PQ=PQ'$, where $Q$, $Q'$ are the points of contact of $PQ$, $PQ'$ with the respective envelopes, and that the angles between the cusp tangents of the envelopes and the included cusp tangent of the given deltoid are $\pm 10°$.

4246. *Proposed by J. A. Greenwood*

Evidently $y'=x'Dy$, $y''=(x''D+x'^2D^2)y$, $\cdots$, where $D\equiv d/dx$ and primes indicate differentiation with respect to $t$. Find an explicit expression for $y^{(n)}$.

4247. *Proposed by Leon Recht and Martin Rosenman, New York City*

The sequence

$$\{a\} = 1;\, 2;\, 1,\, 2;\, 2,\, 1,\, 2;\, 1,\, 2,\, 2,\, 1,\, 2;\, \cdots$$

is formed by writing down in succession sets of terms starting with 1 and such that every subsequent set is obtained from the preceding set by the substitution $1\rightarrow 2;\, 2\rightarrow 1,\, 2$. Show that: (1) the $n$th term $a_n$ is

$$[k(n+1)] - [kn],$$

where $k=\frac{1}{2}(\sqrt{5}+1)$ and $[x]$ is the greatest integer not exceeding $x$; (2) Each set of terms in $\{a\}$, beginning with the third, is a repetition of the terms of the two preceding sets; (3) $\{a\}$ consists of sets of 2's separated by 1's, such that there are $a_n$ 2's in the $n$th set of 2's.

4248. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $ABCD$ be a given tetrahedron. Place a sphere $(S)$ of given radius in such

a manner that the volume of the polar tetrahedron of $ABCD$ with respect to $(S)$, will be a minimum.

## SOLUTIONS

### The Tangential Triangle

4179 [1945, 523]. *Proposed by J. R. Musselman, Western Reserve University*

The poles of the medians of a triangle $A_1A_2A_3$ as to its circumcircle are three points of a line; those points where the external bisectors of the angles of the tangential triangle of $A_1A_2A_3$ meet the opposite sides of this tangential triangle.

*Solution by H. E. Fettis, Dayton, Ohio.* The problem is not quite accurately stated unless $A_1A_2A_3$ is an acute-angled triangle; for the statement to hold in general, the words "external bisectors" should be replaced by "angle bisectors (of the tangential triangle) not passing through the circumcenter of $A_1A_2A_3$."

Let the vertices of the tangential triangle be designated by $B_1$, $B_2$, $B_3$. Then $B_1M_1$, where $M_1$ is the midpoint of $A_2A_3$, passes through the circumcenter of $A_1A_2A_3$, and $B_1$ is the pole of $A_2A_3$. If $P_1$ is the pole of $A_1M_1$, then $B_1M_1$ and $B_1P_1$ are polar conjugate lines relative to $B_1A_2$ and $B_1A_3$, and are therefore harmonically separated from $B_1A_2$ and $B_1A_3$.

Therefore, since $B_1M_1$ is a bisector of angle $A_2B_1A_3$, $B_1P_1$ must be the conjugate bisector. That is, the bisectors of angles of the tangential triangle not passing through the circumcenter of the given triangle pass respectively through the poles of the medians of the given triangle. These poles are obviously on the sides of the tangential triangle and since the medians are concurrent, their poles are necessarilly collinear.

Solved also by J. W. Clawson, Howard Eves, R. Goormaghtigh, A. S. Howard, Ou Li, and A. Sisk.

*Editorial Note.* Goormaghtigh calls attention to the following generalization: If $A_1A_2 \cdots A_n$ is a polygon inscribed in a circle $\Gamma$ having $O$ as center, the poles of the joins of any of the vertices $A_i$ to the centroids $G_i$ of the remaining vertices are on a straight line; those points are the intersections of the tangents at the points $A_i$ to the circle $\Gamma$ with the perpendiculars erected at the inverse points $T_i$ of the points $G_i$ as to $\Gamma$ on the lines $OT_i$.

Goormaghtigh also refers to the remarkable equation given by J. H. Weaver (this MONTHLY, 1933, 91) for the straight line $g$ containing the points of intersection of the sides of the triangle $B_1B_2B_3$ with the external bisectors of that triangle. If, in a system of complex coördinates, $A_1A_2A_3$ is the base-circle and if $\sigma_1$ is the sum of the coördinates of $A_1$, $A_2$, $A_3$, the equation of $g$ is

$$x\bar{\sigma}_1 + \bar{x}\sigma_1 = 6,$$

$\bar{a}$ being conjugate to $a$. But the polar of a point $\alpha$ as to the base-circle $x\bar{x}=1$ is

$$x\bar{\alpha} + \bar{x}\alpha = 2;$$

hence $g$ is the polar of the centroid $\sigma_1/3$. Corresponding to his generalization for

the $n$-gon we have the following generalization of Weaver's equation: If $A_1A_2 \cdots A_n$ is a polygon inscribed in a circle $\Gamma$ having $O$ as center, and if $G_i$ is the centroid of the vertices other than $A_i$ and $T_i$ the inverse of $G_i$ as to $\Gamma$, the tangents to $\Gamma$ at the points $A_i$ meet the perpendiculars erected at the points $T_i$ to the lines $OT_i$ on the straight line

$$x\bar{\sigma}_1 + \bar{x}\sigma_1 = 2n,$$

$\sigma_1$ being the sum of the coördinates of the vertices $A_i$ in a system of complex coördinates in which $\Gamma$ is the base-circle.

The proposer suggests the following extension: The poles of the symmedians of a triangle are the centers of the Appolonian circles and lie on the Lemoine axis.

### Envelope of Conics

4182 [1945, 582]. *Proposed by Cezar Coşniţa, Focşani, Roumania*

Show that the envelope of the conics circumscribing a given triangle and such that the angle between the asymptotes is constant, is a curve of the fourth degree bitangent to the line at infinity at the circular points and having the vertices of the triangle for double points.

I. *Solution by G. A. Williams, Oregon State College.* Let the equations of the asymptotes be $y-m_1x+\alpha=0$ and $y-m_2x+\beta=0$ with the condition

$$\frac{m_2 - m_1}{1 + m_1m_2} = \text{constant} = \frac{1}{k}, \tag{1}$$

and let the vertices of the given triangle be $(0, 0)$, $(a, 0)$, $(b, c)$. Then the equation of the conic may be written

$$(y - m_1x + \alpha)(y - m_2x + \beta) = \lambda. \tag{2}$$

Upon substituting the coördinates of the given points in (2) and then eliminating $\lambda$, $\alpha$, $\beta$, and $m_1$, a quadratic in $m_2$ is obtained:

$$Am_2^2 + Bm_2 + C = 0, \tag{3}$$

where

$$\begin{aligned} A &= kcx^2 - cxy - kacx + b(ka - kb + c)y, \\ B &= - cx^2 - 2kcxy + cy^2 + acx + (b^2 - ab - c^2 + 2kbc)y, \\ C &= cxy + kcy^2 - c(kc + b)y. \end{aligned}$$

The eliminant of this quadratic and its partial derivative with respect to $m_2$ equated to zero is $\phi\equiv B^2-4AC=0$. Then as each of $A$, $B$, $C$ vanishes for $(0, 0)$, $(a, 0)$, $(b, c)$, it follows that $\phi=0$, $\partial\phi/\partial x=0$, $\partial\phi/\partial y=0$ are all satisfied at each point and hence the curve has a double point at each.

Upon expanding $\phi$ we see that $\phi=c^2(x^2+y^2)^2$ plus terms of lower degree. Hence the line at infinity meets the curve twice at each of the circular points.

As the curve can have no more double points than the three already obtained, the line at infinity must be tangent at each of the circular points.

II. *Solution by L. M. Kelly, University of Missouri.* The proposed theorem is merely the result of applying the transformation by isogonal conjugates with respect to the given triangle to the following: The envelope of equal chords of the circumcircle is a circle concentric with the circumcircle.

Under such a transformation the circumcircle goes over into the line at infinity. A family of equal chords goes into a family of similar and circumscribed conics. Since the transformation is a contact transformation the circle which is the envelope of the chords will go into the envelope of the conics. But a circle concentric with the circumcircle goes into a curve of the fourth degree. Since the circular points are isogonal conjugates with respect to any triangle and since two concentric circles are tangent at the circular points, we see that the desired locus is a fourth degree curve tangent to the line at infinity at the circular points. Finally, since the envelope of the chords cuts each side of the triangle in two points, the fourth degree curve must have double points at each vertex.

Also solved by H. E. Fèttis, R. Goormaghtigh, and G. B. Huff.

**Sums of Powers with Binomial Coefficients**

4183 [1945, 582]. *Proposed by Cezar Coşniţa, Focşani, Roumania*

Let $p$ and $q$ be non-negative integers and $x$ a variable. Define

$$f(x, p, q) = \sum_{i=0}^{p} (-1)^i {}_pC_i(x-i)^q,$$

where ${}_pC_i$ are binomial coefficients. Prove that $f(x, p, q)$ equals zero if $p>q$; equals $p!$ if $p=q$; and is a polynomial in $x$ of degree $q-p$ if $p<q$.

I. *Solution by F. Underwood, University College, Nottingham, England.* Put

$$A = (e^{at} - e^{bt})^p = e^{pat} - {}_pC_1e^{\{(p-1)a+b\}t} + {}_pC_2e^{\{(p-2)a+2b\}t} - \cdots$$
$$= e^{xt} - {}_pC_1e^{(x-1)t} + {}_pC_2e^{(x-2)t} - \cdots,$$

in which we have substituted $pa=x$, $a-b=1$, (whence $pb=x-p$). The coefficient of $t^q$ in the expansion of $A$ in powers of $t$ is

$$\frac{1}{q!}\left[x^q - {}_pC_1(x-1)^q + {}_pC_2(x-2)^q - \cdots\right] = \frac{f(x, p, q)}{q!}.$$

On the other hand $A$ may be put in the form

$$A = \left[e^{(a-b)t} - 1\right]^p e^{pbt} = (e^t - 1)^p e^{(x-p)t}$$
$$= \left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]^p e^{(x-p)t}$$
$$= t^p\left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right]^p \left[1 + (x-p)t + \frac{(x-p)^2t^2}{2!} + \cdots\right].$$

In this form of $A$ the coefficient of $t^q$ is zero when $p>q$, unity when $p=q$, a polynomial of degree $q-p$ in $x$ when $p<q$. The equality of the corresponding coefficients completes the proof.

II. *Solution by E. A. Hedberg, University of South Carolina.* Let $a_0=0$, $a_1, a_2, \cdots, a_p$ be any set of distinct numbers, real or complex, and put

$$ {}_pG_i = (-1)^{i+p}\prod_{j=1}^{p}\frac{a_j}{a_i - a_j}, \qquad j \neq i,\ i = 1, 2, \cdots, p. $$

If we define

$$ F(x, p, q) = \sum_{i=0}^{p}(-1)^i\, {}_pG_i(x - a_i)^q, $$

then $F(x, p, q)$ is zero when $p>q$, equals $\prod_{i=1}^{i=p}a_i$ when $p=q$, and is a polynomial of degree $q-p$ in $x$ when $p<q$. (The ${}_pG_i$ enjoy many of the properties of the ${}_pC_i$ and indeed reduce to them if we take $a_i=i$. Thus the ${}_pG_i$ may be considered as a natural generalization of binomial coefficients.) The proof follows.

$F(x, p, q)=\sum_{i=0}^{i=p}(-1)^i{}_pG_i(x-a_i)^q$ is identically equal to $D_1/D_2$, where $D_1$ is the determinant

$$ \begin{vmatrix} x^q & (x-a_1)^q & (x-a_2)^q & (x-a_3)^q & \cdots & (x-a_p)^q \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & a_1 & a_2 & a_3 & \cdots & a_p \\ 0 & a_1^2 & a_2^2 & a_3^2 & \cdots & a_p^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_1^{p-1} & a_2^{p-1} & a_3^{p-1} & \cdots & a_p^{p-1} \end{vmatrix} $$

and $D_2$ is the determinant of Vandermonde which results when the first row and first column of $D_1$ are deleted.

In case $p>q$, all elements of the first row of $D_1$ become zero if we subtract from it the products of the second row by $x^q$, the products of the third row by $-{}_qC_1x^{q-1}$, of the fourth row by ${}_qC_2x^{q-2}$, and so on. Hence $F(x, p, q)=0$.

In case $p=q$, if the foregoing procedure is applied, and the proper interchange of rows is made to bring the first row into the position of the last, $D_1$ becomes

$$ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_1 & a_2 & \cdots & a_p \\ 0 & a_1^2 & a_2^2 & \cdots & a_p^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_1^{p-1} & a_2^{p-1} & \cdots & a_p^{p-1} \\ 0 & a_1^p & a_2^p & \cdots & a_p^p \end{vmatrix} = \prod_{i=1}^{p} a_iD_2. $$

That is, $F(x, p, p) = \prod_{i=1}^{i=p} a_i$.

When $p<q$, it is seen that the $(q-p)$th derivative of $F(x, p, q)$ reduces to $q(q-1) \cdots (p+1)F(x, p, p)$ which is a constant not equal to zero. Hence $F(x, p, q)$ is of degree $(q-p)$ in $x$.

Also solved by David Alling, W. J. Combellack, Sidney Glusman, H. A. Luther, and the Proposer.

### Squares of Special Form

4184 [1945, 582]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what systems of numbers with the base less than 10000 are there the greatest number of squares of four digits of the form $aabb=(cc)^2$? Dedicated to E. P. Starke.

*Solution by Fritz Herzog, Michigan State College.* Let $m$ denote the base $(m \geqq 2)$. Assume $a$, $b$, $c$ to be non-negative integers less than $m$ such that

$$am^3 + am^2 + bm + b = (cm + c)^2. \tag{1}$$

We disregard the trivial solution $0000=00^2$ and assume $a+b$ and $c$ to be positive. (1) may be put in the form

$$a(m - 1) + (a + b)/(m + 1) = c^2. \tag{2}$$

Since $0<a+b<2m+2$ we conclude from (2) that

$$a + b = m + 1. \tag{3}$$

From (2) and (3) we obtain

$$a(m - 1) + 1 = c^2, \tag{4}$$

$$c^2 \equiv 1 \pmod{m - 1}. \tag{5}$$

From (3) we conclude that $a \neq 0, 1$, and hence by (4)

$$c \neq 1, \quad \sqrt{m}. \tag{6}$$

It is easily seen, on the other hand, that any integer $c$ $(1 \leqq c \leqq m-1)$ which satisfies (5) and (6), together with $a$ and $b$ determined by (4) and (3), yields a solution of the problem. Therefore, if we denote by $A(m)$ the number of solutions of the problem for the base $m$, we have

$$A(m) = B(m - 1) - \delta, \tag{7}$$

where $B(n)$ denotes the number of solutions of the congruence $x^2 \equiv 1 \pmod{n}$, and $\delta$ equals 2 or 1 according as $m$ is or is not a perfect square (see (6)). If $s$ denotes the number of distinct odd prime factors of $n$, the value of $B(n)$ is given by

$$B(n) = 2^{s+2}, \ 2^{s+1}, \text{ or } 2^s, \tag{8}$$

according as $n \equiv 0 \pmod 8$, $\equiv 4 \pmod 8$, or $\not\equiv 0 \pmod 4$, respectively.* With (7)

* See, for instance, Landau, *Vorlesungen über Zahlentheorie*, vol. 1, p. 47, Satz 88.

and (8) the number of solutions of the problem for each base is completely determined.

For $n<9999$ we have $s\leqq 4$ (since $3\cdot 5\cdot 7\cdot 11\cdot 13>10000$) and hence the greatest value of $B(n)$ is $2^{4+2}=64$ corresponding to $n=8\cdot 3\cdot 5\cdot 7\cdot 11=9240$. For $m=n+1=9241$, we have $\delta=1$ and $A(m)=63$. Evidently all other $m<10000$ give $A(m)\leqq 31$.

Also solved by A. Barriol, Free Jamison, and E. P. Starke.

*Editorial Note.* The original proposal set the upper limit of 100000 for $m$. However Barriol showed that there is no new value of $m$ for which $A(m)$ exceeds 63. For the following numbers $A(m)=63$: 60061, 78541, 87781, 92421.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*Mathematics of Finance.* By J. A. Northcott. New York, Rinehart & Co., Inc., 1946. 9+252 pages. $3.00.

In the author's own estimate, this text is a "brief and simple development of a subject of great importance to students of banking and finance." He feels that a defense is necessary for his presumption in adding another text to the field. The defense offered is that most of the standard texts place undue emphasis on a multiplicity of formulas—a policy that the present text seeks to supplant by the systematic use of a small body of fundamentals.

The reviewer frequently overhears students discussing mathematics of finance. Their attitude is characterized by one admonition: "Don't take that course! It's all story problems and they have formulas this long!" To students, then, and to their instructors as well, an apology for a text which achieves, to any appreciable degree, brevity, simplicity and system, must seem pointless.

The first two chapters comprise a résumé of progressions and logarithms. The treatment covers—with plenty to spare—all possible applications of these two topics to the subject at hand.

The treatment of simple interest, which many writers draw out to needless length, covers less than six pages, and only one formula is introduced. It is noteworthy that the so-called simple discount rate $d$, that is, the interest per year per dollar of amount, is omitted. From the standpoint of minimum confusion to the student, this trifling detail $d$, whose sole excuse for existence is its aid to computation, might be profitably ignored. One usually hesitates, however, to deal so summarily with a concept as widely used as this one. Some authors

include, as exercises in simple interest, transactions involving cash payments made at different dates. The treatment of these problems by the methods of simple interest is both cumbersome and unnatural; the only satisfactory way of handling them is through equation of value with compound interest. Our author is to be commended for adhering to the latter policy.

The author defines compound interest as simple interest in which the principal is converted at the end of each conversion period. Historically, this is probably the way that compound interest came into being, and it is certainly the popular conception of the topic. It is far neater, however, to define the compound interest law as a value-*vs*-time relation $V(t_2)=V(t_1)k^{t_2-t_1}$, where $V(t)$ is the value at date $t$ and $k$ is constant. Then we need give no thought to "principal" and "amount"; to whether we are going forward or backward in time; to whether the time interval is a whole number of conversion periods. Furthermore, when we speak of two equivalent rates $(j, m)$ and $(j', m')$, we are actually referring to the same compound interest law, rather than to two laws which agree only at certain periodic dates. The popularly conceived form of compound interest is then easily introduced as a computational detail. The reviewer has found that this method of presentation, aided by graphical study, gives the student a better insight than the other approach.

The powerful method of equation of value is given moderate emphasis—but still not enough. The author tacitly assumes that this method of solving a problem (in which all items of investment are brought, at compound interest, to an arbitrary reference date, summed, and equated to a like sum for the returns) will give the same result as analysis by amortization schedule (a progressive, item-by-item study). These methods are conceptually distinct (the amortization schedule being prior to the other approach) and their equivalence should be demonstrated. It is surprising how many students sense this point and demand just such an explanation.

The author foregoes the double superscript notation for annuity symbols. As a result, the user of his text is thrown back almost entirely upon the geometric progression formula. This "roughing it" is undoubtedly healthy for the student—if he doesn't lose heart and quit. Why not show the student once and for all how the progression formula gives rise to the more refined, mechanical procedure of the double superscript notation, and then let him put his progressions back on the reserve shelf?

The author introduces no separate symbolism or formulas for the treatment of annuities due and deferred annuities. Unfortunately, however, he allots a section to each. This invariably deludes the student into thinking that here are two new topics which he must learn (with consequent injury to his morale). How is he to know that there is just as much point in classifying annuities as "due," "immediate" and "deferred" as in distinguishing between the numbers 16, 17-1, and 24-8?

The treatment of bonds is typical of the average text. In order to correlate this topic with the mathematical tools previously developed, it is important to

prevent the reader from imagining that a bond is essentially different from any other set of investments and returns. The present text may fall slightly short in this respect.

Sinking funds and amortization are presented together in the traditional way. This presentation effectively conceals an important characteristic of situations involving sinking funds—namely, that they involve the interrelation of several distinct investors, each with his own investor's rate and his own set of payments.

A short chapter on depreciation and a chapter of miscellaneous review problems complete the text portion. There are 105 pages of tables, including 5 place logarithms and tables for compound interest and annuity computations.

The exercises are generally well fitted to the text, and they are stated with better than average clarity. Some of them are taken from actuarial examinations. Answers are furnished for the odd-numbered exercises.

An instructional device that our author has not fully utilized is graphical representation. The line diagram, which he uses freely, is admirably adapted to the itemizing of investments and returns; but the concept of *variation of invested value*, which underlies the whole subject of investment mathematics, can be best imparted to the student through a series of two-dimensional graphs. For example, the student should be invited to draw a few graphs of items at simple interest, to show the linearity of the value law; of items at compound interest, to distinguish the true and the approximate value; of bond price, to show the periodic variation and the general trend.

On the whole, the author has succeeded moderately well in achieving his set purpose of systematizing finance calculations. However, there are still many important principles that need enunciation. For example, what is the general method of treating commercial paper in analyzing a transaction? Perhaps the only way to change finance mathematics from an art to a science is to put it on a mildly postulational basis, and to effect a small degree of formalization. Most authors, including the present one, make desultory attempts along this line, but invariably their precepts remain tacit.

G. F. ROSE

*Mathematician's Delight.* By W. W. Sawyer. New York, Penguin Books, Inc., 1946. 215 pages. $0.25.

This book is the eighth in the new series of Pelican Books, a series of small inexpensive books planned to treat a wide variey of subjects. The book under review provides, in many repects, an excellent general introduction to elementary mathematics. The title of the book is indicative of the author's ingenuity and originality in departing from traditional and time-worn patterns. In his own words, the book is intended to "try to show what mathematics is about, how mathematicians think, when mathematics can be of some use." Although intended chiefly for beginners, there is much in the book that will be of interest to the teacher as well.

In discussing the fear of mathematics which is held by so many laymen, the author contends that this is due not to the nature of the subject itself, but to the dull way in which it is so often taught. The thesis is developed that in many cases the student does not learn mathematics, but only an imitation of mathematics, which destroys his power to enjoy the real subject. In this connection Dr. Sawyer pays his respects to parrot-learning by quoting the pupil who wrote, "The abdomen contains the stomach and the vowels, which are A, E, I, O, and U!"

The book contains an interesting chapter on geometry in which the value of experimentation is emphasized and a number of rather ingenious geometrical experiments are proposed. A chapter, *The Nature of Reasoning*, leads gradually to mathematical reasoning and to the nature of mathematical abstraction. In making generalizations the author may occasionally be guilty of over-simplification. For example, the statement that "The more one studies the methods of the great, the more common-place do these methods appear," may require considerable qualification. An interesting discussion contrasting pure and applied mathematics is given and the point is made that all mathematics is tied, however remotely, to problems which arose originally in the study of the real world. This will seem rather academic to those mathematicians who believe that many of these ties are very remote indeed.

A chapter entitled *The Strategy and Tactics of Study* contains many interesting and sensible remarks. Subsequent chapters deal with arithmetic, logarithms, algebraic operations, functions, graphs, calculus, and trigonometry. All of these topics are introduced in an intuitive and discursive way, and in many cases quite unusual illustrative examples are used. The reviewer believes that in a few instances the ingenuity with which the author has rigged up his introductory illustrative examples may tend to make obscure to the beginner the mathematical principles involved. This is believed to be particularly true in connection with the way in which logarithms are introduced, where the mathematical principle involved seems much simpler than the mechanical device used by the author to provide a setting for the principle.

The chapter on trigonometry goes as far as the differentiation of the sine and cosine functions and applies the results to the study of the motion of a particle in a circle. It seems unfortunate that the important application of trigonometry to wave motion is not discussed. Following the chapter on trigonometry are two concluding chapters, one entitled *On Backgrounds*, and the other *The Square Root of Minus One*. The former chapter emphasizes the importance of understanding what lies behind the steps of a formal proof and in knowing what it is all about. In the last chapter imaginary numbers are treated from an operational point of view.

While in various places throughout the book the reader may not agree wholly with the author's contentions, nevertheless the reviewer regards this little book as a worthy addition to the growing list of survey type books, which deal in an elementary and intuitive way with the nature and the scope of mathe-

matics while leaving its rigorous elaboration and methodology to the text-books and treatises.

H. P. EVANS

## NEW BOOKS RECEIVED

*Advanced Mathematics for Engineers.* Second Edition. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1947. 12+508 pages. $5.00.

*An Introduction to Mathematical Genetics.* By Lancelot Hogben. New York, W. W. Norton and Co., Inc., 1946. 12+260 pages. $5.00.

*Mathematics of Finance.* By F. S. Harper. Scranton, Pa., International Textbook Company, 1946. 10+327 pages.

*Science Since 1500.* By H. T. Pledge. New York, Philosophical Library, 1947. 357 pages. $5.00.

# CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

EDITOR'S NOTE.—In line with the policy of this department to suggest methods or ways of creating interest in mathematics among students and to print suggestions or ideas which might be used as projects for clubs, the following contribution by Mr. Frank Hawthorne is presented. Mr. Hawthorne has been a frequent contributor to the Problems and Solutions sections of this MONTHLY.

The editor of this department welcomes other suggestions which may be passed on to members of Mathematics Clubs.

## INCREASING STUDENT INTEREST IN MATHEMATICS

One of the objectives stated or implied in the constitutions of most collegiate mathematics clubs is the fostering of increased interest in mathematics among the general student body. On most campuses there are many students who do not take mathematics courses or belong to mathematics clubs. Of course, some of these students contend that they "don't like mathematics" and will resist any attempt to expose them to it but there are usually others who have some interest and ability along that line. An attempt to increase the interest of these particular students may result in a few pleasant surprises.

As a possible means of reaching these people, selected problems whose solution requires some thought but only very elementary technique may be posted upon the general bulletin board at regular intervals. Specific times and places for discussion of each set of problems should be arranged. The number and diversity of people who attend these meetings may indicate that mathematics does have a certain general appeal. Perhaps some of the problems might even interest a few faculty members from non-mathematical departments. At the very least

one may hope that a few students will realize that such problems compare favorably with crossword puzzles in providing mental exercise.

Suitable problems are available in numerous books on recreational mathematics and additional problems of a sufficiently elementary nature appear from time to time in mathematical journals. Consideration might be given to the following problems which have appeared recently in this MONTHLY:

E576 (1944, 95), E581 (1944, 166), E608 (1944, 531), E622 (1945, 96),
E631 (1945, 219), E651 (1945, 397), E671 (1946, 41), E751 (1947, 38).

E. J. Moulton's *A Problem in Geography* (1944, 216 and 220) intrigues many ex-navigators. The solution need not be complicated by the necessity for an argument about the non-existance of bears of any color in the Antarctic.

In selecting problems, care should be taken to avoid using too many of the same type. There is little danger that problems will be "old stuff" to those for whom they are particularly intended. Certainly those people will not have seen them in the MONTHLY.

Some problems simpler than those noted above should be used while an occasional relatively difficult problem might be included. Some examples (by no means new) follow:

1. An electric toaster will toast two slices of bread on one side each simultaneously. If three or any greater odd number of slices is desired, what procedure will use the least electricity?

2. What digits can be final digits of perfect squares? Cubes?

3. A cube three inches on an edge is made from white wood and all faces are painted black. It is then sawed into one inch cubes.
   a. How many saw cuts were necessary?
   b. How many small cubes will there be?
   c. How many small cubes will be all white?
   d. How many small cubes will have only one, only two, three black faces?

4. A board has three holes, one circular, the second square, the third equilaterally triangular. The diameter of the circle, the side of the square, and the altitude of the triangle are equal. Design a solid which will pass through each and fill each completely.

Occasionally a mildly facetious problem might be proposed. Such "problems" as "If one man can see five miles, how far can three men see?" or "If one man can do a job in an hour, how long will it take five hundred men to do it?" might season the selection. Certain students might even answer the second of these on the basis of recent experience. Of course, trick problems whose solutions hinge on a double meaning of words or some other equally obnoxious device should be avoided.

A general invitation to students to submit problems for posting and discussion at a later date should produce some results. Many of the problems so proposed may be unsuitable but some of them may prove helpful. It might be well to recall that many significant additions to mathematics have been made by persons whose interest was first aroused by some particular problem.

## NEWS AND NOTICES

Edited by B. W. Jones, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.*

### THE INSTITUTE OF MATHEMATICAL STATISTICS

This society announces the election of the following officers: President, William Feller; Vice-presidents, J. H. Curtiss and M. H. Hanson; Secretary, Paul Dwyer.

### SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1947:

*Brown University* (Graduate Division of Applied Mathematics). From July 7 to August 1 the following courses will be offered: by Professor Carrier, theory of structural stability; by Professor Lin, dynamics of viscous fluids. From August 5 to August 30: by Professor Goldstein, dynamics of compressible fluids; by Professor Prager, theory of plasticity.

*The Catholic University of America.* From June 27 to August 9 the following graduate courses will be offered: by Professor E. J. Finan, higher algebra; by Dr. V. McBrien, theory of equations; by Professor O. J. Ramler, college geometry, differential equations and analytic projective geometry; by Professor J. N. Rice, statistics and advanced calculus.

*Columbia University.* From July 7 to August 15 the following graduate courses will be offered: by Professor Eilenberg, homotopy theory; by Professor Kasner, survey of modern mathematics, geometry and dynamics; by Professor Lorch, theory of groups, Fourier series; by Professor Murray, differential equations; by Professor Smith, theory of functions of a real variable.

*Northwestern University.* From June 23 to August 23 the following advanced courses will be offered: definite integrals, functions of a complex variable, geometry for teachers, history and teaching of mathematics, introduction to the theory of groups, introduction to the theory of numbers, seminar in analysis, theory of equations, theory of statistics, theory of tensors.

*Ohio State University.* From June 17 to August 29 the following advanced courses will be offered: by Professor Alden, introduction to the theory of functions of a complex variable, differential equations; by Professor Helsel, advanced calculus, advanced geometry; by Professor Mann, fundamental ideas in algebra and geometry, theory of fields.

*Stanford University.* From June 19 to August 30 the following advanced courses will be offered: by Professor Rademacher (of the University of Pennsylvania), elementary mathematics from a higher point of view, elliptic functions; by Professor Szegö, partial differential equations of physics and engineering, another course to be announced.

*Teachers College, Columbia University.* From July 7 to August 15 the following courses will be offered: by Professor Clark, teaching algebra in secondary

schools, teaching arithmetic in the elementary schools; by Professor Fehr, professionalized subject matter in junior high school mathematics, professionalized subject matter in advanced secondary school mathematics; by Dr. Lazar, history of mathematics, logic for teachers of mathematics; by Mr. Mirick, elementary mechanics (statics), observation and participation in the teaching of geometry; by Professor Reeve, teaching and supervision of mathematics—junior high school, teaching and supervision of mathematics—senior high school; by Professor Schlauch, business mathematics; by Professor Shuster, teaching geometry in secondary schools, field work in mathematics. In addition, on consecutive Thursdays beginning on July 10, there will be given five special lectures and discussions in which all the instructors above and others will take part.

*The University of Colorado.* In both terms (June 16 to July 17 and July 21 to August 22 respectively) the following advanced courses will be offered: by Professor Kempner, teachers' course in mathematics (not a methods course); by Professor Hutchinson, functions of a real variable; by instructors not yet determined, theory of equations, vector analysis.

*The University of Michigan.* From June 23 to August 15 the following advanced courses will be offered in addition to the standard courses in differential equations, theory of equations, advanced calculus, mechanics and statistics: by Professor Bartels, vector analysis and hydrodynamics; by Professor Brauer, elementary matrices, theory of group representations; by Professor Coburn, operational mathematics; by Professor Copeland, foundations of mathematics, mathematical probability; by Professor Craig, significance tests and analytic sampling theory; by Professor Dushnik, theory of integration; by Professor Dwyer, computational methods; by Professor Hay, advanced mechanics; by Professor Kaplan, elementary functions of a complex variable with applications; by Professor Karpinski, teaching of geometry, history of algebra; by Professor Myers, functions of a real variable; by Professor Nesbitt, mortality studies; by Professor Rainich, introduction to differential geometry, higher geometry; by Professor Rainville, intermediate differential equations; by Professor Rothe, partial differential equations; by Professor Samelson, general spaces; by Professor Thrall, algebraic theory.

*The University of Minnesota.* From June 16 to July 25 the following advanced courses will be offered: by Professor Cameron, Fourier, Bessel and Legendre series, course in reading and research; by Professor Hatfield, differential equations, theory of numbers; by Professor Wegner, advanced calculus, solid analytic geometry. From July 28 to August 29: by Professor Olmsted, vector analysis; by Professor Wegner, advanced calculus, theory of matrices.

*The University of North Carolina.* From June 12 to July 22 the following advanced courses will be offered: by Professor Browne, introduction to the theory of matrices; by Professor Cameron, introduction to higher algebra; by Professor Garner, history of mathematics; by Professor Henderson, finite groups; by Professor Hill, elementary mathematical statistics; by Professor Hobbs, theory of equations; by Professor Linker, differential equations; by Professor Mackie, ad-

vanced calculus. From July 23 to August 29: by Professor Brauer, theory of equations (continued); by Professor Hoyle, differential equations (continued); by Professor Lasley, synthetic projective geometry; by Professor Winsor, college geometry; by Professor Wong, advanced calculus (continued).

*The University of Pennsylvania.* From June 30 to August 23 the following graduate courses will be offered: by Professor Beal, analytic geometry of three dimensions; by Professor Caris, diophantine analysis; by Professor Clarkson, theory and practice of approximation; by an instructor not yet decided, higher calculus.

*The University of Virginia.* From June 30 to August 23 the following advanced courses will be offered: by Professor Hedlund, advanced calculus and applied mathematics, foundations of geometry; by Professor Harrold, transformation topology.

*The University of Wyoming.* From June 9 to August 16 the following semigraduate and graduate courses will be offered: by Professor Barr, methods of teaching mathematics; by Dr. Bristow, projective geometry; by Dr. Schwid, ordinary differential equations; by Dr. Varineau, advanced college algebra. From June 23 to August 25: by Miss Neubauer, the history of mathematics; by Dr. Smith, college geometry.

## EXAMINATION ANNOUNCED FOR STATISTICIAN POSITIONS

An examination has been announced by the Civil Service Commission for filling high-grade professional Statistician positions in Washington, D. C., and vicinity. The salaries range from $5,905 to $9,975 a year.

To qualify, candidates must have had progressively responsible professional experience in statistical research. Graduate study with major work in statistics will be credited as being equivalent to professional work. No written test is required; applicants will be rated on their experience and training relevant to the duties of the positions. The grade or salary level for which applicants are considered qualified will be determined by the quality of their experience as shown by the scope and level of the responsibilities involved, their influence on policy and program, and the complexity of the problems handled. The age limit of 62 years is waived for persons entitled to veteran preference.

Applications will be accepted in the Commission's Washington office until further notice. However, persons interested in being considered for positions which are to be filled immediately should apply within one month. Application forms may be obtained from first- and second-class post offices, from the Commission's regional offices, or direct from the U. S. Civil Service Commission, Washington 25, D. C.

## PERSONAL ITEMS

President L. R. Ford and his wife represented the Association at the Rockford College Conference on February 21–23, 1947.

Professor P. K. Rees of Louisiana State University represented the Association at the installation of Robert Cecil Cook as president of Mississippi Southern College.

R. H. Beard, of the New York Telephone Company has received the 1947 award of the Duodecimal Society of America.

W. D. Lambert of the U. S. Coast and Geodetic Survey has been elected Corrèspondent of the Paris Academy of Sciences (Institut de France) in the Section of Geography and Navigation.

Associate Professors Garret Birkhoff and Saunders MacLane of Harvard University have been promoted to professorships.

Associate Professor J. J. Barron of Marquette University has been appointed to a professorship at Marshall College, Huntington, West Virginia.

Dr. O. K. Bower of the University of Illinois has been promoted to an assistant professorship.

A. H. Bowker has been appointed to an assistant professorship at Stanford University.

Professor Gregory Breit of the University of Wisconsin has been appointed to a professorship at Yale University.

Dr. J. W. Calkin of the California Institute of Technology has been appointed to an associate professorship at Rice Institute.

Associate Professors E. A. Cameron and V. A. Hoyle of the University of North Carolina have been promoted to professorships.

Dr. Harold Chatland of the University of Montana has been appointed to an assistant professorship at Ohio State University.

Paul Cramer of Huron College, South Dakota, has been appointed to an assistant professorship at Monmouth College, Illinois.

Professor Tobias Dantzig of the University of Maryland has retired.

Professor J. L. Doob of the University of Illinois has been appointed to a visiting professorship at Columbia University.

Dr. R. H. Downing of Fleetwings, Inc., Bristol, Pennsylvania, has been appointed to a professorship at the Army Air Forces Institute of Technology, Wright Field.

Associate Professor C. M. Erikson of Michigan State Normal College has been promoted to a professorship.

Dr. B. E. Gatewood has been appointed to an associate professorship at the Army Air Forces Institute of Technology, Wright Field, Dayton, Ohio.

J. B. Greeley has been appointed chairman of the mathematics department at Utica College, Syracuse University.

Assistant Professor J. F. Heyda of Franklin and Marshall College has been appointed research mathematician at the Naval Ordnance Plant, Indianapolis.

Professor Theodore Lindquist of Michigan State Normal College has retired.

Dr. A. N. Milgram has been appointed to an associate professorship at Syracuse University.

G. J. O'Boyle of the Catholic University of America has been promoted to an assistant professorship.

Associate Professor E. K. Paxton of Washington and Lee University has resigned.

Dr. Paul Reichelderfer has been appointed to an associate professorship at Ohio State University.

Dr. R. W. Shephard of the University of California has been appointed to an assistant professorship at New York University.

Dr. F. C. Smith of the Lincoln National Life Insurance Company has been appointed to an associate professorship at the College of St. Thomas, St. Paul, Minnesota.

Henry E. Smith of Dickinson College has been promoted to an assistant professorship.

Professor H. E. Spencer of Presbyterian College, Clinton, South Carolina, has been appointed to an assistant professorship at Virginia Polytechnic Institute.

Dr. George Tunell, on leave from the Carnegie Institution of Washington, has been appointed acting associate professor of mineralogy and metalliferous geology at California Institute of Technology.

Professor H. S. Vandiver of the University of Texas will be visiting professor at the University of Indiana during the present term.

Assistant Professor E. L. Welker of the University of Illinois has been promoted to an associate professorship.

Assistant Professor L. B. Williams of Hamilton College has been appointed to an assistant professorship at Reed College.

The following appointments to instructorships are announced:

Catholic University of America: R. W. Moller, S. J. Rosenfeld

Hunter College: Paul Brock

Iowa State College: Mrs. R. S. Banton, Mrs. J. V. Carr, Bette L. Flatland, J. W. Markey, J. H. Watson

Ohio State University: Dr. Marjorie Alden

United States Naval Academy: B. H. Buikstra

University of Buffalo: Helen M. Mazzuca, L. O. Ramer

University of Illinois: W. A. Ferguson, J. E. Schubert, B. E. Meserve, Vivian R. Nuess

University of Missouri: Dr. P. B. Burcham

University of Oregon: Mrs. Ethel Lawrence, W. G. Scobert

Professor H. M. Ackley of Western Michigan College died February 8, 1947.

H. G. Avers of the United States Coast and Geodetic Survey died January 19, 1947.

Professor B. F. Finkel of Drury College died February 5, 1947. An account of his life and many services to the Association will appear in a later issue of this MONTHLY.

Professor J. V. Uspensky of Stanford University died January 27, 1947.

Reverend G. L. Winkelmann of St. John's University, Collegeville, Minnesota, died January 23, 1947.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.
Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS, Wichita, April 19, 1947
KENTUCKY, Lexington, May 10, 1947
LOUISIANA-MISSISSIPPI, Hattiesburg, Mississippi, April 25–26, 1947
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May 3, 1947
METROPOLITAN NEW YORK, Brooklyn, April 19, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C, April 18–19, 1947
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS, Lubbock, April 25–26, 1947
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

### REPORT OF THE TREASURER FOR THE YEAR 1946

The following report of the Secretary-Treasurer as Treasurer for the year, 1946, has been approved by the Finance Committee and accepted by vote of the Board of Governors.

#### I. TOTAL FUNDS OF THE ASSOCIATION ON DECEMBER 31, 1945

(See Treasurer's report, pp. 237–240 of the MONTHLY for April, 1946)

| | | |
|---|---|---|
| Current Fund (checking account) | | $ 3,347.26 |
| Savings Account (Ithaca Savings Bank) | | 1,007.50 |
| Savings Account (OberlinSavings Bank) | | 648.96 |
| Invested Funds (Cleveland Trust Company) | | |
| Carus Fund | $ 9,455.75 | |
| Chace Fund | 9,733.84 | |
| Houck Fund | 9,279.29 | |
| Chauvenet Fund | 658.85 | |
| Life Membership Fund | 718.09 | |
| General Fund | 27,547.10 | 57,392.92 |
| | | $62,396.64 |

## II. Current Fund Account for 1946

| Receipts | | Expenditures | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $ 3,347.26 | Monthly | |
| Dues | 9,786.02 | Publication | $ 6,029.46 |
| Initiation fees | 442.00 | Reprints | 227.92 |
| Subscriptions | 3,139.45 | Editor-in-Chief's office | 458.68 |
| Sale of back numbers Monthly | 295.27 | Secretary-Treasurer's office | |
| Advertisements | 726.00 | Clerical help | 3,502.35 |
| Interest on General Fund | 796.87 | Postage | 414.95 |
| Interest on Carus Fund | 273.61 | Printing | 305.01 |
| Interest on Chace Fund | 281.65 | Office supplies | 220.55 |
| Interest on Houck Fund | 268.49 | Bank fee | 100.00 |
| Interest on Chauvenet Fund | 19.08 | Exec. and Finance Committees | 477.35 |
| Interest on Life Membership Fund | 20.78 | Regional Governors | 200.76 |
| Interest from Hardy Fund | 120.00 | Sections | 213.43 |
| Sale of Archibald's Outline | 231.40 | Subventions | |
| Sale of monographs (Carus) | 952.75 | Amer. Math. Society | 100.00 |
| Sale of Papyrus (Chace) | 158.00 | Mathematical Reviews | 350.00 |
| Sale of exchange periodicals | 40.00 | Back numbers Monthly | 114.60 |
| From Oberlin Savings Bank | 27.04 | Meetings | 21.09 |
| Miscellaneous sources | 2.28 | Coordinating Committee | 121.80 |
| Transferred from General Fund | 1,974.36 | Coop. Committee on Sci. Teaching | 106.82 |
| | | Amer. Council on Education | 215.36 |
| | | To General Fund | 500.00 |
| | | Subscriptions to Annals | 15.00 |
| | | B. F. Finkel (Hardy Fund) | 120.00 |
| | | Register | 753.85 |
| | | Bank charges | |
| | | Exchange fees | 9.55 |
| | | Bad checks | .50 |
| | | Transferred | |
| | | To Carus Fund | 1,226.36 |
| | | To Chace Fund | 439.65 |
| | | To Houck Fund | 268.49 |
| | | To Chauvenet Fund | 19.08 |
| | | To Life Membership Fund | 20.78 |
| | | Balance, Dec. 31, 1946 | 6,348.92 |
| | $22,902.31 | | $22,902.31 |

## III. Savings Account, Ithaca Savings Bank

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $1,007.50 | Balance, Dec. 31, 1946 | $1,022.66 |
| Interest | 15.16 | | |
| | $1,022.66 | | $1,022.66 |

## IV. Savings Account, Oberlin Savings Bank

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $ 648.96 | Final Payment | $ 27.04 |
| | | Final loss on the account | 621.92 |
| | $ 648.96 | | $ 648.96 |

## V. Invested Funds, Cleveland Trust Company

| | | | |
|---|---|---|---|
| Cash Balance, Jan. 1, 1946 | $ 279.92 | Decrease in value of securities | $ 1,022.62 |
| Market value of securities, Dec. 31, 1945 | 57,113.00 | Market value of securities, Dec. 31, 1946 | 56,696.00 |
| From Current Fund | 500.00 | Cash balance, Dec. 31, 1946 | 174.30 |
| | $57,892.92 | | $57,892.92 |

## List of Securities

| | Par Value | Market Value Dec. 31, 1946 |
|---|---|---|
| U. S. Savings Bonds, 1947 | $ 700.00 | $ 686.00 |
| U. S. Treasury Bonds, 2%, 1950 | 3,000.00 | 3,030.00 |
| U. S. Treasury Bonds, 2%, 1954 | 2,000.00 | 2,060.00 |
| U. S. Treasury Bonds, $2\frac{1}{2}$%, 1969 | 2,000.00 | 2,080.00 |
| U. S. Treasury Bonds, $2\frac{1}{2}$%, 1972 | 1,000.00 | 1,030.00 |
| U. S. Treasury Bonds, $2\frac{1}{4}$%, 1962 | 5,000.00 | 5,100.00 |
| U. S. Treasury Bonds, $1\frac{3}{4}$%, 1948 | 2,000.00 | 2,020.00 |
| U. S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1953 | 3,000.00 | 2,850.00 |
| U. S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1954 | 8,200.00 | 7,790.00 |
| U. S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1958 | 3,000.00 | 2,970.00 |
| Canadian Nat. Ry. Co. Bonds, $4\frac{1}{2}$%, 1956 | 2,000.00 | 2,360.00 |
| C. and O. Ry. Co. Ref. Bonds, Ser. D, $3\frac{1}{2}$%, 1996 | 3,000.00 | 3,180.00 |
| Columbus and So. Ohio Elec. Co. Bonds, $3\frac{1}{4}$%, 1970 | 2,000.00 | 2,200.00 |
| New York Steam Corp. 1st Mort. Bond, $3\frac{1}{2}$%, 1963 | 1,000.00 | 1,060.00 |
| Amer. Tobacco Co. Bonds, 3%, 1969 | 4,000.00 | 4,240.00 |
| C. and O. Ry. Co. common stock, 25 shares | | 1,350.00 |
| Amer. Tel. & Tel. Co. common stock, 30 shares | | 5,160.00 |
| Standard Oil Co. New Jersey common stock, 20 shares | | 1,380.00 |
| Atch. Top., Santa Fe R.R. non cum. pfd. stock, 15 sh. | | 1,590.00 |
| Commonwealth Edison Co. common stock, 80 shares | | 2,720.00 |
| Dana Corp. cum. pfd. stock, 20 shares | | 1,840.00 |
| | | $56,696.00 |

## VI. Carus Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $ 9,455.75 | Decrease in value of securities | $ 168.46 |
| Sale of monographs | 952.75 | | |
| Interest | 273.61 | Balance, Dec. 31, 1946 | 10,513.65 |
| | $10,682.11 | | $10,682.11 |

## VII. Chace Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1. 1946 | $ 9,733.84 | Decrease in value of securities | $ 173.43 |
| Sale of Papyrus | 158.00 | | |
| Interest | 281.65 | Balance, Dec. 31, 1946 | 10,000.06 |
| | $10,173.49 | | $10,173.49 |

### VIII. Houck Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $9,279.29 | Decrease in value of securities | $ 165.15 |
| Interest | 268.49 | Balance, Dec. 31, 1946 | 9,382.63 |
| | $9,547.78 | | $9,547.78 |

### IX. Chauvenet Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $ 658.85 | Decrease in value of securities | $ 12.35 |
| Interest | 19.08 | Balance, Dec. 31, 1946 | 665.58 |
| | $ 677.93 | | $ 677.93 |

### X. Life Membership Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $ 718.09 | To General Fund | $ 76.82 |
| Interest | 20.78 | Decrease in value of securities | 13.45 |
| | | Balance, Dec. 31, 1946 | 648.60 |
| | $ 738.87 | | $ 738.87 |

### XI. General Fund

| | | | |
|---|---|---|---|
| Balance, Jan. 1, 1946 | $27,547.10 | Decrease in value of securities | $ 489.78 |
| From Current Fund | 500.00 | Transferred to Current Fund | 1,974.36 |
| From Life Membership Fund | 76.82 | Balance, Dec. 31, 1946 | 25,659.78 |
| | $28,123.92 | | $28,123.92 |

### XII. Total Funds of the Association, December 31, 1946

| | | |
|---|---|---|
| Current Fund (checking account) | | $ 6,348.92 |
| Savings Account (Ithaca Savings Bank) | | 1,022.66 |
| Invested Funds (Cleveland Trust Company) | | |
| Carus Fund | $10,513.65 | |
| Chace Fund | 10,000.06 | |
| Houck Fund | 9,382.63 | |
| Chauvenet Fund | 665.58 | |
| Life Membership Fund | 648.60 | |
| General Fund | 25,659.78 | $56,870.30 |
| | | $64,241.88 |

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN

MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 5

CONTENTS



MAY 1947

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)



EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association and advertising in the MONTHLY, should be sent to the Secretary-Treasurer, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the first of the month in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

5. R. Brauer, On the Nilpotency of the Radical of a Ring, Bulletin of the American Mathematical Society, vol. 48, 1942, pp. 752–758.

6. Roy Dubisch and Sam Perlis, On the Radical of a Non-associative Algebra (abstract), Bulletin of the American Mathematical Society, vol. 50, 1944, p. 53.

7. N. Jacobson, The Radical and Semi-simplicity for Arbitrary Rings, American Journal of Mathematics, vol. 67, 1945, pp. 300–320.

8. N. Jacobson, Rational Methods in the Theory of Lie Algebras, Annals of Mathematics, vol. 36, 1935, pp. 987–991.

9. N. Jacobson, Simple Lie Algebras of Type A, Annals of Mathematics, vol. 39, 1938, pp. 181–188.

10. P. Jordan, J. von Neumann, and E. Wigner, On an Algebraic Generalization of the Quantum Mechanical Formalism, Annals of Mathematics, vol. 35, 1934, pp. 29–64.

11. Sam Perlis, A Characterization of the Radical of an Algebra, Bulletin of the American Mathematical Society, vol. 48, 1942, pp. 128–132.

12. J. H. M. Wedderburn, On a Theorem in Hypercomplex Numbers, Royal Society of Edinburgh. Proceedings, vol. 26 (1905), pp. 1–3.

13. J. H. M. Wedderburn, On Hypercomplex Numbers, London Mathematical Society. Proceedings, vol. 6 (1907), pp. 77–118.

14. M. Zorn, Theorie der Alternativer Ringe, Hamburg Mathematisches Seminar. Abhandlungen, vol. 8, 1930, pp. 123–147.

15. M. Zorn, Alternative Rings and Related Questions. I: Existence of the Radical, Annals of Mathematics, vol. 42, 1941, pp. 676–686.

# UNION TORSION OF A CURVE ON A SURFACE

C. E. SPRINGER, University of Oklahoma

**1. Introduction.** The definitions of the well known geodesic curvature and geodesic torsion of a curve on a surface involve the notion of the congruence of normals to the surface. One may ask for the analogues of these definitions relative to a congruence of straight lines which are not normal to the surface.

In a previous paper [1] the author studied union curves and gave a definition of *union curvature* of a curve on a surface relative to a given rectilinear congruence. The purpose of the present note is to exhibit a definition of the *union torsion* of a curve on a surface. The union torsion reduces to the geodesic torsion for the case in which the congruence is normal to the surface. It will be shown that a union curve is a plane curve if, and only if, it is tangent to one of the curves in which the developables of the congruence intersect the surface.

The notation of Eisenhart [2] will be employed. Greek indices will always take the range 1, 2, and Latin indices the range 1, 2, 3. The summation convention of the tensor analysis as to repeated indices will be observed.

**2. Analytical development.** Let the surface be represented by $x^i = x^i(u^1, u^2)$, $(i = 1, 2, 3)$, referred to a rectangular cartesian system of coördinates. The functions $x^i(u^1, u^2)$ together with their partial derivatives to the second order, are to

be regarded as continuous at any point $P$ of the surface under consideration. A unique line of the congruence is specified at each point $P$ of the surface by the direction cosines

$$\lambda^i = \lambda^i(u^1, u^2), \qquad (\lambda^i\lambda^i = 1), \tag{1}$$

where the functions $\lambda^i(u^1, u^2)$, with their first partial derivatives, are continuous at the point $P$. The functions $\lambda^i(u^1, u^2)$ may be expressed in the form

$$\lambda^i = p^\alpha x^i{}_{,\alpha} + qX^i, \tag{2}$$

where $p^\alpha$, $(\alpha=1, 2)$, are the contravariant components of a surface vector at $P$, $q$ is a positive scalar function, $x^i{}_{,\alpha}$ denotes the covariant derivative of $x^i$ with respect to $u^\alpha$ based on the fundamental tensor $g_{\alpha\beta}=x^i{}_{,\alpha}x^i{}_{,\beta}$, and $X^i$ are the direction cosines of the normal to the surface at $P$.

If $\theta$ is the angle between the lines with directions $X^i$ and $\lambda^i$, multiplication of equations (2) by $X^i$ shows that $q=\cos\theta$. For the angle $\phi$ between the tangent to a curve $C$: $u^\alpha=u^\alpha(s)$ through $P$ on the surface and the line of the congruence through $P$ with direction $\lambda^i$, we have

$$\cos\phi = \lambda^i\frac{dx^i}{ds} = (p^\alpha x^i{}_{,\alpha} + qX^i)x^i{}_{,\beta}u'^\beta = g_{\alpha\beta}p^\alpha u'^\beta, \tag{3}$$

where the prime indicates differentiation with respect to arc length $s$ on $C$.

Covariant differentiation of $\lambda^i$, and use of the Gauss and Weingarten equations,

$$x^i{}_{,\alpha\beta} = d_{\alpha\beta}X^i, \qquad X^i{}_{,\beta} = -d_{\beta\gamma}g^{\gamma\sigma}x^i{}_{,\sigma},$$

lead to the expression

$$\lambda^i{}_{,\alpha} = \mu^\gamma{}_\alpha x^i{}_{,\gamma} + \nu_\alpha X^i, \tag{4}$$

where $\mu^\gamma{}_\alpha$ and $\nu_\alpha$ are defined by

$$\mu^\gamma{}_\alpha \equiv p^\gamma{}_{,\alpha} - qd_{\alpha\sigma}g^{\sigma\gamma}, \qquad \nu_\alpha \equiv q_{,\alpha} + p^\beta d_{\alpha\beta}. \tag{5}$$

The curve $C$ is a union curve relative to the congruence of lines with directions $\lambda^i$ in case the osculating plane to $C$ at every point $P$ of $C$ contains the line of the congruence at $P$. The differential equation of the union curves on the surface may be written in the form [1, p. 688]

$$e_{\sigma\tau}(p^\sigma K_n - q\rho^\sigma)u'^\tau = 0, \tag{6}$$

where $e_{12}=1$, $e_{21}=-1$, $e_{11}=e_{22}=0$, $K_n$ is the normal curvature of $C$ at $P$, and $\rho^\sigma$ are the contravariant components of the curvature vector of $C$ at $P$.

**3. Torsion of a union curve.** Let the direction cosines of the tangent, principal normal, and binormal to the curve $C$ at $P$ be denoted by $\alpha^i$, $\beta^i$, $\gamma^i$. Then, the determinant $\epsilon_{ikl}\alpha^i\beta^k\gamma^l$ is equal to unity, and

$$\gamma^i = \epsilon_{ikl}\alpha^k\beta^l. \tag{7}$$

If $C$ is a union curve, the principal normal to $C$ lies in the plane of the lines with directions $\lambda^i$ and $\alpha^i = dx^i/ds$. Hence

$$\beta^i = a\frac{dx^i}{ds} + b\lambda^i, \tag{8}$$

where $a$ and $b$ are to be determined. Multiplication of equations (8) in turn by $dx^i/ds$ and $\beta^i$, and use of equation (3) yield $a = -\cot\phi$, $b = \csc\phi$, so that

$$\beta^i = \csc\phi\left(\lambda^i - \cos\phi\frac{dx^i}{ds}\right). \tag{9}$$

Use of $\beta^i$ from (9) in (7) gives

$$\gamma^i = \csc\phi\epsilon_{ikl}\frac{dx^k}{ds}\left(\lambda^l - \cos\phi\frac{dx^l}{ds}\right) = \csc\phi\epsilon_{ikl}\frac{dx^k}{ds}\lambda^l. \tag{10}$$

By the Frenet formulas,

$$\frac{d\gamma^i}{ds} = \tau\beta^i, \tag{11}$$

where $\tau$ is the torsion of the curve $C$ at $P$. Multiplication of equations (11) by $\beta^i$ yields

$$\tau = \beta^i\frac{d\gamma^i}{ds}\cdot \tag{12}$$

Use of $\beta^i$ from (9) and of $d\gamma^i/ds$ as computed from (10) in (12) gives

$$\tau = \csc^2\phi\epsilon_{ikl}\left(\lambda^i - \cos\phi\frac{dx^i}{ds}\right)\left(-\cot\phi\frac{dx^k}{ds}\lambda^l\frac{d\phi}{ds} + \frac{dx^k}{ds}\frac{d\lambda^l}{ds} + \frac{d^2x^k}{ds^2}\lambda^l\right). \tag{13}$$

On dropping the vanishing determinants in (13), one finds

$$\tau = \csc^2\phi\epsilon_{ikl}\left(\lambda^i\frac{dx^k}{ds}\frac{d\lambda^l}{ds} - \cos\phi\frac{dx^i}{ds}\frac{d^2x^k}{ds^2}\lambda^l\right) \tag{14}$$

or, by a permutation of the indices $i$, $k$, $l$,

$$\tau = \csc^2\phi\epsilon_{ikl}\lambda^i\frac{dx^k}{ds}\left(\frac{d\lambda^l}{ds} - \cos\phi\frac{d^2x^l}{ds^2}\right). \tag{15}$$

Now, by use of equations (3), (4), and the fact that

$$\frac{d^2x^k}{ds^2} = \rho^\gamma x^k{}_{,\gamma} + K_nX^k,$$

the expression in parentheses in equation (15) can be written in the form

$$[(\mu^\gamma{}_\alpha - p_\alpha\rho^\gamma)x^l{}_{,\gamma} + (\nu_\alpha - p_\alpha K_n)X^l]u'^\alpha. \tag{16}$$

On using the expressions (2) and (16) in (15), and on observing that [2, pp. 135, 213]

$$\epsilon_{ikl}X^i x^k{}_{,\alpha} x^l{}_{,\beta} = e_{\alpha\beta}\sqrt{g} \equiv \epsilon_{\alpha\beta},$$

one finds that the formula for $\tau$ becomes

$$\tau = \csc^2 \phi \epsilon_{\tau\alpha}[(p^\tau \nu_\beta - q\mu^\tau{}_\beta)u'^\alpha u'^\beta + p_\beta(q\rho^\tau - p^\tau K_n)u'^\alpha u'^\beta. \tag{17}$$

Because $C$ is a union curve, equation (6) holds, and the formula for $\tau$ reduces to

$$\tau = \csc^2 \phi \epsilon_{\tau\alpha}(p^\beta \nu_\beta - q\mu^{\tau\beta})u'^\alpha u'^\beta. \tag{18}$$

**4. Union torsion of a curve.** If the congruence is required to be normal to the surface, $p^\tau = 0$, $(\tau = 1, 2)$; $q = 1$, $\phi = \pi/2$, the union curve $C$ becomes a geodesic curve on the surface, and $\tau$ becomes $\tau_g$, the geodesic torsion of the geodesic curve through $P$. By equations (5), equation (18) becomes, in the present case of a normal congruence,

$$\tau_g = \epsilon_{\beta\alpha} d_{\beta\sigma} g^{\sigma\tau} u'^\alpha u'^\beta, \tag{19}$$

which is equivalent to the formula for $\tau_g$ as given by Eisenhart [2, p. 247].

Because the expression for $\tau$ in (18) depends only upon a point and a direction through the point on the surface, it is the same for all curves through a point with a common surface tangent. Thus, the *union torsion* of a curve $C$ on a surface may be defined as the torsion of the union curve on the surface in the direction of the curve $C$.

The net of curves in which the developables of the congruence (2) intersect the surface is represented by the differential equation [3, p. 992]

$$\epsilon_{\alpha\tau}(p^\tau \nu_\beta - q\mu^\tau{}_\beta)du^\alpha du^\beta = 0. \tag{20}$$

The net given by equation (20) may be styled the *intersector net.*

On comparing equations (18) and (20), one concludes that *a union curve relative to a congruence is a plane curve if, and only if, it is tangent to a curve of the intersector net of the congruence.* This theorem is a generalization of a classical theorem which states that a geodesic is a plane curve if, and only if, it is a line of curvature, and it also provides a metric analogue of a theorem stated by Lane [4, p. 107] for projective space.

**References**

1. C. E. Springer, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 686–691.
2. L. P. Eisenhart, Differential Geometry, Princeton University Press, 1940.
3. C. E. Springer, Rectilinear congruences whose developables intersect a surface in its lines of curvature, Bull. Amer. Math. Soc. vol. 51 (1945) p. 992.
4. E. P. Lane, Projective differential geometry of curves and surfaces, University of Chicago Press, 1931, p. 107.

# THE CURVATURES OF THE POLAR CURVES OF A GENERAL ALGEBRAIC CURVE*

EDWARD KASNER AND JOHN DE CICCO, Columbia University and Illinois Institute of Technology

**1. Introduction.** In this paper we present a new theorem in the polar theory of a general algebraic curve $C_n$ of degree $n$. If the first polar $C_{n-1}$ of degree $(n-1)$ with respect to the algebraic curve $C_n$ of an ordinary point $O$ on $C_n$ is constructed, then $C_n$ and $C_{n-1}$ are initially tangent. It is found that the curvatures at $O$ are in general distinct. We study the ratio $\rho_1$ of the curvature of the first polar $C_{n-1}$ to that of $C_n$; it is proved that this ratio $\rho_1$ is given by a simple rational formula involving the degree $n$ alone. Thus $\rho_1$ is independent of the nature of the algebraic curve $C_n$.

In particular, it follows from our theory that the polar conic $C_2$ of any ordinary point $O$ of a cubic curve $C_3$ not only touches $C_3$ but also has its curvature equal to one half of the curvature of $C_3$ at $O$.

Since the $r$th polar $C_{n-r}$ of degree $(n-r)$ of a point with respect to the algebraic curve $C_n$ may be defined by induction as the first polar of the same point with respect to the $(r-1)$ polar $C_{n-r+1}$ of degree $(n-r+1)$, our discussion is extended quite readily to the case of the $r$th polar curve $C_{n-r}$ of an ordinary point $O$ on $C_n$. The two algebraic curves $C_n$ and $C_{n-r}$ are tangent at $O$. We show that the ratio $\rho_r$ of the curvature of the $r$th polar $C_{n-r}$ to that of $C_n$ is given by a simple rational formula involving the positive integers $n$ and $r$ only. Therefore $\rho_r$ is independent of the polynomial equation defining $C_n$.

Although our new theorem is stated in metric terms, it is essentially a theorem of differential projective geometry. This is a consequence of the theorem of Mehmke-Segre which states that if any two curves are tangent at a given point $O$, then the ratio $\rho$ of their curvatures at $O$ is a projective invariant.

**2. The polar curves $C_{n-r}$ of an algebraic curve $C_n$.** If $(x)=(x_1, x_2, x_3)$ denote homogeneous coördinates of a point in the plane, then an algebraic curve $C_n$ of degree $n$ is defined by the equation

$$(1) \qquad C_n: a_x^n = f(x) = f(x_1, x_2, x_3) = \sum_{i+j+k=n} a_{ijk} x_1^i x_2^j x_3^k = 0,$$

where the summation extends over the indices $(i, j, k)$ such that $i+j+k=n$.

The first polar curve $C_{n-1}$ of degree $n-1$ of the point $(y)=(y_1, y_2, y_3)$ with respect to the algebraic curve $C_n$ is defined by the equation [1]

$$(2) \qquad C_{n-1}: a_y a_x^{n-1} = \Delta_y f(x) = \sum_i y_i \frac{\partial f}{\partial x_i} = y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} + y_3 \frac{\partial f}{\partial x_3} = 0.$$

By induction, the $r$th polar curve $C_{n-r}$ of degree $n-r$ of a point with respect to the algebraic curve $C_n$ may be defined as the first polar of the same point with respect to the $(r-1)$ polar $C_{n-r+1}$ of degree $n-r+1$. The $r$th polar curve

* Presented to the American Mathematical Society, February, 1947.

$C_{n-r}$ is given by the equation

$$
(3)\qquad \begin{aligned} C_{n-r}: \overset{r}{a_y}\overset{n-r}{a_x} &= \overset{r}{\Delta_y} f(x) = \Delta_y(\overset{r-1}{\Delta_y} f(x)) \\ &= \sum_{i+j+k=r} \frac{\lfloor r}{\lfloor i \lfloor j \lfloor k} y_1^i y_2^j y_3^k \frac{\partial^r f}{\partial x_1^i \partial x_2^j \partial x_3^k} = 0. \end{aligned}
$$

For the operator $\Delta_y^r f(x)$ defined by equations (2) and (3), the identity

$$
(4)\qquad \frac{1}{\lfloor r} \overset{r}{\Delta_y} f(x) = \frac{1}{\lfloor n-r} \overset{n-r}{\Delta_x} f(y)
$$

may be proved by means of the Taylor series expansion of $f(ax+by)=f(ax_1+by_1, ax_2+by_2, ax_3+by_3)$.

From relations (3) and (4), it is seen that the $r$th polar curve $C_{n-r}$ may be defined by either of the equations

$$
(5)\qquad C_{n-r}: \overset{r}{a_y}\overset{n-r}{a_x} = \overset{r}{\Delta_y} f(x) = 0, \quad \text{or} \quad \overset{n-r}{a_x}\overset{r}{a_y} = \overset{n-r}{\Delta_x} f(y) = 0.
$$

In geometric language, this means that if $(x)$ is on the $r$th polar $C_{n-r}$ of $(y)$ with respect to the algebraic curve $C_n$, then $(y)$ is on the $(n-r)$th polar $C_r$ of $(x)$ with respect to $C_n$.

From (2), it is evident that the first polar passes through all the singular points of the algebraic curve $C_n$. Hence any polar passes through all the singular points of the preceding polar.

**3. The non-homogeneous equations of the polars $C_{n-r}$.** In order to give a simple demonstration of our theorem, it is found advisable to obtain the equations of the polar curves $C_{n-r}$ in non-homogeneous coördinates [2].

There is no loss in generality by assuming that $(x, y)$ are cartesian coördinates of a point where $x=x_1/x_3$ and $y=x_2/x_3$. The equation of the algebraic curve $C_n$ as given in the homogeneous form (1) is now of the form

$$
(6)\qquad C_n: \phi(x, y) = x_3^{-n} f(x_1, x_2, x_3) = f(x, y, 1) = 0.
$$

Of course, $\phi(x, y)$ is a general non-homogeneous polynomial of degree $n$.

From (6), we may obtain the partial derivatives of first order of $f(x) = f(x_1, x_2, x_3)$ with respect to $(x_1, x_2, x_3)$ in terms of $\phi(x, y)$ and its partial derivatives of first order with respect to $(x, y)$. The appropriate relations are

$$
(7)\qquad \frac{\partial f}{\partial x_1} = x_3^{n-1}\phi_x, \quad \frac{\partial f}{\partial x_2} = x_3^{n-1}\phi_y, \quad \frac{\partial f}{\partial x_3} = x_3^{n-1}(n\phi - x\phi_x - y\phi_y).
$$

Upon placing $X=y_1/y_3$ and $Y=y_2/y_3$ and using (7), we find that the first polar $C_{n-1}$ of $(X, Y)$ with respect to the algebraic curve $C_n$ is

$$
(8)\qquad C_{n-1}: n\phi - (x-X)\phi_x - (y-Y)\phi_y = 0.
$$

This is also the polar line $C_1$ of $(x, y)$ with respect to $C_n$.

By finding the first polar of $(X, Y)$ with respect to the algebraic curve (8), we find that the second polar $C_{n-2}$ of $(X, Y)$ with respect to the algebraic curve $C_n$ is

$$\begin{aligned} C_{n-2}: n(n-1)\phi - 2(n-1)[(x-X)\phi_x + (y-Y)\phi_y] \\ + [(x-X)^2\phi_{xx} + 2(x-X)(y-Y)\phi_{xy} + (y-Y)^2\phi_{yy}] = 0. \end{aligned} \tag{9}$$

This is also the polar conic of $(x, y)$ with respect to $C_n$.

By repeating the above process, we see that the $r$th polar $C_{n-r}$ of $(X, Y)$ with respect to $C_n$ is

$$\begin{aligned} C_{n-r}: {}^nP_r\phi - {}^rC_1^{\,n-1}P_{r-1}\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(1)}\phi \\ + {}^rC_2^{\,n-2}P_{r-2}\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(2)}\phi - \cdots \\ + (-1)^k\,{}^rC_k^{\,n-k}P_{r-k}\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(k)}\phi + \cdots \\ + (-1)^r\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(r)}\phi = 0. \end{aligned} \tag{10}$$

This is also the $(n-r)$ polar $C_r$ of $(x, y)$ with respect to $C_n$.

**4. The ratio $\rho_1$ of the curvatures of $C_{n-1}$ and $C_n$.** The slope $dy/dx = y'$ at any point $(x, y)$ of $C_{n-1}$ as given by (8), is determined by the equation

$$(n-1)(\phi_x + y'\phi_y) - (x-X)(\phi_{xx} + y'\phi_{xy}) - (y-Y)(\phi_{xy} + y'\phi_{yy}) = 0. \tag{11}$$

If $O(X, Y)$ is an ordinary point of the algebraic curve $C_n$, then from (8), it is seen that by taking $x = X$ and $y = Y$, the point $O(X, Y)$ is on the first polar curve $C_{n-1}$. By (11), it is found that the slope $y'$ of $C_{n-1}$ at $O$ is $-\phi_x/\phi_y$, which is also the slope $Y'$ of $C_n$ at $O$. Therefore $C_n$ and $C_{n-1}$ are tangent at $O$. Hence all the polar curves of an ordinary point $O$ of $C_n$ with respect to $C_n$ are tangent to $C_n$ to $O$. From this follows the well-known result that the polar line of an ordinary point $O$ of an algebraic curve $C_n$ is the tangent line of $C_n$ at $O$.

Now we are in a position to state and prove our new proposition.

THEOREM. *The ratio $\rho_1$ of the curvature of the first polar $C_{n-1}$ to that of the algebraic curve $C_n$, constructed at an ordinary point $O(X, Y)$ of $C_n$, is*

$$\rho_1 = \frac{n-2}{n-1}. \tag{12}$$

For example, the curvature of the polar conic section $C_2$ with respect to a cubic curve $C_3$ of an ordinary point $O$ of $C_3$, is $1/2$ the curvature of $C_3$.

As another example, the curvature of the polar cubic $C_3$ with respect to a quartic curve $C_4$ at an ordinary point $O$ of $C_4$ is $2/3$ that of $C_4$.

To prove our theorem, we proceed in the following manner. The second derivative $d^2y/dx^2 = y''$ at any point $(x, y)$ of the first polar $C_{n-1}$ is found by differentiating the equation (11) with respect to $x$. The result is

$$\begin{aligned}(n-2)(\phi_{xx} + 2y'\phi_{xy} + y'^2\phi_{yy}) + (n-1)\phi_y y'' \\ - (x - X)(\phi_{xxx} + 2y'\phi_{xxy} + y'^2\phi_{xyy} + y''\phi_{xy}) \\ - (y - Y)(\phi_{xxy} + 2y'\phi_{xyy} + y'^2\phi_{yyy} + y''\phi_{yy}) = 0.\end{aligned} \tag{13}$$

At the point $O$ where $x = X$, $y = Y$, the second derivative $y''$ of $C_{n-1}$ is given by the equation

$$(n-2)(\phi_{xx} + 2y'\phi_{xy} + y'^2\phi_{yy}) + (n-1)\phi_y y'' = 0. \tag{14}$$

We recall that the second derivative $Y''$ of the algebraic curve $C_n: \phi(x, y) = 0$, is determined by the equation

$$\phi_{xx} + 2Y'\phi_{xy} + Y'^2\phi_{yy} + \phi_y Y'' = 0. \tag{15}$$

Hence using the fact that $y' = Y'$ at the point 0 where $x = X$, $y = Y$, we deduce from the equations (14) and (15), the following result

$$(n-1)y'' = (n-2)Y''. \tag{16}$$

Finally since $y' = Y'$, the ratio $\rho_1$ of the curvature of the first polar curve $C_{n-1}$ to that of $C_n$ at the point $O(X, Y)$ is

$$\rho_1 = \frac{y''}{Y''} = \frac{n-2}{n-1}. \tag{17}$$

This completes the proof of our theorem.

**5. The polar curves $C_{n-r}$ of the origin.** Before giving an alternate proof of our theorem by the application of infinite series expansions, we shall develop the equations of the various polar curves $C_{n-r}$ of the origin $O(0, 0)$ (which need not be on $C_n$) with respect to the algebraic curve $C_n$.

By (1) and (6), any algebraic curve $C_n$ of degree $n$ can be written in the form

$$C_n: \phi(x, y) = \sum_{k=0}^{n} P_k(x, y) = 0, \tag{18}$$

where $P_k(x, y)$ is a homogeneous polynomial in $(x, y)$ of degree $k$.

By (8), it is found that the first polar $C_{n-1}$ of the origin $O$ with respect to $C_n$ is

$$C_{n-1}: \sum_{k=0}^{n-1} (n-k)P_k(x, y) = 0. \tag{19}$$

Upon finding the first polar of the origin with respect to this curve (19), we find that the second polar $C_{n-2}$ of the origin with respect to $C_n$ is

$$C_{n-2}: \sum_{k=0}^{n-2} (n-k)(n-k-1)P_k(x, y) = 0. \tag{20}$$

By repeated application of the above process, we find that the $r$th polar curve $C_{n-r}$ of the origin with respect to $C_n$ is

$$C_{n-r}: \sum_{k=0}^{n-r} (n-k)(n-k-1)\cdots(n-k-r+1)P_k(x, y) = 0. \tag{21}$$

Upon placing $r=n-2$ in the preceding equation, the polar conic $C_2$ of the origin is

$$C_2: n(n-1)P_0 + (n-1)P_1(x, y) + P_2(x, y) = 0. \tag{22}$$

The polar line $C_1$ is

$$C_1: nP_0 + P_1(x, y) = 0. \tag{23}$$

By (18) and (23), it is deduced that the polar line $C_1$ may be defined geometrically as follows. Draw any line $L$ through the point $O$. This line $L$ intersects the algebraic curve $C_n$ in the $n$ points $(P_1, P_2, \cdots, P_n)$ which may be coincident or imaginary. Let $P$ be the generalized harmonic mean of these $n$ points; that is, let $P$ be such that $1/OP=(1/n)\sum_{i=1}^{n} 1/(OP_i)$. As $L$ rotates about the point $O$, the point $P$ describes the polar line $C_1$ of $O$.

Having defined the polar line $C_1$ of the point $O$ geometrically, the first polar $C_{n-1}$ of the point $O$ is seen by (8) to be the locus of points whose polar lines pass through the point $O$. By induction, the $r$th polar $C_{n-r}$ of the point $O$ is the locus of points whose polar lines with respect to the $(r-1)$ polar $C_{n-r+1}$ pass through the point $O$.

If $C_{n-r}$ is the $r$th polar of a point $O$ with respect to $C_n$, then on any line $L$ passing through $O$, the generalized harmonic mean of the $n$ intersections of $L$ with $C_n$ coincides with that of the $(n-r)$ intersections of $L$ with $C_{n-r}$ [3].

**6. Alternate proof of the theorem.** Let $O$ be an ordinary point of the algebraic curve $C_n$. Then by (18), $P_0=0$ and the equation of the tangent line of $C_n$ at $O$ is $P_1(x, y)=0$. By the formulas of the preceding section, it is verified that all the polar curves of $O$ with respect to $C_n$ are tangent to $C_n$ at $O$.

There is no loss of generality if we assume $P_1(x, y) = -y$. Let $\phi_{xx}$ have the value $2a_{20}$ at the origin. By equations (18) and (19), we find that the expressions for $y$ as power series in $x$ of the algebraic curve $C_n$ and the first polar curve $C_{n-1}$ are

$$C_n: y = a_{20}x^2 + \cdots; \quad C_{n-1}: y = \frac{n-2}{n-1}a_{20}x^2 + \cdots. \tag{24}$$

Therefore at the origin $O$, the ratio $\rho_1$ of the curvature of $C_{n-1}$ to that of $C_n$ is

$$\rho_1 = \frac{\dfrac{n-2}{n-1}a_{20}}{a_{20}} = \frac{n-2}{n-1}. \tag{25}$$

**7. The ratio of the curvature of the $r$th polar curve $C_{n-r}$ to that of $C_n$.** By using the fact that the $r$th polar curve $C_{n-r}$ of a point $O$ with respect to the algebraic curve $C_n$ is the first polar curve of $O$ with respect to the $(r-1)$ polar curve $C_{n-r+1}$ of $O$ with respect to $C_n$, we can extend our theorem in the following way.

THEOREM. *The ratio $\rho_r$ of the curvature of the rth polar curve $C_{n-r}$ to that of the algebraic curve $C_n$, constructed at an ordinary point $O$ of $C_n$, is*

$$\rho_r = \frac{n-r-1}{n-1}. \tag{26}$$

For example, consider a general quintic curve $C_5$. At an ordinary point $O$ of $C_5$, construct the polar quartic $C_4$, the polar cubic $C_3$, the polar conic $C_2$, and the polar line $C_1$. The corresponding ratios of the curvatures are 3/4, 1/2, 1/4, and 0.

Another proof of this theorem is the following one. In equation (18), $P_0=0$, and we can take $P_1(x, y) = -y$. Let $\phi_{xx}$ have the value $2a_{20}$ at the origin. By (18) and (21), the expressions for $y$ as power series in $x$ of the algebraic curve $C_n$ and the $r$th polar curve $C_{n-2}$, are

$$C_n: y = a_{20}x^2 + \cdots, \qquad C_{n-r}: y = \frac{n-r-1}{n-1} a_{20}x^2 + \cdots. \tag{27}$$

Hence at the origin $O$, the ratio $\rho_r$ of the curvature of $C_{n-r}$ to that of $C_n$ is

$$\rho_r = \frac{\dfrac{n-r-1}{n-1} a_{20}}{a_{20}} = \frac{n-r-1}{n-1}. \tag{28}$$

**8. Concluding remarks.** We have also studied the ratio $\rho_r$ of the departure of the polar curve $C_{n-r}$ to that of the algebraic curve $C_n$ from their common tangent line in the case where the order of contact is $p$. We have found that this ratio $\rho_r$ is given by a simple rational formula involving only the positive integers $(n, r, p)$. In particular, this applies to inflections.

Also we have given some consideration to the case where the point $O$ on $C_n$ is a singularity. Both $C_n$ and $C_{n-r}$ have a singularity of the same qualitative nature at $O$. An analogous ratio $\rho_r$ can be constructed. We find that in many cases, this ratio $\rho_r$ depends on the coefficients of the polynomial defining the algebraic curve $C_n$.

## REFERENCES

1. Snyder and Sisam, Analytic geometry of space, Henry Holt and Company, New York (1928).

2. Kasner, Determination of the algebraic curves whose polar conics are parabolas, Amer. Journ. Math. 26, 164–168 (1903).

3. Salmon, Treatise on higher plane curves, Dublin: Hodges and Smith, Grafton Street (1852).

## THE DETERMINATION OF THE COMPLEX ZEROS OF A POLYNOMIAL

G. C. BEST, San Diego, California

**1. Introduction.** In this paper a method is explained for the determination of the real parts of complex zeros of a polynomial once their moduli are known. The method discussed is intended for use with Graeffe's method [**1**, **2**, **3**] of root squaring, and could very easily be treated in a class in the Theory of Equations after Graeffe's method for the real roots and the moduli of complex roots of real polynomials had been considered. Graeffe's method, though very convenient when real roots alone are involved, begins to become awkward when complex roots appear, due to the difficulty in determining the arguments of such roots. The difficulty arises largely because of the lack of a method which can be extended easily to cover cases involving many complex roots. The following procedures are offered in the hope of reducing some of these difficulties.

**2. The general method.** Let us consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \cdots + a_nx^n = 0,$$

in which $a_0 \neq 0$. Moreover, let us assume that the modulus $r$ of a pair of complex roots is known. Let $r^2 = \beta$. Then we consider the factor $x^2+\alpha x+\beta$, where $-\alpha/2$ equals the real part of either complex root in question. Let $u = -\alpha/\beta$ and $k = -1/\beta$, and expand the fraction $P(x)/(1-ux-kx^2)$ in ascending powers of $x$. The coefficient of $x^r$, $r = 1, 2, \cdots$, in this expansion is a polynomial $p_r(u)$ of degree $r$ in $u$. A study of the process of synthetic division applied to the above fraction shows that the $p_r$'s are connected by the relationship:

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_0u + a_1 \\ &\cdots\cdots\cdots \\ p_r &= up_{r-1} + kp_{r-2} + a_r \qquad (r = 2, 3, \cdots). \end{aligned} \tag{1}$$

THEOREM. *A necessary and sufficient condition that the coefficients of $x^n$ and $x^{n-1}$ be zero is that $1-ux-kx^2$ or, what amounts to the same thing, $\beta+\alpha x+x^2$ is a factor of $P(x)$.*

To show that this condition is necessary, consider the coefficients of $x^n$ and $x^{n-1}$ zero. Then by relation (1), all coefficients $p_r$ $(r>n)$ also equal zero, it being understood that $a_r = 0$ for $r>n$, that is, there is no remainder. Therefore, $\beta+\alpha x+x^2$ is a factor.

To show that the condition is sufficient, assume that $\beta+\alpha x+x^2$ is a factor of $P(x)$. In this case the quotient will be a polynomial of degree $(n-2)$, and there will be no remainder; hence all coefficients $p_r$ for $r>(n-2)$ will be zero, in particular, those of $x^n$ and $x^{n-1}$.

The greatest common divisor of $p_n$ and $p_{n-1}$ is then obtained by the usual

process. This common factor, set equal to zero, will then give the desired value of $u$, for the argument of the complex number involved is easily determined from $u$.

If, when determining the zeros of $P(x)$ by Graeffe's method, it is found that $s$ pairs of complex roots have the same modulus $r$, then the greatest common divisor of $p_n$ and $p_{n-1}$ will be a polynomial of degree $s$, and it will have real zeros. The determination of these zeros can be made by Graeffe's method if necessary. If the formulas for the determination of $\alpha$ shown in Table I are employed, consideration must be given to multiple moduli, for if an $s$-fold modulus exists, the formulas for $(s-1)$-fold or less will give the result $0/0$.

TABLE I. FORMULAS FOR $\alpha$ FOR KNOWN $\beta$

| Degree $n$ of $P(x)$ | Number of identical moduli | Formula from which to determine $\alpha$ in factor $x^2+\alpha x+\beta$ |
|---|---|---|
| 3 | 1 | $\alpha = \dfrac{a_1\beta - a_3\beta^2}{a_0}$ |
| 4 | 1 | $\alpha = \dfrac{a_1\beta - a_3\beta^2}{a_0 - a_4\beta^2}$ |
| | 2 | $a_0\alpha^2 - (a_1\beta)\alpha - (a_0\beta - a_2\beta^2 + a_4\beta^3) = 0$ |
| 5 | 1 | $\alpha = \dfrac{a_0a_1\beta - a_0a_3\beta^2 + a_2a_5\beta^4 - a_4a_5\beta^5}{+a_0 - a_0a_4\beta^2 + a_1a_5\beta^3 - a_5^2\beta^5}$ |
| | 2 | $a_0\alpha^2 - (a_1\beta - a_5\beta^3)\alpha - (a_0\beta - a_2\beta^2 + a_4\beta^3) = 0$ |

**3. The derivation of formulas for $n=3, 4, 5$.** Consider the first division in the usual algorithm for the determination of the greatest common divisor of $p_n(u)$ and $p_{n-1}(u)$. Since $p_n(u) = up_{n-1}(u) + kp_{n-2}(u) + a_n$, evidently $kp_{n-2}(u) + a_n$ is the first remainder. (The first quotient is $u$.) Any common factor of $p_n$ and $p_{n-1}$ is a factor of

$$kp_{n-2}(u) + a_n. \tag{2}$$

If $n=3$, the polynomial (2) is linear, and must give the common root $u$ that is desired. This gives

$$\alpha = \frac{\beta a_1 - \beta^2 a_3}{a_0}.$$

The next step in the greatest common divisor algorithm, unnecessary when $n=3$, calls for dividing $p_{n-1}(u)$ by the polynomial (2). Since

$$p_{n-1}(u) = up_{n-2}(u) + kp_{n-3}(u) + a_{n-1},$$

we have

$$kp_{n-1}(u) - u[kp_{n-2}(u) + a_n] = k^2p_{n-3}(u) + ka_{n-1} - a_nu, \tag{3}$$

where the dividend $p_{n-1}(u)$ has been multiplied by the constant $k$, a procedure that is permissible in the algorithm. The right member of (3) is the desired linear

common factor when $n=4$, $s=1$, and is a polynomial of degree $(n-3)$, divisible by the greatest common divisor of $p_n$ and $p_{n-1}$ when $n=5$. The next step in the algorithm results in

$$k^3p_{n-4}(u) + k^2a_{n-2} + ka_n - ka_{n-1}u + a_nu^2. \tag{4}$$

The process will not be carried farther as the resulting formulas become impracticably complicated. Equations (1) to (4) yield the results shown in Table I for polynomials up to the fifth degree. The $p_r$'s can be developed from equation (1) with the substitutions $u=-\alpha/\beta$ and $k=-1/\beta$ made to obtain the final forms indicated.

When determining the arguments for complex zeros of polynomials of degree higher than the fifth, it is suggested that each time a pair of complex zeros is completely determined $P(x)$ be divided by the associated quadratic factor, thus reducing its degree and the amount of labor involved in determining the remaining zeros. When the resulting polynomial is eventually reduced to one of fourth or fifth degree, use can be made of the equations of Table I, thereby greatly facilitating the completion of the work.

**4. Tabular method.** As an illustrative example, consider the polynomial

$$x^5 + 8x^4 + 9x^3 - 22x^2 - 36x - 72 = 0.$$

Assume it is known that the moduli of one pair of complex zeros of this polynomial have the product 2. Then $\beta=2$ and $k=-1/2$.

A tabular method based upon equation (1) is illustrated in Table II. The coefficients of $P(x)$ are arranged in ascending powers of $x$ across the top of the table. In column 3 the degree $r$ of the resulting polynomial in $u$ is indicated. The coefficients for the polynomials in $u$ are arranged in horizontal rows for descending powers of $u$. For example, for $r=3$, reference to row 6 reveals the polynomial

$$-72u^3 - 36u^2 + 50u + 27.$$

This polynomial is the coefficient of $x^3$ in the expansion of

$$\frac{x^5 + 8x^4 + 9x^3 - 22x^2 - 36x - 72}{1 - ux + .5x^2}.$$

Mention should be made of the shift of columns horizontally in connection with the explanatory equations shown in column 2. For instance, the elements of row 3, designated as $R_3$, are equal to $kp_1$ or $-\frac{1}{2}R_1$, moved two columns to the right. A similar shift is involved in $R_5$, $R_7$, and $R_9$. Each number in a row for $p_r$ is the sum of the two numbers above it. The coefficients $a_r$ across the top line are added to the elements in the corresponding lines $p_r$. The elements of row 10 correspond to $p_5$; at that point the procedure for building $p_r$'s is stopped. The process thereafter is that of determining the greatest common divisor of $p_4$ and $p_5$. Row 11, for example, is $-up_4$ so as to match the highest

powers of $u$. Row 12 $=p_5-up_4$. Subsequent processes are similar, namely, that of shifting columns to match highest powers of $u$ and multiplying by constants to equate corresponding coefficients, continually reducing the degree of the difference. In row 21 the equation $2u+1=0$ is indicated. Hence $u=-1/2$, and since $u=-\alpha/\beta$,

$$\alpha = -\beta u = -2(-\tfrac{1}{2}) = +1.$$

Therefore the quadratic factor sought is $x^2+x+2$, and the complex roots finally obtained are

$$x = \frac{-1 \pm \sqrt{7}\,i}{2}.$$

TABLE II. COMPUTATION OF $u$ FOR KNOWN $k$

| 1 | 2 | 3 | 4<br>$a_0$ | 5<br>$a_1$ | 6<br>$a_2$ | 7<br>$a_3$ | 8<br>$a_4$ | 9<br>$a_5$ | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| Item | Equation | $p_r$ | −72 | −36 | −22 | + 9 | + 8 | + 1 | | |
| | | | 0 | | | | | | | |
| 1 | | $p_0$ | −72 | | | | | | | |
| | | | 0 | 0 | | | | | | |
| 2 | | $p_1$ | −72 | −36 | | | | | | |
| 3 | $k\times R_1$* | | 0 | 0 | +36 | | | | | |
| 4 | $R_2+R_3$ | $p_2$ | −72 | −36 | +14 | | | | | |
| 5 | $k\times R_2$* | | 0 | 0 | +36 | +18 | | | | |
| 6 | $R_4+R_5$ | $p_3$ | −72 | −36 | +50 | +27 | | | | |
| 7 | $k\times R_4$* | | 0 | 0 | +36 | +18 | − 7 | | | |
| 8 | $R_6+R_7$ | $p_4$ | −72 | −36 | +86 | +45 | + 1 | | | |
| 9 | $k\times R_6$* | | 0 | 0 | + 36 | +18 | −25 | −13.5 | | |
| 10 | $R_8+R_9$ | $p_5$ | −72 | −36 | +122 | +63 | −24 | −12.5 | | |
| 11 | $-R_8$ | | +72 | +36 | −86 | −45 | − 1 | 0 | | |
| 12 | $R_{10}+R_{11}$ | | | | +36 | +18 | −25 | −12.5 | | |
| 13 | $2\times R_{12}$ | | | | +72 | +36 | −50 | −25 | 0 | |
| 14 | $-R_{11}$ | | | | −72 | −36 | +86 | +45 | + 1 | |
| 15 | $R_{13}+R_{14}$ | | | | | | +36 | +20 | + 1 | |
| 16 | $-R_{12}$ | | | | | | −36 | −18 | + 25 | + 12.5 |
| 17 | $R_{15}+R_{16}$ | | | | | | | + 2 | + 26 | + 12.5 |
| 18 | $18\times R_{17}$ | | | | | | | +36 | +468 | +225 |
| 19 | $-R_{15}$ | | | | | | | −36 | − 20 | − 1 |
| 20 | $R_{18}+R_{19}$ | | | | | | | | +448 | +224 |
| 21 | $R_{20}/224$ | | | | | | | | 2 | 1 |

* This result is to be moved two rows to the right.

The method, though occupying considerable space, is fairly simple to follow and proceeds rather rapidly. If the zeros are required to many decimal places, a vertical arrangement would be superior. Also if a calculating machine is used, the operations corresponding to rows 3, 5, 7, 9, and so on, in Table II may be made upon the machine and omitted from the table.

If $\beta$ is known only approximately, the assumption that the coefficients of $x^{n-1}$ and all higher powers of $x$ are zero is slightly in error. Consequently, a precise common factor will not exist; since, however, the degree of the desired common factor is known, that is, the multiplicity of the modulus, there is no ambiguity as to when to stop the greatest common divisor algorithm. The resulting $\alpha$ will, in such cases, also be approximate.

**References**

1. E. T. Whittaker and G. Robinson, The Calculus of Observations, Blackie & Son, 2nd ed. p. 106.

2. C. Runge and H. Konig, Numerisches Rechnen, Julius Springer, p. 164.

3. Th. v. Kármán and T. A. Biot, Mathematical Methods in Engineering, McGraw-Hill, 1st ed. p. 194.

---

# MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

## A SLOWLY DIVERGENT SERIES

R. P. AGNEW, Cornell University

It is well known, and is easily proved by the integral test for convergence and divergence of series, that the series

$$\sum \frac{1}{n}, \quad \sum \frac{1}{n \log n}, \quad \sum \frac{1}{n \log n \log \log n}, \cdots \tag{1}$$

are all divergent. These are classic examples of slowly divergent series, each one after the first diverging more slowly than its predecessor. The following theorem, which is a simple corollary of a theorem of B. Pettineo,* gives a series which diverges more slowly than any of those in (1).

* B. Pettineo. Estensione di una classe di serie divergenti. Atti della Reale Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali. Series 8, vol 1 (1946), pp. 680–685.

THEOREM. *For each $x>0$, let $P(x)$ denote the product of $x$ and all of the numbers*

$$\log x, \quad \log\log x, \quad \log\log\log x, \cdots \tag{2}$$

*which are greater than* 1. *Then the series*

$$\sum_{n=1}^{\infty} \frac{1}{P(n)} \tag{3}$$

*is divergent.*

Following a method different from that of Pettineo we shall use the integral test, proving that (3) is divergent by proving that

$$\int_1^{\infty} \frac{1}{P(x)}\,dx = \infty.$$

The function $1/P(x)$ is continuous and decreasing over the infinite interval $x \geqq 1$. Over the interval $1 \leqq x \leqq e$, $1/P(x)$ is $1/x$; hence

$$\int_1^{e} \frac{1}{P(x)}\,dx = \log x \Big]_1^{e} = 1.$$

Over the interval $e \leqq x \leqq e_2 = e^e$, $1/P(x)$ is $1/(x \log x)$; hence

$$\int_e^{e_2} \frac{1}{P(x)}\,dx = \log\log x \Big]_e^{e_2} = 1.$$

Over the interval $e_2 \leqq x \leqq e_3 = \exp e_2$, $1/P(x)$ is $1/[x(\log x)\log\log x]$; hence

$$\int_{e_2}^{e_3} \frac{1}{P(x)}\,dx = \log\log\log x \Big]_{e_2}^{e_3} = 1.$$

Continuation of this process gives an infinite set of intervals (of rapidly increasing lengths) over each of which the integral of $1/P(x)$ is 1. The results follow from this.

## RESIDUE OF $\sigma_k(n)$ MODULO 2

A. R. NASIR, Talim-ul-Islam College, India

Let $\sigma_k(n) = \sum_{d/n} d^k$, so that $\sigma_k(n)$ denotes the sum of the $k$th powers of the divisors of $n$. By entirely elementary methods we prove two theorems which completely determine the residue of $\sigma_k(n)$.

THEOREM A. *If $n = 2^a \prod_{p/n} p^{2b_i}$, $a \geqq 0$, $b_i \geqq 0$ for every $i$, $1 \leqq i \leqq m$, and $p$'s are distinct primes $>2$, then $\sigma_k(n) \equiv 1$ (mod 2).*

It is sufficient to consider the case when $a>0$ and $b_i>0$ for every $i$. Write

$$\sigma_k(n) = \left(1 + \sum_{x=1}^{x=a} 2^{xk}\right) \prod_{i=1}^{i=m} \left(1 + \sum_{y=1}^{y=2b_i} p^{yk}\right).$$

Now $\sum_{y=1}^{y=2b_j} p^{yk}$, for $i=j$, represents the sum of an even number of integers each $\equiv 1 \pmod 2$.

Therefore the factor of $\sigma_k(n)$ corresponding to any $p$ is congruent to 1 (mod 2). The factor corresponding to 2 evidently is congruent to 1 (mod 2). Hence our result follows.

THEOREM B. *If* $n = 2^a p^b \prod_{q/n} q^{c_i}$, $a \geqq 0$, *$b$ an odd integer* $>0$, $c_i \geqq 0$ *for every* $i$, $1 \leqq i \leqq m$, *$p$ and $q$'s all distinct primes* $>2$, *then* $\sigma_k(n) \equiv 0$ (*mod* 2).

Here the factor corresponding to $p$ is congruent to 0 (mod 2). For, $\sum_{b=1}^{b_j} p^{bk}$ represents the sum of an odd number of integers each $\equiv 1 \pmod 2$.

## NAGEL POINT IN THE TETRAHEDRON

V. THÉBAULT, Tennie, Sarthe, France

**1. A well known property.** In a triangle $ABC$, the lines joining the vertices $A$, $B$, $C$ to the contact points $D_a$, $E_b$, $F_c$ of the ex-circles $I_a$, $I_b$, $I_c$ with the sides $BC$, $CA$, $AB$, respectively, between $B$ and $C$, $C$ and $A$, $A$ and $B$, meet at the Nagel point $N$ of the triangle; and the points $D_a$, $E_b$, $F_c$ divide the perimeter of the triangle in equivalent parts

$$\begin{aligned} AB + BD_a &= D_aC + CA = BC + CE_b = E_bA + AB \\ &= CA + AF_c = F_cB + BC \\ &= \frac{BC + CA + AB}{2}, \end{aligned}$$

and conversely. Further, the Nagel point is the in-center of the anticomplementary triangle $A_1B_1C_1$, and its associates $N_a$, $N_b$, $N_c$ have similar properties.

**2. Analogue in space.** Let $T \equiv ABCD$ be any tetrahedron, $a$ and $a'$, $b$ and $b'$, $c$ and $c'$ the lengths of the edges $BC$ and $DA$, $CA$ and $DB$, $AB$ and $DC$; $A$, $B$, $C$, $D$ the areas of the faces $BCD$, $CDA$, $DAB$, $ABC$; $h_a$, $h_b$, $h_c$, $h_d$ the altitudes from $A$, $B$, $C$, $D$; $r$ the radius of the in-sphere; $V$ the volume. A point $A'$ in the plane $BCD$ is such that

$$D + BCA' = B + CDA' = C + DBA' = \frac{(A + B + C + D)}{3}, \tag{1}$$

if and only if

$$D + \frac{ax}{2} = B + \frac{c'y}{2} = C + \frac{b'z}{2} = \frac{A + B + C + D}{3},$$

$x$, $y$, $z$ being the distances from $A'$ to the edges $BC$, $CD$, $DB$. It follows that

$$ax = \frac{A + B + C + D - 3D}{3} = 2V\left(\frac{1}{2} - \frac{3}{h_d}\right) = \frac{2A(h_d - 3r)}{3r};$$

by circular permutations we find similar values for $c'y$, $b'z$; and we have also similar relations for the points $B'$, $C'$, $D'$ in the planes $CDA$, $DAB$, $ABC$.

These barycentric coördinates $(ax, c'y, b'z), \cdots$, of the points $A'$, $B'$, $C'$, $D'$, in the triangles $BCD$, $CDA$, $DAB$, $ABC$, are proportional to the barycentric coördinates $A(h_a-3r)$, $B(h_b-3r)$, $C(h_c-3r)$, $D(h_d-3r)$ with respect to $T$, of the center $I_1$, of the in-sphere of the tetrahedron $T_1 \equiv A_1B_1C_1D_1$, anticomplementary to $T$, and whose distances to the faces of $T$ are $h_a-3r$, $h_b-3r$, $h_c-3r$, $h_d-3r$. The lines $AI_1$, $BI_1$, $CI_1$, $DI_1$ meet the faces $BCD$, $CDA$, $DAB$, $ABC$, of $T$ at the points $A'$, $B'$, $C'$, $D'$ corresponding to the above relations (1) and to the other similar ones. Hence we have the following theorem.

THEOREM. *In any tetrahedron $T \equiv ABCD$, the lines $AA'$, $BB'$, $CC'$, $DD'$ joining the vertices to the points $A'$, $B'$, $C'$, $D'$, corresponding to the relations* (1) *and to the other similar ones, meet at a point* (*Nagel point*) *which is also the center, $I_1$, of the in-sphere of the anticomplementary tetrahedron $T_1$.*

To the point $I_1$ correspond *seven associated points* coinciding with the centers of the four escribed spheres contained in the trunks and the three escribed spheres contained in the roofs. These seven points correspond to the sets of points $(A'', B'', C'', D'')$ and $(A''', B''', C''', D''')$ in the planes $BCD$, $CDA$, $DAB$, $ABC$. Thus we have the relations between areas,

$$-D + BCA'' = -B + CDA'' = -C + DBA'' = \frac{A - B - C - D}{3}, \cdots$$

similar to (1), corresponding to the center of the escribed sphere of $T_1$ contained in the trunk opposite to the vertex $A_1$.

It is to be noted that, generally, the lines $AA'$, $BB'$, $CC'$, $DD'$ do not pass through the contact points of the escribed spheres with the faces of the tetrahedron $T$.

## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.*

### FOUR USEFUL BLACKBOARD AIDS

R. M. SUTTON, Haverford College

In the teaching of physics, I am constantly aware of the enormous part played by mathematics in the development of that science. It is therefore with satisfaction that I make a partial payment of my debt to mathematics in these suggested simple, physical teaching aids.

**1. A protractor and slope indicator.** Next to a piece of chalk and an eraser, I find the most useful article for blackboard use is a straight edge with a plumb-bob and protractor. That may sound complicated, but it is really very simple. For the past five years I have used a homemade gadget like the one shown in Figure 1. This figure is approximately one fifth the actual size. It consists of a

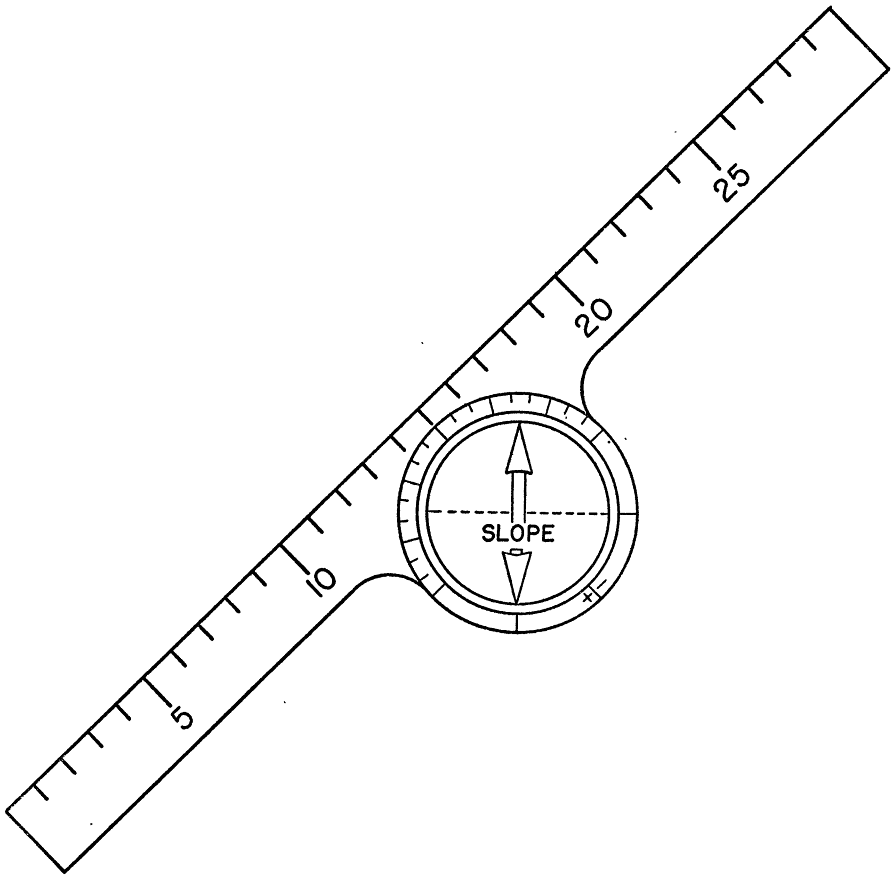



FIG. 1

straight edge about 30 inches long, equipped with a circular scale. At the center of the scale is a pivot on which turns a loaded disk whose center of gravity is well below the pivot. As the straight edge is tilted, this disk remains fixed with respect to the earth and it therefore turns relative to the zero point on the scale. A vertical arrow on the disk points to 0° on the scale when the edge is horizontal. It likewise indicates the inclination of the edge in any other orientation.

The upper half of the scale is graduated from $-90°$ to $+90°$. For analytic geometry, a second arrow pointing downward traverses another scale on the

lower half of the circle. This second scale is graduated directly in slope, positive on the left, negative on the right.

The protrusion on the side of the straight edge makes a convenient handle by which to grip the instrument while chalk is being run along the drawing edge. If the disk rotates freely, the error in setting the instrument does not exceed one degree.

Among the uses of the instrument are: drawing accurately parallel, vertical or horizontal lines, or lines at any desired inclination, drawing parallelograms, equilateral triangles, or other regular polygons in any orientation; drawing orthogonal coördinate axes in any orientation, especially good for showing transformation of coördinates; and drawing components of a vector resolved along arbitrarily chosen axes. With a few minutes practice, excellent five-pointed stars can be drawn. The ease and accuracy with which the instructor can produce accurate figures is sufficient justification for the expenditure of the few extra seconds required to set the directed straight edge on the blackboard.

**2. Slides for projecting coördinate systems.** In the teaching of coördinate geometry, the instructor either wastes time drawing accurate coördinate systems on which to locate specified points or lines (Instrument #1 may improve this part of his work), or he may draw a slap-dash figure that later embarrasses him and leaves the class with a low estimate of his muscular ability. A simple technique is available for obtaining a coördinate system almost instantly: use a projection lantern and an appropriate slide. Simply prepare a transparent slide for the kind of coördinate system desired and project it on the blackboard. Curtains do not have to be drawn. Like the use of black chalk on a blackboard (which sounds absurd, but is actually quite feasible), the use of a slide makes the coördinate system show up clearly. The lines of the system are not erasable, a feature which is very convenient when changes in the figure are to be made. The only disadvantage is that the system disappears from view under the shadow of the instructor's hand or body, but this may help to remind him to stand out of the way of his work whenever he wants students to see what he has drawn.

Suggested slides for use in the projection lantern are: plane cartesian coördinates with origin at center of field of view; cartesian system for first quadrant only; plane polar coördinates; semi-logarithmic coördinates; log-log coördinates; plane representation of three-dimensional cartesians or polars, and so forth. Slides may be either made photographically from systems drawn on paper, or they may be prepared by drawing with India ink on cellophane.

**3. An improved compass.** The simplest and least expensive blackboard compass is a string tied to a piece of chalk. For many purposes this is quite satisfactory. My previous experience with mechanical blackboard compasses leaves me with a low opinion of those that I have seen. The one described here has some advantages not offered by others, and it is recommended where precise drawings are desired.

The basic element of the compass shown in Figure 2 (approximately full size), is an ordinary 6-foot steel tape-measure which rolls into a circular holder about 2 inches in diameter. This holder is modified by the addition of a handle, pivot, and suction cup. The suction cup establishes the center on the blackboard. Moreover, the holder is provided with a narrow channel through which the tape is pulled out and pushed in. This channel offers enough friction to prevent the

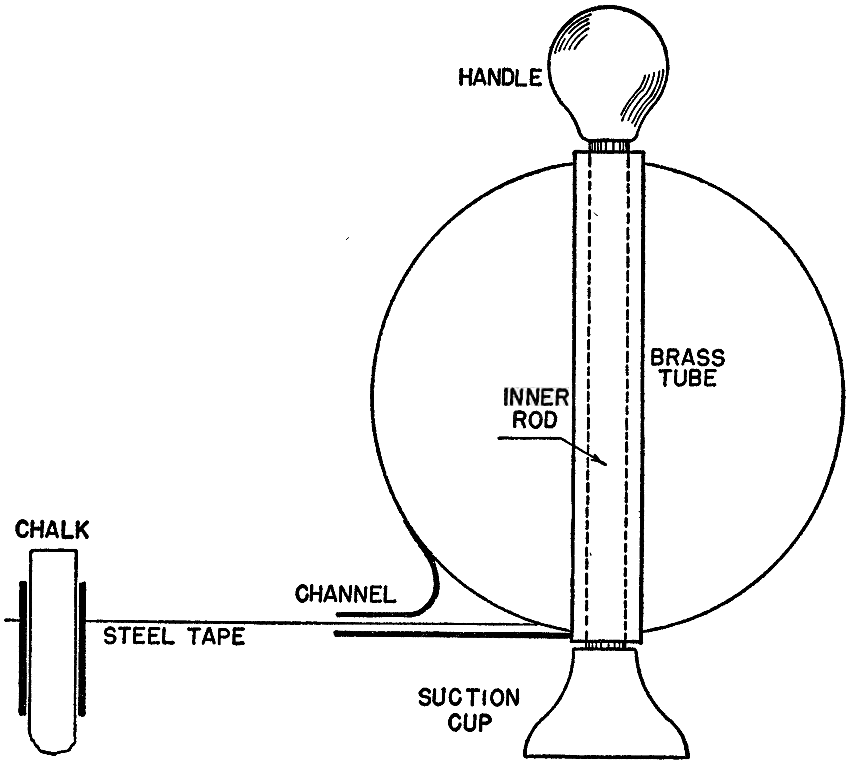



FIG. 2

tape from moving unless the operator chooses to change its length. A tubular fixture is soldered to the end of the tape to hold a piece of chalk. The radius of the circle for which the tape is set can be read immediately from a fixed reference point.

The instrument offers compactness, ease and precision of setting, convenience in drawing, and a long radius (arm's length). It is not very good for circles under five or six inches in radius but it is excellent for larger circles. For drawing a series of concentric circles of specified radii, as in diagrams pertaining to gravitational or electric potential or to spherical wave patterns, it is invaluable.

Only one precaution is needed in using the instrument. The tape may be pulled out easily, but care must be exercised in pushing it back into the holder.

It is not very rigid under compression and may be damaged by sharp bends. Therefore, the operator should push back only two or three inches at a time to prevent buckling under compression.

**4. Templates for drawing sine curves.** Almost too trivial to mention, yet very useful, is a set of templates cut from thick cardboard or from $\frac{1}{4}$-inch plywood for drawing sine and cosine curves. I have seen only one mechanical blackboard sine curve machine. It was complicated, expensive, and not very satisfactory. It would be desirable to have such an instrument constructed in such a manner that there would be no limitation to the possible range of amplitude of the curve and in the scale of length per cycle, but there does not seem to be any simple and satisfactory design. However, it is easy to make a set of templates, any one of which can be held against the blackboard to serve as a pattern for a portion of a sine or cosine curve. These templates may differ in amplitude and length of a cycle. I find them useful when drawing figures to accompany the discussion of simple harmonic motion and wave motions. Particularly when matters of phase difference or interference are involved, they are far superior to free-hand drawings.

Three styles will cover most needs:

(a) a full cycle of amplitude $A$ and length $l$.

(b) two half cycles of different $A$ and $l$; these may conveniently form the upper and lower edges of the template.

(c) a half cycle of $A \sin \theta$ and a full cycle of $A \sin^2 \theta$, again forming the upper and lower edges of the same template.

It is convenient to give each template a handle, to indicate on it the value of $A$ and $l$, and to mark the ordinate at every 15 or 30 degrees.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 771. *Proposed by C. C. Carter, Bluffs, Illinois*

A die bearing the numbers 0, 1, 2, 3, 4, 5 on its faces is repeatedly thrown until the total of the throws first exceeds 12. What is the most likely total that will be thus obtained?

E 772. *Proposed by D. H. Browne, Buffalo, N. Y.*

What is the number of shortest paths between two points in an $n$-dimensional lattice?

E 773. *Proposed by A. L. Rubinoff, University of Toronto*

Suppose that noughts and crosses are played on an $n$-dimensional cube of side $k$. Show that there are precisely

$$\frac{(k+2)^n - k^n}{2}$$

rows, columns, diagonals, $\cdots$ on which a win may be scored.

E 774. *Proposed by Norman Anning, Ann Arbor, Michigan*

Consider points on the median of a triangle. Through the centroid no straight line can be drawn which will cut off one-third of the area. Through a point four-fifths of the distance from vertex to base, four such lines can be drawn. Find points on the median at which the number of possible lines changes.

E 775. *Proposed by R. P. Boas, Jr., Brown University*

Consider a determinant of order $n$ whose elements are $x$ on the main diagonal, $\pm 1$ elsewhere. Find the smallest positive number $a$ such that for $x > a$ the determinant is positive for all choices of the $\pm$ signs.

## SOLUTIONS

### Cupid's Problem

E 740 [1946, 462]. *Proposed by Esther Szekeres, Shanghai, China*

Let there be given five points in the plane. Prove that we can select four of them which determine a convex quadrilateral.

I. *Solution by P. T. Bateman, Yale University.* Let $W$ be the boundary of the convex hull of the five points. If all five points lie on $W$, then either all five points are collinear or else any non-collinear set of four of the points has the desired property. If exactly four of the given points lie on $W$, then these four points have the desired property.

If exactly three of the given points $A$, $B$, $C$ lie on $W$, then $A$, $B$, $C$ are the vertices of a triangle enclosing the other two points $D$ and $E$. Now the line $DE$ cuts the interior of at most two sides of triangle $ABC$. Consider a side whose interior $DE$ does not cut. Then $D$, $E$, and the two vertices on this side have the desired property.

The general case of this problem is discussed by Erdös and G. Szekeres in an article in *Compositio Mathematica*, vol. 2 (1935), pp. 463–470. From nine

points in the plane we can select five which determine a convex pentagon, but whether or not it is possible to select $n$ points from any given $2^{n-2}+1$ points in the plane in such a way that the $n$ points determine a convex $n$-gon is unknown.

II. *Solution by D. K. Pease, University of Connecticut.* We will consider the generalized problem: Let there be given $(n+3)$ points in an $n$-flat. Prove that we can select $(n+2)$ of them which determine a convex polytope.

We exclude cases where any $(m+1)$ points lie in an $m$-flat.

Choose any $(n+1)$ of the points. These determine an $n$-simplex. Now choose either of the remaining points. There will be three cases: $(A)$ The point is within the simplex. $(B)$ The point is within a vertical angle of the simplex. $(C)$ The point determines, with the $(n+1)$ points of the simplex, a convex polytope.

For $(C)$ there is nothing further to show.

$(B)$ is in the same situation as $(A)$ since the point at the vertex angle is within a simplex determined by the $(n+2)$nd point and the other $n$ points.

We now have a simplex with a point inside. The point inside, together with any $n$ points of the simplex, determines $(n+1)$ sub-simplexes.

Now consider the $(n+3)$rd point. It will also fall into cases $(A)$, $(B)$ and $(C)$ with respect to the simplex. For case $(A)$ the point will be within one of the sub-simplexes. This sub-simplex is made up of the vertical angles of the $n$ other sub-simplexes. For case $(B)$ the vertical angle of the simplex is made up of the vertical angles of $n$ of the sub-simplexes. So, for $(A)$ or $(B)$ the $(n+3)$rd point will be in case $(C)$ for $(n-1)$ of the sub-simplexes.

Also solved by Paul Brock, William Gustin, Frank Hawthorne, J. B. Kelly, Norman Miller, and C. R. Perisho.

### A Pythagorean Inequality

E 741 [1946, 532]. *Proposed by William Scott, Columbus, Ohio*

Prove that in a (non-degenerate) right spherical triangle with hypotenuse $c$ and legs $a$, $b$ we have $a^2+b^2>c^2$.

I. *Solution by the Proposer.* We may assume $0<a\leqq b<\pi$, $0<c<\pi$. Also, we have $\cos c=\cos a \cos b$.

If $c\geqq\pi/2$, then $\cos c\leqq 0$. Therefore, for this case, $\cos b\leqq 0$, $\cos c\geqq\cos b$, $c\leqq b$, and $a^2+b^2>c^2$.

Thus we may take $c<\pi/2<\sqrt{8}$. Let $b$ be fixed and note that

$$dc/da = (\sin a \cos b)/\sin c > 0.$$

Let $b<c\leqq\sqrt{2}b$. Then, since $0<c<\sqrt{8}$,

$$(\sin c)/c > 1 - c^2/6 > 1 - c^2/4 + c^4/96 > \cos (c/\sqrt{2}) \geqq \cos b.$$

Therefore

$$d(a^2 + b^2)/da = 2a > 2c \cos b \sin a/\sin c = d(c^2)/da.$$

Since $c=b$ when $a=0$, this shows that $a^2+b^2>c^2$ for $b<c\leqq\sqrt{2}b$. Hence if

$c=\sqrt{2}b$, then $a>b$. Therefore, since $da/dc>0$, if $c>\sqrt{2}b$, then again $a>b$.

Hence $a^2+b^2>c^2$ for any $b$ and all $a$ such that $0<a\leqq b$, and the proof is complete.

II. *Solution by L. M. Kelly, University of Missouri.* The announced relation follows quite nicely from a theorem of L. M. Blumenthal: *If a spherical triangle is reproduced congruently in the plane* (*that is, with length of sides preserved*), *each angle of the plane triangle is less than the corresponding angle of the spherical triangle.* In the particular case at hand, reproduction of the right spherical triangle in the plane yields a plane triangle in which angle $C$ is acute, from which it follows, by the cosine law, that $c^2<a^2+b^2$.

It might be well to indicate a simple proof of Blumenthal's theorem which, it would seem, should be better known than is the case.

Let $ABC$ be the spherical triangle and $A'B'C'$ the corresponding plane triangle, where $BC=B'C'=a$, $CA=C'A'=b$, $AB=A'B'=c$. Now

$$\sin\frac{A'}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

Since $A/2$ and $A'/2$ are certainly less than or equal to $\pi/2$, the theorem will be proved if we can show that the right member of the first equation is less than that of the second. This we can do in the following way. First note that $(\sin x)/x$ is a montone decreasing function for $0<x<\pi/2$. Furthermore, since $a<b+c$, $s-b<c$. Similarly $s-c<b$. Thus

$$\frac{\sin(s-b)}{s-b}\cdot\frac{\sin(s-c)}{s-c}>\frac{\sin b}{b}\cdot\frac{\sin c}{c},$$

or

$$\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}>\frac{(s-b)(s-c)}{bc}.$$

In a precisely analogous fashion we may prove that reproduction in the plane of a pseudo-spherical triangle (a geodesic triangle on the pseudo sphere) results in correspondingly larger angles. These results may be phrased, of course, in terms of congruent imbedding in spherical and hyperbolic spaces. Furthermore, these two results have been extended by Blumenthal as follows: *If a "shortest distance" triangle on a surface having everywhere positive curvature be reproduced congruently in the plane, the angles of the plane triangle will each be less than the corresponding angle of the surface triangle. A similar result holds for surfaces of everywhere negative curvature.* It is necessary to make a careful distinction between geodesic and shortest distance triangles, since they are not always the same.

**Some Tangent Circles of a Triangle**

E 742 [1946, 532]. *Proposed by B. F. Laposky, Cherokee, Iowa*

Let $ABC$ be a triangle, $LMN$ the median triangle, $DEF$ the orthic triangle, $O$ the circumcenter, $J$ the nine-point center, and $T$, $U$, $V$ the other intersections of the medians $AL$, $BM$, $CN$ with the nine-point circle $(J)$. Now there are two sets of circles tangent to the circumcircle at the vertices $A$, $B$, $C$ and also tangent to $(J)$. Show that the circles of one set have their centers at the intersections of $OA$, $OB$, $OC$ with the corresponding sides of $LMN$ and touch $(J)$ at $D$, $E$, $F$; the circles of the other set have their centers at the intersections of $OA$, $OB$, $OC$ with the lines $JT$, $JU$, $JV$ and touch $(J)$ at $T$, $U$, $V$.

*Solution by B. R. Leeds, Brooklyn, N. Y.* Let $S_1$ be the intersection of $DJ$ with $OA$, and $P$ the other intersection of $(J)$ with $AD$. Then, if $H$ is the orthocenter, $HJ=JO$, and $PJ$ is parallel to $OA$. Therefore $\sphericalangle OAD=\sphericalangle JPD=\sphericalangle JDP$, whence $S_1A=S_1D$. Now $MN$ is the perpendicular bisector of the altitude $AD$, and therefore passes through $S_1$. Thus $S_1$ is the center of a circle $(S_1)$ passing through $A$ and $D$. Since $S_1$, $J$, $D$ are collinear, $(S_1)$ is tangent to $(J)$ at $D$. Similarly, since $S_1$, $O$, $A$ are collinear, $(S_1)$ is tangent to $(O)$ at $A$. In a similar manner circles $(S_2)$, $(S_3)$ can be constructed tangent to $(J)$, $(O)$ at $E$, $B$ and $F$, $C$ respectively.

Let $C_1$ be the intersection of $JT$ with $OA$. Since $\sphericalangle ADB$ is a right angle, $PJ$, which is parallel to $OA$, passes through $L$, and $\sphericalangle C_1AT=\sphericalangle TLP=\sphericalangle LTJ=\sphericalangle C_1TA$. Thus $C_1$ is the center of a circle $(C_1)$ passing through $A$ and $T$. Since $C_1$, $T$, $J$ are collinear, $(C_1)$ is tangent to $(J)$ at $T$. Similarly, since $C_1$, $O$, $A$ are collinear, $(C_1)$ is tangent to $(O)$ at $A$. In a similar manner circles $(C_2)$, $(C_3)$ can be constructed tangent to $(J)$, $(O)$ at $U$, $B$ and $V$, $C$ respectively.

Also solved by E. R. Stabler and the proposer.

**Number of Real Solutions of a Transcendental Equation**

E 743 [1946, 532]. *Proposed by E. P. Starke, Rutgers University*

Determine the conditions on the constants $a$ and $b$ such that

$$\log x - ax + b = 0$$

shall have two real solutions, one real solution, or no real solutions.

I. *Solution by Norman Miller, Queen's University.* The question is that of the the points of intersection of the line $y=ax-b$ with the curve $y=\log x$. Inspection of the curve shows that any line with negative or zero slope cuts it in one point only. If $a>0$, three cases arise. The tangent with $y$-intercept $-b$ is found to be $y=xe^{b-1}-b$. It follows that, if $a=e^{b-1}$, the given equation has one real root, which may be regarded as a double root; if $a>e^{b-1}$, the equation has no real root; if $0<a<e^{b-1}$, the equation has two real roots; if $a\leqq 0$, the equation has one real root.

II. *Solution by C. F. Pinzka, North Plainfield, N. J.* Let

$$y = \log x - ax + b,$$

then $y' = 1/x - a$ and $y'' = -1/x^2$, the latter being always positive. If $a \leqq 0$, $y'$ is always positive and, since $y$ increases from $-\infty$ to $+\infty$, there is one real solution. If $a > 0$, $y' = 0$ for $x = 1/a$, and a maximum exists at $y = b - 1 - \log a$. Hence there are no, one, or two real solutions if $a > 0$ according as $b - 1 - \log a$ is less than, equal to, or greater than zero.

Also solved by D. W. Alling, P. T. Bateman, R. G. Blake, Paul Brock, F. L. Celauro, N. J. Fine, I. M. Gardoff, R. T. Hood, Meyer Karlin, L. M. Kelly, R. J. Koch, Sidney Kravitz, LeRoy Pietsch, C. F. Pinzka (also like I), A. Sisk, C. W. Topp, and the proposer.

### A Set of $n$ Positive Integers

E 744 [1946, 532]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots < a_n \leqq 2n$ be $n$ positive integers such that the least common multiple of any two is greater than $2n$. Then $a_1 > [2n/3]$.

I. *Solution by N. J. Fine, Washington, D. C.* It is clear that no one of the numbers can divide another. Hence, writing $a_i = 2^{p_i} A_i$, $A_i$ odd, we see that the $A_i$ are all different. Since there are $n$ of them, they coincide in some order with the set of all odd numbers less than $2n$.

Now consider $a_1 = 2^{p_1} A_1$. If $a_1 \leqq [2n/3]$, then $3a_1 = 2^{p_1} \cdot 3A_1 \leqq 2n$ and $3A_1 < 2n$. Hence $3A_1 = A_j$ for some $j$, and $a_j = 2^{p_j} \cdot 3A_1$. The least common multiple of $a_1$ and $a_j$ is either $2^{p_1} \cdot 3A_1 = 3a_1 \leqq 2n$ or $2^{p_j} \cdot 3A_1 = a_j \leqq 2n$. This contradiction proves the theorem.

II. *Solution by the Proposer.* Suppose $a_1 \leqq [2n/3]$, then $3a_1 \leqq 2n$. Consider $2a_1, 3a_1, a_2, \cdots, a_n$, a set of $n+1$ integers no one of which divides another. This is impossible, whence the theorem.

Also solved by Murray Barbour, P. T. Bateman, N. Kaufman, and J. M. Zucker.

### A Theorem on Arithmetic Progressions

E 745 [1946, 532]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If, in an arithmetic progression of $n$ terms and common difference $d$, $n$ is prime to each term, then $n$ is not prime to $d$.

*Solution by P. T. Bateman, Yale University.* Suppose $n$ were prime to $d$. Then, if the first term of the arithmetic progression is $a$, the numbers $a, a+d, a+2d, \cdots, a+(n-1)d$ would have different residues modulo $n$, and thus one of them would be divisible by $n$, which contradicts the hypothesis.

Also solved by Murray Barbour, Paul Brook, N. J. Fine, William Gustin, R. T. Hood, V. L. Klee, Jr., C. F. Pinzka, Robert Seall, E. D. Schell, David Wellinger, and the proposer. A number of solvers pointed out that this is merely a rewording of a theorem known to Euler. See Dickson, *History of the Theory of Numbers*, p. 113, and *Introduction to the Theory of Numbers*, Theorem 9, p. 6.

# ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins of at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.*

## PROBLEMS FOR SOLUTION

4249. *Proposed by W. B. Campbell, Philadelphia Textile Institute*

A body is projected from a point $O$ in a plane making an angle $A$ with the horizontal, the direction of projection being in a vertical plane containing a line of greatest slope of the plane, and making an angle $B$ with the upward direction of that line. If the plane be smooth and the body perfectly elastic, derive expressions for $t_n$, the time consumed in the $n$th flight, and for $x_n$, the coördinate of the point of impact at the end of the $n$th flight. Will it ever strike $O$ again, and will any of its flights be vertical? What is maximum $x_n$?

4250. *Proposed by Richard Bellman, Princeton University*

If

$$s_n = \sum_{k=1}^{n} a_k, \qquad \sigma_n = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k,$$

and

$$\sum_{n=1}^{\infty} \left| s_n - \sigma_n \right|^k < \infty,$$

for any $k>0$; prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

4251. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron the lines joining the symmedian points (Lemoine points) of the faces to the midpoints of the corresponding altitudes, are concurrent at a point such that the sum of the squares of its distances to the planes of the faces is a minimum.

4252. *Proposed by Paul Erdös, Syracuse University*

It is well known that $2n!/n!(n+1)!$ is always an integer. Prove that for every $k$ there are infinitely many $n$'s such that $2n!/n!(n+k)!$ is an integer.

4253. *Proposed by G. T. Williams, Cambridge, Massachusetts*

Given two tangent unit circles, $C_1$ and $C_2$, and their common external tangent, $T$. A third circle, $C_3$, is drawn tangent to $C_1$, $C_2$, and $T$; $C_4$ is then drawn

tangent to $C_1$, $C_2$ and $C_3$; and so on, each $C_{j+1}$ being tangent to $C_1$, $C_2$, and $C_j$. Find the total area of the aggregate of circles, $C_1, C_2, C_3, \cdots$.

## SOLUTIONS

### Polars and a Family of Circles

3962 [1940, 323]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A circle $(C)$ with a given radius rolls on a fixed circle $(C')$. Find the form of the locus of the points of intersection of $(C)$ with the polar of a fixed point with respect to $(C)$. Consider the cases where $(C')$ reduces to a point, or a straight line.

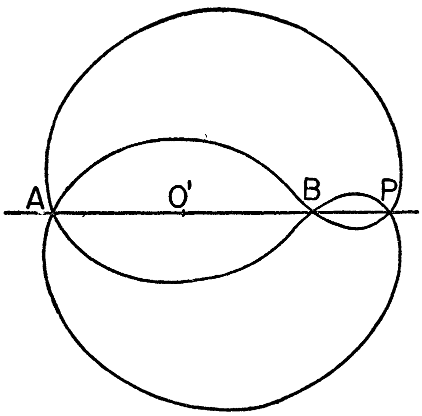



FIG. 1

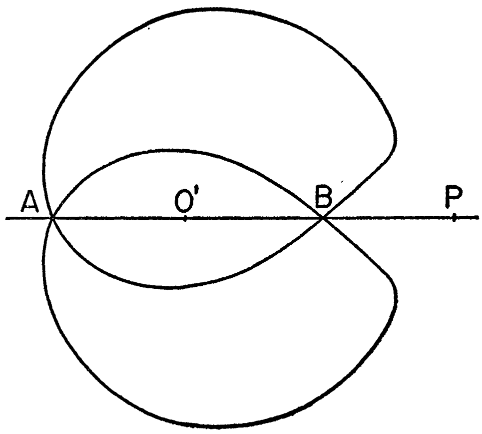



FIG. 2

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.* The origin is the center $O'$ of the fixed circle $(C')$. The axis $O'X$ is directed through $P$, with $O'P=a$. The locus of the center of $(C)$ is a circle with center $O'$: let $R$ be its radius, and let $r$ be the radius of $(C)$. The family of circles $(C)$ is then given by

$$(x - R\cos\phi)^2 + (y - R\sin\phi)^2 = r^2.$$

The points of contact of the tangents to $(C)$ through $P$ are on the circle of diameter $PO$ ($O$ being the center of $(C)$) whose equation is

$$(x - a)(x - R\cos\phi) + y(y - R\sin\phi) = 0,$$

whence the desired locus has the parametric representation

$$\text{(1)} \qquad \frac{x - R\cos\phi}{y} = \frac{y - R\sin\phi}{a - x} = \frac{\pm r}{[(x-a)^2 + y^2]^{1/2}}.$$

Translating the origin to $P$, and using polar coördinates, we obtain the equation of the locus in the form

$$(\rho + a\cos\theta)^2 = (R - r \pm a\sin\theta)(R + r \mp a\sin\theta).$$

This equation defines a tricircular sextic having three double-points at a finite distance on $O'X$: $P$ and the points $A$ and $B$ symmetric with respect to $O'$ and such that $O'A = O'B = (R^2 - r^2)^{1/2}$. Furthermore the curve has $O'X$ as axis of symmetry. It is not difficult to sketch. Figure 1 shows such a curve for the case $r < R < a < R + r$. $P$ becomes an isolated double-point when either $a < R - r$ or $a > R + r$; the latter case is shown in Figure 2. If the radius of $(C')$ is zero, so that $R = r$, the two double-points $A$ and $B$ are brought into coincidence at $O'$ so that the two branches of the curve are tangent to $O'X$ at $O'$. If $R < r$, all three double-points are isolated and the locus degenerates into two separate ovals. The resulting figures are easy to visualize from those shown.

When the circle $(C')$ is replaced by a line $(L)$, analysis similar to the above yields the equation

$$(x^2 + y^2)(x - r + a)^2 = r^2y^2$$

or

$$\rho \cos \theta = r(1 \pm \sin \theta) - a,$$

where the origin is taken at $P$ and $(L)$ is the line $x + a = 0$. This circular quartic has double-points at $P$ and at $A: (r - a,\ 0)$, and has the lines $x + a = 0$ and $x + a = 2r$ as asymptotes. Figure 3 shows the locus for the case $r < a < 2r$. If $a > 2r$ or $a < 0$, then $P$ becomes an isolated double-point.

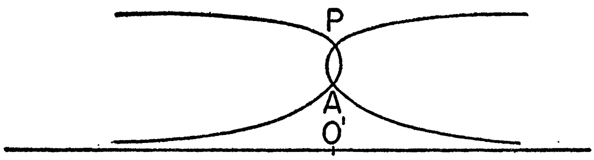



FIG. 3

**The Lost Hunter**

4185 [1946, 44]. *Proposed by B. M. Stewart, Michigan State College*

In an unexplored region known as Wild Basin, a hunter found himself lost. But he had on hand a compass, and there were visible on two distant peaks fire ranger stations, $A$ and $B$, whose bearings from his own cabin, $O$, he knew. From one observation point, $C$, the hunter took bearings on $A$ and $B$; walking to another observation point, $D$, nearby, he took bearings on $A$, $B$, and $C$. Somehow he felt these seven bearings ought to enable him to find the direction homeward, that is, the bearing from $D$ to $O$.

Show that the hunter's problem may be solved if he has either (1) mathematical tables or (2) a straightedge, in this case using the compass card as a protractor.

*Note.* Suggested by a correspondent of M. H. Ingraham, University of Wisconsin.

*Solution by E. S. Keeping, University of Alberta, Edmonton.* Let $\theta$ be the bearing of $O$ from $D$; $\alpha$, $\beta$, the bearings of $A$, $B$ from $O$; $\alpha_1$, $\beta_1$, the bearings of $A$, $B$ from $C$; $\alpha_2$, $\beta_2$, $\gamma_2$, those of $A$, $B$, $C$ from $D$. We seek to determine $\theta$.

(1) All the angles of the triangles $ODA$, $DAC$, $DCB$, $ODB$ are determined in terms of the seven known bearings and $\theta$. The sine law and the identity

$$\frac{OD}{DB} = \frac{OD}{DA}\cdot\frac{DA}{DC}\cdot\frac{DC}{DB}$$

give

$$\frac{\sin(\beta - \beta_2)}{\sin(\theta - \beta)} = \frac{\sin(\alpha - \alpha_2)}{\sin(\theta - \alpha)}\cdot\frac{\sin(\alpha_1 - \gamma_2)}{\sin(\alpha_1 - \alpha_2)}\cdot\frac{\sin(\beta_1 - \beta_2)}{\sin(\beta_1 - \gamma_2)},$$

whence

$$\sin(\theta - \alpha) = k \sin(\theta - \beta),$$

$$k = \frac{\sin(\beta - \beta_2)\sin(\alpha_1 - \alpha_2)\sin(\beta_1 - \gamma_2)}{\sin(\alpha - \alpha_2)\sin(\beta_1 - \beta_2)\sin(\alpha_1 - \gamma_2)}.$$

From these results the required bearing $\theta$ may be obtained. We find

$$\tan\theta = \frac{\sin\alpha - k\sin\beta}{\cos\alpha - k\cos\beta}.$$

(2) Without tables, but with a protractor and straight edge, the problem may be solved by the following easy construction. Lay off $CD$ of any convenient length. The lines $CA$, $DA$, $CB$, $DB$ locate points $A$ and $B$. Then the lines $AO$ and $BO$ determine $O$, whereupon the bearing of $DO$ can be read.

In two cases, there is no solution by construction, and the trigonometric equation becomes indeterminate: (1) if $CD$ is collinear with $A$ (or $B$), or (2) if $O$ is collinear with $AB$. If no further facts are available, the hunter may still take the long way home: follow the known bearing to $A$ (or $B$) and then from there home to $O$.

Solved also by M. H. Ingraham, Vladimir Karapetoff, W. A. Rees, R. H. Urbano, and the Proposer.

**Monge Point and Circumcenter of a Tetrahedron**

4187 [1946, 45]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron the ratio of the distances of the Monge point and of the circumcenter to the common perpendicular to two opposite edges is equal to $\cos\theta$, where $\theta$ is the angle between the two edges.

*Solution by P. D. Thomas, Navy Department, Washington, D. C.* If $ABCD$ is the given tetrahedron, let the feet of the common perpendicular to the sides $AB$, $CD$ be $P$, $Q$, respectively, and take $PQ=2a$, $AP=d$, $PB=b$, $DQ=e$, $QC=c$. At $S$, the midpoint of $PQ$, construct a plane $t$ perpendicular to $PQ$. Let the inter-

nal and external bisectors of the angle $\theta$ between the projections of $AB$, $CD$ upon $t$ be the $x$- and $y$-axes, respectively. $PQ$ is then the $z$-axis. The coördinates of the vertices are then

$$A(-d\cos R, -d\sin R, a), \qquad B(b\cos R, b\sin R, a),$$
$$C(c\cos R, -c\sin R, -a), \qquad D(-e\cos R, e\sin R, -a),$$

where $R=\frac{1}{2}\theta$.

Let the Monge point and the circumcenter be respectively $M(x_1, y_1, z_1)$, $O(x_2, y_2, z_2)$. Then $d_1$, $d_2$, the distances to $PQ$ (the $z$-axis) from $M$, $O$, respectively, are given by

$$d_1^2 = x_1^2 + y_1^2, \qquad d_2^2 = x_2^2 + y_2^2.$$

The coördinates of $O$ and $M$ are easily determined as

$$O\left(\frac{b+c-e-d}{4}\sec R, \frac{b+e-c-d}{4}\csc R, \frac{bd-ce}{4a}\right),$$
$$M\left(\frac{b+c-e-d}{4}\sec R\cos 2R, \frac{c+d-b-e}{4}\csc R\cos 2R, \frac{ce-bd}{4a}\right).$$

Hence $x_1=x_2\cos 2R$, $y_1=y_2\cos 2R$, and the desired result follows at once.

Solved also by M. R. Blanchard, R. Bouvaist, and the Proposer.

*Editorial Note.* The proposer remarks that the distance from the circumcenter to the common perpendicular of the two edges $AB$ and $CD$, multiplied by $\sin\theta$, is equal to the projection on a plane parallel to $AB$ and $CD$ of the line joining the midpoints of $AB$ and $CD$. A similar statement holds in which we replace the circumcenter by the Monge point and the factor $\sin\theta$ by $\tan\theta$. In case the tetrahedron is orthocentric, so that $\cos\theta=0$, we come back to a known property. (See N. A. Court, *Modern Pure Geometry*, p. 63.)

### Orthogonal Sets of Integers

4190 [1946, 103]. *Proposed by Norman Anning, University of Michigan.*

If $a, b, c, R$ are integers such that $a^2+b^2+c^2=R^2$, solve in integers the simultaneous equations

$$x^2+y^2+z^2=R^2$$
$$ax+by+cz=0.$$

There are at least four solutions

*Solution by Fritz Herzog, Michigan State College* Since the stated equations are homogeneous it is evidently sufficient to consider the case in which $a$, $b$, $c$ have no common divisor, that is,

$$(1) \qquad (a, b, c) = 1.$$

We may as well assume that $abc\neq 0$, for if one, say $c$, vanishes, we have at once the desired solutions, namely, $x=\pm b$, $y=\mp a$, $z=0$ and $x=0$, $y=0$, $z=\pm R$.

Let $x$, $y$, $z$ be a solution of the problem. We have then

$$z = -(ax+by)/c, \tag{2}$$

$$c^2(x^2+y^2)+(ax+by)^2 = c^2R^2. \tag{3}$$

Multiplying (3) by $b^2+c^2$, we obtain, after simple algebraic changes,

$$c^2R^2x^2+[abx+(b^2+c^2)y]^2 = (b^2+c^2)c^2R^2.$$

We put

$$X = x, \qquad Y = \frac{abx+(b^2+c^2)y}{cR}, \tag{4}$$

so that

$$X^2+Y^2 = b^2+c^2. \tag{5}$$

It is easily seen that, on the other hand, any solution in integers $X$, $Y$ of (5) leads to values $x$, $y$, $z$, determined by (4) and (2), which will solve the problem provided only that the value of $y$ from (4) is integral, that is,

$$abX \equiv cRY \pmod{b^2+c^2}. \tag{6}$$

By (2) and (3), $z$ will then be an integer. It is also clear from (4) that two different solutions $X$, $Y$ of (5) and (6) will lead to two different pairs of values $x$, $y$ and thus to two different solutions $x$, $y$, $z$ of the problem. It remains therefore to show that there are at least four pairs of integers $X$, $Y$ which satisfy (5) and (6).

To show this, we begin with the congruence

$$a^2b^2 \equiv -c^2R^2 \pmod{b^2+c^2}, \tag{7}$$

which follows at once from $a^2\equiv R^2$ and $b^2\equiv -c^2 \pmod{b^2+c^2}$. Now the greatest common divisor of $a^2b^2$ and $b^2+c^2$ must be a square. For if this were not the case there would exist a prime $p$ and a positive integer $j$ such that $p^{2j-1}$ is the highest power of $p$ which divides $b^2+c^2$, while $p^{2j}$ divides $a^2b^2$. Since by (1), $p$ could not be a prime divisor of both $a$ and $b$, we would have $p^{2j}$ divides either $a^2$ or $b^2$. However, the latter would imply that $p^{2j-1}$ would be the highest power of $p$ that divides $c^2$, while the former (together with $b^2+c^2=R^2-a^2$) would imply that $p^{2j-1}$ is the highest power of $p$ that divides $R^2$. Both of these alternatives are impossible.

We may therefore write $(a^2b^2, b^2+c^2)=d^2$ and obtain from (7)

$$(ab/d)^2 \equiv -(cR/d)^2 \left(\operatorname{mod} \frac{b^2+c^2}{d^2}\right). \tag{8}$$

Since $(ab/d)$ is relatively prime to the modulus, we can determine $\lambda$ such that

$$\lambda(ab/d) \equiv (cR/d) \left(\operatorname{mod} \frac{b^2+c^2}{d^2}\right). \tag{9}$$

From (8) and (9) we conclude that

$$\lambda^2 \equiv -1 \left(\mathrm{mod}\, \frac{b^2+c^2}{d^2}\right).$$

Therefore, according to a well-known theorem,* it is possible to find integers $u$ and $v$ such that

$$u^2+v^2 = \frac{b^2+c^2}{d^2}, \tag{10}$$

$$u \equiv \lambda v \left(\mathrm{mod}\, \frac{b^2+c^2}{d^2}\right). \tag{11}$$

From (9) and (11) we have

$$(ab/d)u \equiv (cR/d)v \left(\mathrm{mod}\, \frac{b^2+c^2}{d^2}\right). \tag{12}$$

Multiplying (10) and (12) throughout by $d^2$ and putting $ud=X$ and $vd=Y$, we obtain the fact that $X$, $Y$ is a solution of (5) and (6). Finally, since the four pairs $(u, v)$, $(-u, -v)$, $(v, -u)$ and $(-v, u)$ are easily seen to be four different solutions of (10) and (11), we obtain from them the four solutions $(X, Y)$, $(-X, -Y)$, $(Y, -X)$ and $(-Y, X)$, respectively, of (5) and (6).

Solved also, partially, by Murray Barbour, Mary A. English, and the Proposer.

*Editorial Note.* From equations (2) and (4) of the above solution we easily derive

$$Y = \frac{cy-bz}{R}.$$

Put $Y=-x'$, and by analogous argument obtain the integers

$$y' = \frac{cx-az}{R}, \qquad z' = \frac{ay-bx}{R}.$$

It is now easy to show that the determinant

$$\begin{vmatrix} a & b & c \\ x & y & z \\ x' & y' & z' \end{vmatrix} \tag{13}$$

has the following properties: the sum of the squares of the elements in any row or column is $R^2$, any two rows or two columns are orthogonal, the cofactor of each element $a_{ij}$ equals $Ra_{ij}$, the value of the determinant is $R^3$. It follows that $\pm x$, $\pm y$, $\pm z$ and $\pm x'$, $\pm y'$, $\pm z'$ provide four solutions of the original problem.

* See, for instance, Landau, Vorlesungen über Zahlentheorie, vol. 1, p. 101, Theorem 160.

The general solution in integers of the equation

$$a^2+b^2+c^2=R^2$$

is given by

$$a=\tfrac{1}{2}(p^2+q^2-m^2-n^2), \qquad b=np-mq,$$
$$c=mp+nq, \qquad R=\tfrac{1}{2}(p^2+q^2+m^2+n^2).$$

(See Carmichael, *Diophantine Analysis*, pp. 35–44.) In terms of $m$, $n$, $p$, and $q$, it is now easy to find explicit expressions for the elements of (13). They are

$$x=np+mq, \qquad y=\tfrac{1}{2}(n^2+q^2-m^2-p^2), \qquad z=mn-pq$$
$$x'=mp-nq, \qquad y'=mn+pq, \qquad z'=\tfrac{1}{2}(m^2+q^2-n^2-p^2).$$

The Proposer noted that essentially this same determinant is mentioned in Coxeter, *Non-Euclidean Geometry*, p. 126; and in Kovalevski, *Determinanten*, p. 177.

The interesting possibility of extension to higher orders invites further investigation. The determination of $x_i$, $i=1, 2, \cdots, n$, for given $a_i$ where $\sum a_i^2=R^2$, such that $\sum a_ix_i=0$ and $\sum x_i^2=R^2$ is always possible when $n$ is even, in at least a trivial way. Among the examples that have come to hand, the smallest value of $n$ for which $x_i$ do not exist for certain $a_i$, is $n=9$: if the $a_i$ are all odd (*e.g.* 3, 3, 1, 1, 1, 1, 1, 1, 1) there must be an odd number of odd $x_i$, whence $\sum a_ix_i$ cannot be zero.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*Analytic Geometry and Calculus.* By J. F. Randolph and Mark Kac. New York, The Macmillan Co., 1946. 642 pages (including tables and answers). $4.75.

This text for a unified first course is designed to present the basic notions of the calculus as early as possible. After a brief treatment of the straight line and circle it proceeds to functions, derivatives and their applications, differentials, definite integrals, and the fundamental theorem, dealing always with explicit algebraic functions. This development occupies the first third of the book. Only after this does it take up conics, change of coördinates, and even such topics as perpendicularity and the angle between two lines. Parametric equations and polar coördinates come still later.

This arrangement of material has two advantages. It makes the elements

of the calculus available for use in science and engineering courses much earlier than the conventional order, and it permits a more satisfactory treatment of some topics in analytic geometry, notably asymptotes and general properties of graphs. Most of the usual material in analytic geometry is covered, but the treatment is very condensed; for example, tangents and focal properties of conics are left as exercises.

The text is printed in two sizes of type. The sections in large print constitute a fairly formal presentation. More rigorous proofs and a good deal of material commonly left to more advanced courses are included as sections in small print intended for the student who wishes to go deeper. The authors suggest that 10 semester hours should suffice to cover the sections in large print. Chapters on Taylor's theorem, solid analytic geometry, partial derivatives, and multiple integrals are included, but not on infinite series or differential equations.

The exposition is for the most part very clear, and a sincere effort is made to distinguish between what is proved and what is assumed. Problems are grouped according to difficulty and tend to avoid technical complication.

A number of more or less novel features may be noted. The symbols $\Delta x$ and $dx$ are defined as arbitrary numbers, not necessarily equal, which are distinguished by their specific uses, $\Delta x$ as a number to be added to a value of the independent variable, $dx$ as a number to be multiplied by $f'(x)$. Anti-derivatives are introduced very early, but the integral sign is not used until the definite integral is defined. Area and arc length are defined from the outset by limiting processes. The emphasis at the very beginning on algebraic manipulation of inequalities and absolute values is an excellent feature, as is the varied use of the "greatest integer" function $[x]$. The notation $\ln x$ for $\log x$ is adopted.

One feature of the arrangement that may be criticized is the postponement of the formula for the derivative of a composite function until very late (page 207). As a result, the method of integration by substitution cannot be properly justified when it is first used, the rule for differentiating implicit functions is given without its rational explanation, and the formula for the derivative of $u^p$, for fractional $p$, has to be assumed without proof for many chapters. Another questionable feature is the presentation of the notion of definite integral in simplified form, with equal subdivisions and the value of the function taken at an end-point of each. The more general formulation is mentioned in small print but is seldom used. With the less flexible definition one is not free to take the approximating sums equal to the quantity being computed, and this tends to obscure somewhat the precision of integral calculations. The general definition is just as easy to teach and is superior both in theory and application. Of course, these features can be corrected by a teacher if he wishes.

As a whole, this is definitely a superior text that should prove especially useful where early introduction to the calculus is desired. It should also be noted that the text can be adapted to a course in calculus for students who have already had a course in analytic geometry, as the authors point out in the preface.

J. C. OXTOBY

*An Introduction to College Mathematics.* By C. V. Newsom. New York, Prentice-Hall, Inc., 1946. 8+344 pages. $3.50.

The problem of how and what to teach the non-specialist student who takes one year of college mathematics and no more has long rested uneasily upon the conscience of most teachers of mathematics. What information, what ideas should such a student carry away from his last contact with mathematics? Most of us would agree wholeheartedly with the objectives set forth forty years ago in the Meran curriculum:

"1. A scientific survey of the systematic structure of mathematics.

2. A certain degree of skill in the complete handling, numerical and graphical, of problems.

3. An appreciation of the significance of mathematical thought for a knowledge of nature and for modern culture."

To these we might add Godfrey's statement that "we want to make boys believe that mathematics is indispensable in their daily life and not something which they will have to do in hell."

That the traditional course of algebra, trigonometry, and analytic geometry, either seriatim or in any "unified" permutation, leaves something to be desired in aiding the attainment of these objectives can hardly be questioned. Several text-books of the "cultural" type have been written in a laudable attempt to solve the problem of imparting at least the rudiments of a liberal mathematical education in one year. At present, however, even the existence of a solution of this problem has yet to be convincingly demonstrated. Nevertheless, each new text that is written for the non-specialist student does make its peculiar contribution toward the solution of the problem, and the text under review is a welcome and valuable addition to the lists.

Professor Newsom states that his book "has been written for the student and not for the instructor." Knowing that the student's view of the course will depend in large measure upon the problems he works, the author has expended unusual effort and care in devising over 800 problems. The field of application in the many practical problems ranges from cometary orbits to chimney area, from the velocity of ants to that of the circulation of money. Many of the exercises "are important in the development of the argument," and the student is warned not to omit these.

The choice of material is a matter on which there has been, perhaps fortunately, little unanimity among authors of "cultural" texts. Professor Newsom explains that "No topic has been chosen simply because of some special appeal to the author or because of a traditional prejudice among mathematicians in favor of it. In fact, all material introduced was examined critically for its possible value to the non-specializing student. . . . Considerable material proposed for the course was discarded when it was found that students generally were unable to appreciate its significance." About 12 per cent of this rather slim book (301 pages plus tables, answers, index) is devoted to topics such as logic, bases of numeration, infinite sets, that are not ordinarily encountered in the traditional first-year

text. The choice and order of the subject matter are indicated by the chapter headings: 1. The Nature of Mathematics, 2. Number and the Operations of Arithmetic, 3. The Arithmetic of Numbers in the Exponential Form, 4. The Arithmetic of Measurement, 5. Logarithms, 6. Some Topics in the Mathematics of Finance, 7. Progressions of Numbers, 8. Combinations and Probability, 9. Functional Relationships, 10. Variation, 11. The Circular Functions, 12. The Equation, 13. Some Common Curves. Some will quarrel with the author's decision to discuss compound interest and annuities before taking up geometric progressions.

An *Introduction to College Mathematics* has none of the florid rhetoric with which some texts are made obscure. Both chapter and section headings are clear and helpful signposts and not coy conundrums set by the author. Professor Newsom writes lucidly in a pleasingly informal, conversational style; his book will be welcomed by all teachers of non-specialist students.

R. D. SPECHT

## NEW BOOKS RECEIVED

*Cosmic Radiation. Fifteen Lectures.* Edited by W. Heisenberg. Translated from the German by T. H. Johnson. New York, Dover Publications, 1946. 192 pages. $3.50.

*Cours Complet de Mathêmatiques Élêmentaires. Tome* 1. *Arithmetique.* By J. Haag. Paris, Gauthier-Villars, 1945. 6+105 pages. 80 Fr.

*Élêments de Calcul Infinitêsimal.* By Adrien Grosrey. Paris, Gauthier-Villars, 1945. 192 pages. 280 Fr.

*Élêments de Mathêmatiques Supêrieures.* By P. Gaudiot. Paris, Editions Leon Eyrolles, Librairie de l'Enseignement Technique, 1944. 381 pages. 182 Fr.

*Essential Business Mathematics.* By L. R. Snyder. New York and London, McGraw-Hill Book Co., Inc., 1947. 12+434 pages. $2.75.

*Introduction to Mathematical Statistics.* By P. G. Hoel. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1947. 10+258 pages. $3.50.

*Matrix and Tensor Calculus with Applications to Mechanics, Elasticity, and Aeronautics.* By A. D. Michal. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1947. 13+132 pages. $3.00.

*Meson Theory of Nuclear Forces.* By W. Pauli. New York, Interscience Publishers, Inc., 1946. 7+69 pages. $2.00.

*Vorlesungen über Differential- und Integralrechnung. Erster Band. Funktionen einer Variablen.* By A. Ostrowski. Basel, Birkhauser, 1945. 12+373 pages.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

EDITOR'S NOTE.—A letter asking for a report of club activities for 1946–47 has been sent by the editor of this department to the sponsors of the various mathematical clubs and societies. A general invitation is hereby extended to all the readers of this section to submit articles, suggestions, and papers to the editor which might be of interest to the members of mathematical clubs. Information on sources of mathematical films, slides, and models as well as bibliographies on topics suitable for discussion in meetings are particularly desired. This material will be collected for dissemination to individuals upon request and will be summarized in this department of the MONTHLY at intervals.

### CLUB REPORTS, 1945-46

#### Pi Mu Epsilon, University of Illinois

The theme for the activities of the *Illinois Alpha* chapter of *Pi Mu Epsilon* was *The use of mathematics in other departments.* Staff members of various colleges of the University of Illinois addressed the forty-five graduate and undergraduate members on this theme as it applied to their respective fields.

The annual Spring banquet and initiation was held on May 16, at which time thirty-two students were initiated. The group was addressed by Professor Neiswanger of the Economics Department as he contributed to the general theme with the topic *The use of statistical methods in price control.* The annual *Pi Mu Epsilon* scholarship award was presented at this banquet to John Schumacher.

The officers for 1945–46 were: President, Clarence Phillips; Vice-President, Mildred Brannon; Secretary, LaVerne Bloomberg; Treasurer, Aileen Hostinsky; Faculty Advisor, Professor E. Welker.

#### Mathematics Club, Mount Mary College

The project for the year of the *Mathematics Club* of Mount Mary College was a survey of prospects in the business, scientific, educational and other fields for graduates with interest and ability in mathematics. Reports on results were given at each monthly meeting of the club. The average attendance at the meetings was twenty-seven. New members were initiated into the club after each had presented a summary of the life of a great mathematician.

The topics of the guest speakers appearing during the year include:

*The atomic bomb*, by Reverend Joseph F. Carroll, head of the Physics Department at Marquette University.

*A search for perfection*, by Dr. Harvey P. Pettit, head of the Mathematics Department of Marquette University. This topic dealt with a phase of the development of mathematics.

At another meeting a motion picture was shown which demonstrated Einstein's theory of relativity. The members of the club were hostesses for the an-

nual meeting of the *Wisconsin Section of The Mathematical Association* which was held at Mount Mary College on May 4. Sister Mary Felice of the Mount Mary College Mathematics Department was the presiding officer at this meeting.

Officers for 1945–46 were: President, Patricia O'Neill; Secretary-Treasurer, Marilyn Welsch; Club Adviser, Sister Mary Felice.

**Mathematics Society, Brooklyn College**

After a period of suspended activities, the *Mathematics Society* of Brooklyn College resumed meetings in February, 1946. The activities of the club centered around lectures on different phases of mathematics and the allied fields given by various members of the society. The topics were:

*Summation of infinite series*, by Dr. H. F. MacNeisch

*Foundation of the number system*, by Arthur Zeichner

*Real sequences*, by Julian Keilson

*Transcendental numbers*, by George Shapiro

*Bruns method in the theory of numbers*, by Gerard Washnitzer

*Aerodynamics*, by Dr. Moses Richardson

*Various topics in mathematics*, by Dr. R. A. Johnson

*Vector and function spaces*, by Gerard Washnitzer

*Finite geometry*, by Dr. James Singer

*Prominence of Brooklyn College graduates in the field of Mathematics*, by Dr. H. F. MacNeisch

*Topology*, by Daniel Waterman

*Symbolic logic*, by Melvin Hausner

*Sense in curves*, by Dr. Samuel Borofsky

*Topics in mathematical logic*, by Dr. Ira Rosenbaum of the Department of Philosophy.

A social evening in the form of a party was held with the Physics Society as one of the meetings. A semi-annual Integration Contest among the students of integral calculus was conducted by the club. The fourteenth edition of *The Math Mirror*, in which the high standard of the previous issues was maintained, was published in the Spring of 1946.

The officers for the February term of 1946 were: President, Gerard Washnitzer; Vice-President, Arthur Zeichner; Secretary, Cecile Pollack.

The officers elected for the September, 1946 term were: President, Julian Deilson; Vice-President, George Shapiro; Secretary, Rita Schwartz.

## NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.*

### CANADIAN CONGRESS

The Canadian Mathematical Congress is planning a Summer Seminar to be held at the University of Toronto from August 15 to September 14, 1947. This year the Seminar will center on Algebra and Theory of Numbers and have as its object the stimulation of research in these fields on the part of younger mathematicians. Professor L. J. Mordell of Cambridge University, Professor Saunders Mac Lane of Harvard University, and Professor Paul Dubreil of the University of Nancy, will each give a series of eight lectures. The seminars in Algebra and Theory of Numbers will be presided over by Professor Richard Brauer of the University of Toronto and Professor Gordon Pall of the Illinois Institute of Technology, respectively.

The Seminar will be open to all qualified mathematicians on payment of a fee of $10.00. Arrangements will be made if possible for the accommodation of mathematicians and their families in one of the residences of the University at a moderate rate. Inquiries may be directed to the Secretariat, Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, or to the secretary of the local committee, Professor G. de B. Robinson, Department of Mathematics, University of Toronto.

### CONFERENCE ON ALGEBRA

The University of Michigan announces a Conference on Algebra to be held from Friday to Monday, July 25–28, 1947. Those who wish to present papers at the Conference, or who wish further information about it, are invited to communicate with R. M. Thrall, Department of Mathematics, University of Michigan. The program will feature papers by E. Artin, R. Brauer, S. Eilenberg, N. Jacobson and S. Mac Lane, and will have room for a considerable number of shorter papers. The time allotted to each paper will depend upon the total number of participants, but will in no case be less than 20 minutes. A limited number of dormitory rooms will be available to those attending the conference.

### SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1947:

*Case School of Applied Science.* From June 30 to August 8 the following advanced courses will be offered: by Professor Agnew (Cornell University), infinite series and summability; by Professor Morris, introduction to higher geometry; by Professor Rinehart, introduction to algebraic theories.

*Iowa State College.* From June 16 to July 23 the following advanced courses

will be offered: by Professor Daniells, teaching of secondary school mathematics; by Professor Homeyer, design of experiments; by Dr. Mood, mathematical statistics; by Dr. Sealander, advanced calculus; by Professor Snedecor, statistical methods I. From July 24 to August 29: by Professor Kempthorne, advanced design of experiments, sampling methods; by Professor Snedecor, statistical methods II. From June 16 to August 29: by Dr. Anderson, advanced mathematics for engineers; by Dr. Gaskell, boundary value problems; by Dr. Rock, vector analysis; by Dr. Thielman, theory of functions of a real variable. During this twelve weeks' term a seminar in non-linear mechanics will be conducted and, if registration warrants, an additional graduate course will be offered.

*Massachusetts Institute of Technology.* From August 4 to September 19 there will be a special summer session program in applied mathematics in which the following graduate courses will be offered: by Professors P. D. Crout, F. B. Hildebrand, W. Prager, H. Reissner, J. L. Synge: advanced topics in applied mathematics; by Professors Hildebrand and E. Reissner, theory of plates and shells; by Professor C. C. Lin and E. Reissner, theoretical hydromechanics; by Professor D. J. Struik, tensors in mechanics.

*Oklahoma Agricultural and Mechanical College.* During the summer session the following series of lectures will be offered: by Professor Neugebauer of Brown University, history of mathematics; by Professor Lonseth of Northwestern University, theory of errors.

*The University of Southern California.* From June 23 to August 1 the following advanced courses will be offered: by Professor D. G. Bourgin (Illinois), Fourier series, partial differential equations; by Dr. Tobias Dantzig, history of mathematics; by Professor D. H. Hyers, differential equations; by Professor Ernst Snapper, theory of equations; by Professor D. V. Steed, tensor analysis; by Professor P. A. White, analytic geometry of space; by Professor R. L. Wilder (Michigan), mathematical analysis, seminar in topology.

*The University of Wisconsin.* From June 2 to September 20 the following graduate courses will be offered: advanced calculus, advanced college algebra, calculus of variations, college geometry, differential equations, higher algebra, higher analysis, modern theory of differential equations, vector analysis. From June 20 to August 15: advanced analytic geometry, differential geometry, higher analysis, introduction to the theory of probability, survey of the foundations of algebra. During the first period (summer semester) the following staff members will teach: Professors Allen, Bing, Everett, Gould, Kleene, Langer, MacDuffee, and Doctors Colvin, Sokolnikoff. During the second period (eight weeks' session) the following staff members will teach: Professors Arnold, Bing, Fullerton, Gould, MacDuffee, Whaples.

## PERSONAL ITEMS

Professor Arnold Dresden of Swarthmore College was the delegate of the Mathematical Association at the meeting of the UNESCO at Philadelphia on March 24, 25, 26.

Professor Cristóbal de Losada y Puga of the Catholic University of Peru has been nominated by the President of the Republic of Peru as Minister of Public Education.

Professor Marston Morse of the Institute for Advanced Study was awarded the degree of Doctor (honoris causa) by the University of Paris on the recommendation of the Faculty of Science of the Sorbonne.

Assistant Professor Gordon Walker of Temple University will be the representative of the Mathematical Association at the meeting of the American Academy of Political and Social Science.

Dr. P. H. Anderson has been appointed economic analyst with the Marketing Division, Office of Domestic Commerce, Department of Commerce, Washington, D. C.

Assistant Professor Herbert Busemann of Smith College has been appointed to a professorship at the University of Southern California.

Assistant Professor Lola M. Christy of Westminster College, New Wilmington, Pennsylvania, has retired.

Assistant Professor B. H. Gere of the United States Naval Academy has been appointed to an associate professorship at Hamilton College.

Professor L. E. Gurney of the University of Southern California has retired.

Professor R. L. Krueger of Wittenberg College, Springfield, Ohio, has been appointed coordinator of the Division of Natural Sciences.

Associate Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has been promoted to a professorship.

Assistant Professor E. N. Oberg of the University of Iowa has been promoted to an associate professorship.

The following appointments to instructorships are announced:

Queens College (tutorship): Jack Moshman

University of Arkansas: E. L. Eagle

Westminster College: W. A. Gibson

William Orange of Los Angeles City College died December 9, 1946.

Professor Emeritus W. A. Moody of Bowdoin College died February 3, 1947. He was a charter member of the Mathematical Association.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### APRIL MEETING OF THE IOWA SECTION

The thirty-third regular meeting of the Iowa Section of the Mathematical Association of America was held at Grinnell College, Grinnell, Iowa, on Friday and Saturday, April 19 and 20, 1946, in conjunction with the regular meeting of the Iowa Academy of Science. Professor E. N. Oberg, Vice-Chairman of the Section, presided.

There were thirty-five persons in attendance, including the following twenty-one members of the Association: E. W. Anderson, T. A. Bancroft, J. W. Beach, F. A. Brandner, E. W. Chittenden, W. M. Davis, Cornelius Gouwens, D. L. Holl, O. C. Kreider, F. M. McGaw, J. V. McKelvey, E. E. Moots, E. N. Oberg, H. V. Price, Fred Robertson, W. J. Rusk, E. R. Smith, L. W. Swanson, H. P. Thielman, Henry Van Engen, Roscoe Woods.

At the business meeting the following officers were elected for the coming year: Chairman, L. W. Swanson, Coe College; Vice-Chairman, H. V. Price, State University of Iowa High School; Secretary-Treasurer, Fred Robertson, Iowa State College.

The following papers were presented:

1. *College entrance requirements and their effect upon high school mathematics teaching*, by Professor Henry Van Engen, Iowa State Teachers College.

The speaker gave an illustration to show how the usual examination emphasizes operational procedure in solving problems. He urged that an understanding of the ideas involved should be stressed.

2. *Progress of demoted and promoted students in mathematics at Iowa State College*, by Professor Fred Robertson, Iowa State College.

Professor Robertson explained how the two groups (the demoted and the promoted) were selected, and presented tables showing the academic progress of these groups at his institution for the years 1936–1944 inclusive.

3. *Notes on some functional equations*, by Professor H. P. Thielman, Iowa State College.

It was shown that the equation

$$f(x)\cdot\phi(y) = X(x + y + nxy), \qquad n > 0,$$

can be reduced to $f(x)\cdot f(y) = f(0)\cdot f(x+y+nxy)$, and that all solutions of these equations, continuous for $x > -1/n$, are of the form $f(x) = a(1+nx)^b$, where $a$ and $b$ are constants. Two functions, $c(x)$ and $t(x)$, were defined as follows:

$$c(x) = \frac{1}{2}\left[f(x) + f\left(\frac{-x}{1+nx}\right)\right], \qquad t(x) = \frac{1}{2}\left[f(x) - f\left(\frac{-x}{1+nx}\right)\right]$$

where $f(x)=(1+nx)^b$. These functions were shown to possess many properties analogous to those of cos $x$ and sin $x$.

4. *Topological Groups*, by Professor Bernard Vinograde, Iowa State College, introduced by the Secretary.

This paper consisted of a description of the axiomatic foundations of topological group theory, and some immediate consequences and basic theorems.

5. *The converse of a certain theorem in analytic geometry*, by Professor Roscoe Woods, University of Iowa.

Professor Woods started with the well known fact that the three lines $a_ix+b_iy+c_i=0$, $i=1, 2, 3$, meet in a point if the determinant $|a_1b_2c_3|$ vanishes, and conversely (if the assumption is made that parallel lines meet in an ideal point). He then showed that by subjecting a given determinant $|A_1B_2C_3|$ to the so-called elementary transformations, there are transformations induced upon the coördinates $x$ and $y$. These transformations represent the general collineation in the plane, so that any point associated with a given determinant $|a_1b_2c_3|=0$ may be transformed into any other point in the plane.

6. *The algebra of four functional operators on Boolean algebra and its characterization*, by Professor E. W. Chittenden, State University of Iowa.

This paper is to be published in the *Proceedings of the Iowa Academy of Science.*

7. *Approximate formulas for radii of circles including a proportion $p$ of errors subject to a normal bivariate distribution*, by Professor E. N. Oberg, State University of Iowa.

This paper dealt with approximate formulas for the determination of the radius of a circle that includes a proportion $p$ of errors from a normal bivariate distribution.

8. *Some new and old properties of the incomplete beta function*, by Professor T. A. Bancroft, Iowa State College.

Professor Bancroft gave several recurrence formulas relating to the incomplete "normalized" beta function. The possibility of their use in the extension of the *Incomplete Beta Function Tables* of Karl Pearson was discussed.

9. *Secondary mathematics in the post-war years*, by Professor H. V. Price, State University of Iowa High School.

It was stated that, in the opinion of the Commission on Postwar Plans of the National Council of Teachers of Mathematics, the first purpose of the high school mathematics program is to guarantee functional competence to all who can possibly achieve it. To attain this goal a combined program of conventional and general mathematics seems to be the answer.

10. *Some properties of differential equations of the Thomas-Fermi type*, by Dr. G. H. Wannier, State University of Iowa, introduced by Professor E. N. Oberg.

The speaker discussed differential equations of the type

$$x^2 \frac{d^2\phi}{dx^2} - (n + m - 1)x \frac{d\phi}{dx} + nm\phi = rx^q\phi^{1+p}.$$

Such equations occur in the study of potential distributions in the presence of space charge. Particular cases are the equation of Thomas-Fermi,

$$\frac{d^2\phi}{dx^2} = \phi^{3/2}x^{-1/2},$$

and the equation of Langmuir,

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} = x^{-1}\phi^{-1/2}.$$

11. *Second solutions of certain differential equations associated with the theory of orthogonal polynomials*, by Professor L. W. Swanson, Coe College.

In solving a certain differential equation associated with orthogonal polynomials, a second solution had been omitted. The paper dealt with this second solution of the differential equation.

12. *Mathematics teaching procedure in the light of our experience with the army and navy schools*, by Professor O. C. Kreider, Iowa State College, Professor T. A. Bancroft, Iowa State College, and Professor W. M. Davis, Cornell College.

The speakers discussed the purpose, organization, difficulties, and successes of the army university centers and the navy educational programs overseas.

FRED ROBERTSON, *Secretary*

## ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-ninth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colorado, on April 19 and 20, 1946.

The attendance was one hundred and twenty, including the following twenty-three members of the Association: H. H. Alden, C. F. Barr, William Betz, J. R. Britton, A. G. Clark, J. R. Everett, J. C. Fitterer, H. T. Guard, D. F. Gunder, Marian S. Gysland, Leota C. Hayward, I. L. Hebel, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, M. L. Madison, A. E. Mallory, W. K. Nelson, Greta Neubauer, O. H. Rechard, A. W. Recht, G. A. Whetstone.

The following papers were presented:

1. *Spherical trigonometry by projection on a plane*, by Professor I. L. Hebel, Colorado School of Mines.

2. *A compatibility relation in the flow of an incompressible ideal fluid*, by Dr. G. A. Whetstone, Amarillo College.

By applying the procedures developed by Riquier for the study of partial differential equations to the usual four equations

$$\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} = g_{x_i} - \frac{1}{\rho} \frac{\partial P}{\partial x_i}, \quad (i = 1, 2, 3)$$

and

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0,$$

under the assumption $\rho = \text{constant}$, the author was lead to a necessary and sufficient compatibility condition.

3. *Teaching mathematics in the army*, by Professor H. T. Guard, Colorado State College.

This paper consisted of a description of the curriculum and the methods of instruction in the United States Military Academy.

4. *On the definition of functions of a complex variable*, by Professor A. J. Kempner, University of Colorado.

5. *Tables for the power function for tests of hypotheses relating to Poisson distributions*, by Professor A. G. Clark, Colorado A. and M. College.

The speaker discussed devices, including recursion formulas, which serve to reduce the labor of computation in constructing tables for the function specified in the title. Such tables are useful in the construction of efficient sampling experiments.

6. *The present educational situation and the crisis in mathematics*, by William Betz, Public Schools of Rochester, N. Y.

The speaker rehearsed the role of mathematics in the recent war effort. He referred to the mathematical deficiencies of millions of our young men, first pointed out by Admiral Nimitz, and later substantiated by selective service tests. It was suggested that there be a re-examination of the controversy between "education" and mathematics. On the basis of significant quotations it was shown that the educational scene is one of confusion bordering on chaos. Mathematics cannot be adjusted to the prevailing educational philosophies without giving up its real purposes. Fortunately, a healthy reaction against the destructive forces in our educational policies is now in the making. In conclusion, the speaker outlined the remedial steps that seem to be necessary if we wish to help improve the situation.

7. *The Laplace transformation*, by Professor J. R. Britton, University of Colorado.

Professor Britton gave an expository talk on the Laplace transformation and its applications to the solution of boundary value and initial value problems. Some of the simpler transforms were derived, and application was made to the problem of a two mass, two spring vibrating system. A mechanical model

served to demonstrate the types of behavior indicated by the previously obtained solution.

8. *The necessary reconstruction of mathematics in the light of war experiences*, by William Betz.

This was an invited address, delivered at a joint session with the National Council of Teachers of Mathematics and the Mathematics Section of the Colorado Education Association. The speaker holds the position of specialist in mathematics for the public schools of Rochester, N. Y. The address dealt with the reports of various committees which have issued pronouncements on the problem of mathematical instruction in the post-war period. He presented a check list of mathematical objectives, and suggested methods for attaining these objectives.

J. R. BRITTON, *Secretary*

## ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-third annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Louisiana Polytechnic Institute, Ruston, Louisiana, on Friday and Saturday, March 22 and 23, 1946. Professor I. C. Nichols was elected temporary chairman and presided at the Friday afternoon and Saturday morning sessions. Professor P. K. Smith presided at the dinner meeting.

There were fifty in attendance, including the following twenty-one members of the association: W. G. Banks, N. A. Court, J. C. Currie, W. L. Duren, Jr., L. M. Garrison, F. C. Gentry, R. V. Guthrie, Jr., J. A. Hardin, W. L. Johnson, H. T. Karnes, C. G. Killen, A. C. Maddox, Dorothy McCoy, B. E. Mitchell, I. C. Nichols, W. V. Parker, P. K. Rees, F. A. Rickey, H. F. Schroeder, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, Marelena White.

At the business meeting the following officers were elected for the coming year: Chairman, W. V. Parker, Louisiana State University; Vice-Chairmen, W. L. Johnson, Mississippi Southern College, Z. L. Loflin, Southwestern Louisiana Institute; Secretary-Treasurer, F. C. Gentry, Louisiana Polytechnic Institute. Invitations to meet at Mississippi Southern College in 1947, and at Southwestern Louisiana Institute in 1948 were accepted.

The following papers were presented at the Friday afternoon program:

1. *Esthetic and moral implications of the ark of the covenant*, by Professor B. E. Mitchell, Millsaps College.

The purport of the speaker's remarks was that if the golden rectangle (length/width $=1.618$) and the Platonic rectangle (length/width $=1.732$) have esthetic value as polygonal forms, then the Mosaic rectangle (length/width $=1.667$) has also, since it differs from the arithmetic, geometric, and harmonic means of the other two by 0.008, 0.007, and 0.006 of a part, respectively.

2. *The smoothing effects of linear oscillators*, by Professor W. L. Duren, Tulane University.

It was recalled that forced linear oscillators, which can be realized as simple $R$-$C$-$L$ electrical circuits, or as mechanical couplings having mass, spring, and dashpot, have a solution whose principal term may be regarded as a smoothed and damped response to the input voltage or force. In contrast to the usual elementary treatment which admits a forcing function which is either a constant or $A \sin \omega t$, the speaker discussed by elementary and intuitive methods the nature of the separate smoothing effects due to inertia, capacitance, and resistance when the forcing function is general in form.

3. *The future importance of secondary mathematics is now in the hands of the college teacher*, by Professor H. T. Karnes, Louisiana State University.

Professor Karnes discussed contemporary problems in the field of secondary mathematics. He commented upon the scarcity of qualified teachers and the shaky place of mathematics in the curriculum. Suggestions were offered for the improvement of the position of mathematics in the schools, and the responsibilities of college teachers in this connection were pointed out.

4. *The presentation of certain application problems in the calculus*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

Professor Smith stated that definitions of mechanical terms in calculus texts are often so presented that the student fails to appreciate the basic physics involved. His paper was primarily concerned with the need of requiring students to understand the physical meaning of the center of mass. A simple procedure based on the law of levers was demonstrated as a means of enabling students to appreciate the physical significance of the center of mass.

5. *Topics in applied mathematics*, by Professor C. D. Smith, Mississippi State College.

In this address it was remarked that editors of magazines should place emphasis both on new mathematics and new uses of mathematics. A recent article, *Mathematics of a Nut Cutter*, in the *National Mathematics Magazine* was used as an illustration.

Professor N. A. Court of the University of Oklahoma was present as the guest of the Section and the Branch of the National Council. At the joint dinner Friday evening he spoke on *The Motionless Arrow*, the address being a discussion of the history and implications of Zeno's paradox. This paper is to be published in the *Scientific Monthly*. At the Saturday morning meeting Professor Court delivered a paper entitled *On a Pencil of Ruled Quadrics*, which was later published in the *Duke Mathematical Journal*.

F. C. GENTRY, *Secretary*

## FALL MEETING OF THE INDIANA SECTION

The twenty-fourth annual meeting of the Indiana Section of the Mathematical Association of America was held at Indiana State Teachers College, Terre Haute, Indiana, on Friday, October 18, 1946, in conjunction with the fall meeting of the Indiana Academy of Science. Professor W. L. Ayres presided.

Thirty-four persons registered at the meeting, including the following sixteen members of the Association: W. L. Ayres, Juna L. Beal, W. H. Carnahan, G. E. Carscallen, Olive M. Draper, W. E. Edington, P. D. Edwards, Rufus Isaacs, M. W. Keller, J. P. LaSalle, P. M. Pepper, J. C. Polley, M. E. Shanks, W. O. Shriner, F. C. Smith, and C. P. Sousley.

At the business meeting the following officers were elected for the coming year: Chairman, G. H. Graves, Purdue University; Vice-chairman, H. E. Wolfe, Indiana University; Secretary-Treasurer, M. W. Keller, Purdue University. It was decided to hold a spring meeting in 1947 at a time and place to be determined by the officers.

The following papers were read:

1. *The American University at Shrivenham Barracks*, by Professor P. D. Edwards, Ball State Teachers College.

Shrivenham American University, although created by army personnel awaiting redeployment, was a true American University operated on foreign soil. The faculty of more than 220 members represented 149 American institutions of higher education. About 150 were civilians who were sent to England for this purpose. The University was divided into eight sections which correspond to the usual division of an American university into schools. The faculty of the mathematics branch included fourteen civilians and seven members of the army, all of whom were college teachers in civil life. The enrollment was approximately 4000 each term. Approximately three-fourths of the student body had had actual combat experience. An elective system prevailed, and under it the mathematics branch was exceeded in size by only one other branch. In spite of the unusual difficulties which prevailed, very gratifying results were obtained.

2. *The American University at Biarritz*, by Professor J. C. Polley, Wabash College.

The speaker discussed the mathematics program in the American University at Biarritz, and his experiences while there.

3. *Applications of the linear transformation*, by Professor J. P. LaSalle, Notre Dame University.

Several applications of the linear transformation $(az+b)/(cz+d)$ to problems in electrical engineering were presented by Professor LaSalle. A clear geometric picture of the variation of power transfer to a load with change of load or generator impedance, and of the condition for maximum power transfer, was obtained by means of the transformation $(1-z)/(1+z)$. The "circle" diagram

which relates impedance to the reflection coefficient can be used for this purpose. Though the concepts of reflection and transmission coefficients appear to be more natural than those of impedance and the resulting equivalent circuits, particularly for wave guides, only limited use of the former concepts have been made. This may be due to difficulties in applying the general linear transformation. Algebraic identities which simplify the application of this transformation were given.

4. *The force of mortality function*, by Dr. F. C. Smith, Lincoln Life Insurance.

In this paper, the author discussed the definition of the force of mortality function $\mu$, and some of its properties. Several methods of approximating the values of this function were also presented. The importance of this function in the field of actuarial mathematics was stressed, and the effects of assuming the Gompertz and Makeham hypotheses were shown.

5. *Recent progress in the theory of compressible fluids*, by Professor Rufus Isaacs, Notre Dame University.

Recent developments make the need for a workable theory of compressible fluids imperative. In the past, progress has been checked, first, by the complexity of the theory, and, second, by the formidable amount of numerical computation needed to apply what theory is extant. The new approach of Bergman to the methods of Chaplygin now yields a usable theory when used in conjunction with such modern computational devices as the Aiken machine at Harvard University. A research program under Professor Von Mises is now under way at Harvard.

In two-dimensional incompressible flows, the stream function (a function whose values completely determine the flow) satisfies the Laplace equation. Thus each flow pattern can be determined from an analytic function of a complex variable by taking the imaginary part. In distinction, for compressible fluids the differential equation satisfied by the stream function is non-linear. But Chaplygin showed that in the hodograph plane (where the velocity components are the independent variables) the equation becomes linear although complicated. Bergman has developed an operator for this equation, wherein a flow can again be obtained for each analytic function. This operator requires knowledge of a certain function sequence which may be (and now is being) calculated once and for all. With this apparatus, all flow patterns may be obtained with comparatively little labor.

6. *The achievement of large classes in mathematics* (preliminary report), by Professors H. F. S. Jonah, and M. W. Keller, Purdue University.

The authors discuss in this paper the achievement of large classes in mathematics in comparison with achievement of small classes, as measured by uniform objective tests. These preliminary results indicate that for mature groups and selected instructors, large classes are as effective as small classes for teaching algebra and trigonometry.

7. *Hodograph methods for compressible flow*, by Professor M. E. Shanks, Purdue University.

Professor Shanks discussed the types of hodographs obtainable from flow past an airfoil, and pointed out problems unsolved even for incompressible flows. The case of supersonic flows and the method of characteristics were also discussed.

8. *Engineering applications of spherical trigonometry*, by Professor P. M. Pepper, Notre Dame University.

In this paper Professor Pepper describes some of the engineering applications of spherical trigonometry. Since its inception, spherical trigonometry has been applied principally to the sciences of astronomy, geodesy and navigation. It is little known that spherical trigonometry can be useful to the tool engineer, first, to derive the usual formulas for "compound angles" and, second, to solve atypical problems of this nature. Certain of the compound angle formulas are identified with Napier's rules for right spherical triangles, whereas certain of the nonstandard problems lead to the laws of oblique spherical triangles.

M. W. KELLER, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.
Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Bryn Mawr, November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 6


CONTENTS




JUNE-JULY 1947

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)



EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions and advertising in the Monthly, should be sent to the Secretary-Treasurer, W. B. Carver, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 20th of the month preceding that in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

BENJAMIN FRANKLIN FINKEL

## BENJAMIN FRANKLIN FINKEL

Professor Benjamin Franklin Finkel, founder of the AMERICAN MATHEMATICAL MONTHLY, was born July 5, 1865, and died February 5, 1947. He studied in the rural schools of Ohio and taught in the schools of Ohio and Tennessee, with short periods of study at a normal school, until he became an instructor in mathematics and astronomy at Kidder Institute, Kidder, Missouri, in 1892. During these years he had become keenly aware of the poor instruction given in elementary mathematics, and he had indulged his natural bent for solving problems by contributing solutions to several mathematical and educational journals. While at Kidder Institute he published his well-known *Mathematical Solution Book*, and proposed to establish a journal "devoted solely to mathematics and suitable to the needs of teachers of mathematics in these schools" (high schools and academies).

In 1895, he was made Professor of Mathematics and Physics in Drury College, Springfield, Missouri, and continued in that position, with periods of study at the University of Chicago and the University of Pennsylvania, until his retirement in 1937. After his retirement he published a history of American mathematical journals in 1940, and he taught army classes in the University of Missouri during 1944. He received the degrees of A.M. and Ph.D. from the University of Pennsylvania in 1904 and 1906, respectively, and was honored with the degree of LL.D. by Drury College in 1923.

The first number of the MONTHLY appeared in January, 1894, and Professor Finkel continued to bear the sole responsibility for the editorship and the business management of the journal until he was joined by L. E. Dickson in October, 1902, by H. E. Slaught in January, 1907, and by G. A. Miller in January, 1909. The MONTHLY was formally transferred in January, 1913, to a board of editors representing fourteen colleges and universities. Then, in January, 1916, the magazine became the official journal of the Mathematical Association of America. Professor Finkel remained one of the editors of the department of *Problems and Solutions* through 1933, and was still a member of the board of editors at the time of his death.

The present writer has known perhaps thirty individuals who are the fortunate possessors of a full set of the volumes of the MONTHLY; thirty-five libaries have the full set, according to the Union List of Periodicals, and a few other libraries have the first volumes. Most of these early issues are accessible, so that those who wish to study the history of the journal may readily do so. A perusal of the first volumes will show the uneven character of the contents but, withal, a steady advance toward a journal of high grade. Quite early it was evident from the list of subscribers that there was little response from high school instructors and that the more fertile field was to be found in the colleges and universities.

The very first issue contained an article by a young graduate student in the University of Texas, who was later to be known as Professor L. E. Dickson. In the first three volumes, to go no further, appeared contributions by such well-

known mathematicians as R. J. Aley, Fletcher Durell, Arnold Emch, George Bruce Halsted, Alexander Macfarlane, Artemas Martin, G. A. Miller, E. H. Moore, R. E. Moritz, I. J. Schwatt, D. E. Smith, and B. F. Yanney.

During the first fifteen or twenty years of the MONTHLY's existence there were great financial and technical obstacles; a picture of these difficulties was vividly presented by Professor Finkel in an address given at the meeting of the Association in Cleveland, January 1, 1931, and published in the MONTHLY for June-July of that year. American mathematics is indebted to him for his imagination and his persistent courage in establishing and promoting a journal which has become most important in the field of mathematics. His zeal, intelligently directed as it was along useful lines, has earned for him a unique place in the history of mathematical publications.

WILLIAM DEWEESE CAIRNS

## THE NEW SECRETARY-TREASURER

At the end of the present calendar year the term of office of our Secretary-Treasurer, Professor Walter B. Carver, will come to an end. Professor Carver accepted this position during the difficult period of the war with the understanding that he would be relieved at the end of his five-year term. After his long service to the Association, which includes five years as Editor-in-Chief and two years as President, he has well merited a respite from its labors.

On behalf of the officers and Board of Governors of the Mathematical Association of America I am happy to announce the election of Professor Harry M. Gehman, Chairman of the Department of Mathematics of the University of Buffalo, as Secretary-Treasurer for the five-year term beginning January 1, 1948.

LESTER R. FORD, *President*

# REMARKS ON ANALYTICITY AND INTEGRATION

A. D. PERRY, Purdue University
J. W. T. YOUNGS, Indiana University

**1. Introduction.** The theory of functions of a complex variable makes implicit use of Stieltjes integration in defining the concept of the integral of a continuous, complex-valued function along a rectifiable curve. To be more precise, suppose that $f(z)$ is a complex-valued function, defined and continuous at the points $z=z(t)=x(t)+iy(t)$, where $x(t)$ and $y(t)$ are two real-valued continuous functions for $a \leqq t \leqq b$. If $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(z)$, let $\bar{u}(t)=u[x(t), y(t)]$ and $\bar{v}(t)=v[x(t), y(t)]$, $a \leqq t \leqq b$. Then the integral of $f(z)$ along $z(t)$, $a \leqq t \leqq b$, written $\int_{z(t)} f(z)dz$, is simply $\int_a^b \bar{u}(t)dx(t)-\int_a^b \bar{v}(t)dy(t)+i[\int_a^b \bar{v}(t)dx(t)+\int_a^b \bar{u}(t)dy(t)]$, these four terms being Stieltjes integrals. Consequently, the functions $x(t)$ and $y(t)$ are normally assumed to be of bounded variation which is the same thing as assuming that the curve $z=z(t)$, $a \leqq t \leqq b$ is rectifiable. It is true that Stieltjes integration is rarely mentioned explicitly in the development; but be that as it may, the condition that the curve is rectifiable is invariably imposed.

In contradistinction to this approach, it is convenient to record the usual process employed. A *partition*, $\pi$, of $a \leqq t \leqq b$, is a sequence of numbers $t_0, \tau_0, t_1, \cdots, t_{i-1}, \tau_i, t_i, \cdots, t_{k-1}, \tau_k, t_k$ such that: (*i*) $a=t_0<t_1< \cdots <t_{i-1}<t_i < \cdots t_{k-1}<t_k=b$, and (*ii*) $t_{i-1} \leqq \tau_i \leqq t_i$ for $i=1, \cdots, k$. The symbol $|\pi|$ is used to indicate the maximum of the numbers $(t_i-t_{i-1})$ for $i=1, \cdots, k$. The complex number $J[\pi, z(t), f]$ is defined to be $\sum_{i=1}^k [z(\tau_i)][z(t_i)-z(t_{i-1})]$. The usual procedure is to say that if $\lim_{|\pi| \to 0} J[\pi, z(t), f]$ exists, then this limit is defined to be $\int_{z(t)} f(z)dz$, and is called the integral of $f(z)$ along the curve $z=z(t)$, $a \leqq t \leqq b$. The assumption that the curve is rectifiable simply guarantees the existence of $\lim_{|\pi| \to 0} J[\pi, z(t), f]$.

It is to be noticed that the remarks to this point merely call for the definition of $f(z)$ along the curve $z=z(t)$. Suppose that $f(z)$ is defined and *continuous* for $z \epsilon G$, where $G$ is a region (=connected open set) on the complex $z$-plane. Then, of course, as soon as a curve $z=z(t)$, $a \leqq t \leqq b$, is given with the property that: (*i*) $z(t) \epsilon G$, $a \leqq t \leqq b$, (*ii*) $z(t)$ is rectifiable, then the integral of $f(z)$ along the curve $z=z(t)$ exists. Moreover, if $f(z)$ is *analytic* in $G$, while $z=z_n(t)$, $a \leqq t \leqq b$, $n=0, 1, 2, \cdots$, is a sequence of curves, each having the above pair of properties, and $z_n(t) \rightrightarrows z_0(t)$, $a \leqq t \leqq b$, ($=z_n(t)$ converges uniformly to $z_0(t)$ for $a \leqq t \leqq b$) then $\lim \int_{z_n(t)} f(z)dz = \int_{z_0(t)} f(z)dz$. This classical result shows that the integral is, so to speak, a continuous functional of the curve of integration.

This note was initiated by an interest in this continuity property, and is an attempt to analyze several details of the situation. For example, it is shown that, relative to the continuity property, the requirement that the limit curve be rectifiable is not necessary. That is, $z_0(t)$ need not have property (*ii*) above, and it is nevertheless true that the sequence of complex numbers $\int_{z_n(t)} f(z)dz$ converges. One can no longer say, of course, that the limit of this sequence of

complex numbers is $\int_{z_0(t)} f(z)dz$, inasmuch as this term is normally undefined unless $z_0(t)$ is rectifiable.

This continuity result is certainly not startling and the proof is recorded here merely for the sake of completeness. On the other hand, it is shown that the continuity property is not only necessary but also sufficient for the analyticity of $f(z)$. In fact stronger theorems are proved, in that several weaker conditions are shown to guarantee the analyticity of $f(z)$.

It will become apparent in the proofs that the necessity argument makes active use of the Cauchy Theorem, while for sufficiency, the Morera Theorem supplies the key. From one point of view, therefore, these results may be described as generalized Cauchy-Morera Theorems.

**2. Preliminary concepts.** It will not have escaped attention that the term *curve* has been used as synonymous with the term *continuous function* $z=z(t)$, $a\leqq t\leqq b$. It is often convenient to use an equivalence relation in the family of continuous functions and consider a curve as an equivalence class. Such a procedure would, in fact, be rather convenient in some of the proofs, and has not been employed here merely because the standard texts on the subject do not explicitly employ this device.

In this paper, therefore, a *curve* $C$ is simply a *continuous function* $z=z(t)$, $a\leqq t\leqq b$. The point $z(a)$ is said to be the *first* point, while $z(b)$ is known as the *last* point of $C$. A second continuous function $z=\bar{z}(t)$, $a\leqq t\leqq b$ is another curve $\overline{C}$, distinct from $C$, unless $\bar{z}(t)=z(t)$, $a\leqq t\leqq b$. The curve $C$: $z=z(t)$, $a\leqq t\leqq b$, is said to be *contained in a point set* $G$, if $z(t)$ is in $G$ for every $t$ in the interval $a\leqq t\leqq b$. (Notation: $C\subset G$ or $G\supset C$.) In the event a curve $C$: $z=z(t)$, $a\leqq t\leqq b$, has the property that $z(t)$ is a constant, $\zeta$, for every value of $t$ in the interval $a\leqq t\leqq b$, then $\zeta$ is sometimes used to denote the curve $C$. A sequence of curves, $\{C_n\}$, is said to converge to a curve $C_0$ if $C_n$ is a continuous function $z=z_n(t)$ defined on the same interval $a\leqq t\leqq b$ for every $n$, and $z_n(t)\rightrightarrows z_0(t)$, $a\leqq t\leqq b$, where $C_0$ is the continuous function $z_0(t)$, $a\leqq t\leqq b$. (Notation: $C_n\rightarrow C_0$.) A curve $C$: $z=z(t)$, $a\leqq t\leqq b$, is said to be *closed* if $z(a)=z(b)$, that is, if the first and last points are the same.

It is convenient to have a specific notation for *rectifiable* curves and these are designated by the German letter $\mathfrak{C}$, in contradistinction to $C$ which stands for a *general curve.*

For convenience in printing, if $f(z)$ is defined and continuous in a region $G$, while $\mathfrak{C}$: $z=z(t)$, $a\leqq t\leqq b$, is a rectifiable curve in $G$, then $I(\mathfrak{C}, f)$ will replace $\int_{z(t)} f(t)dz$. In fact, if the function $f(z)$ is fixed in the course of an argument, then the above notation will be abbreviated to $I(\mathfrak{C})$. As usual, the symbol $|I(\mathfrak{C})|$ stands for the absolute value of the complex number $I(\mathfrak{C})$. In accordance with the usual convention, the symbol $I(C)$ has meaning only for rectifiable curves $C$.

**3. The theorems.** A complex-valued function, $f(z)$, defined and continuous for $z$ in a region $G$ is said to have property:

$\mathcal{A}$ If it is analytic in $G$, that is, if it has a derivative at every point of $G$.

$\mathcal{C}$ If $\mathfrak{C}_n \to C_0$ and $C_0 \subset G \supset \mathfrak{C}_n$, $n=1, 2, 3, \cdots$, imply that $\lim I(\mathfrak{C}_n)$ exists.
$\mathcal{Z}$ If $\mathfrak{C}_n \to \zeta$ and $\zeta \subset G \supset \mathfrak{C}_n$, $n=1, 2, 3, \cdots$, imply that $\lim I(\mathfrak{C}_n)=0$.
$\mathcal{B}$ If $\mathfrak{C}_n \to \zeta$ and $\zeta \subset G \supset \mathfrak{C}_n$, $n=1, 2, 3, \cdots$, imply that there is a real number $B$ associated with the sequence $\{\mathfrak{C}_n\}$ such that $|I(\mathfrak{C}_n)| < B$, $n=1, 2, 3, \cdots$.

It will be shown that these four properties are equivalent.

THEOREM 1. *$\mathcal{A}$ implies $\mathcal{C}$.*

*Proof.* Since $\mathfrak{C}_n \to C_0$, each $\mathfrak{C}_n$ is a function $z=z_n(t)$ on an interval $a \leqq t \leqq b$, while $z_n(t) \rightrightarrows z_0(t)$, $a \leqq t \leqq b$, and $C_0$ is the function $z_0(t)$, $a \leqq t \leqq b$. The set of points given by the formula $z=z_0(t)$, $a \leqq t \leqq b$, is closed, and as $C_0 \subset G$ there is a $\delta > 0$ such that the set of points, $z$, determined by the inequality $|z-z_0(t)| < \delta$ lies in $G$ for $a \leqq t \leqq b$. Moreover the function $z_0(t)$, $a \leqq t \leqq b$, is continuous on a closed interval, and hence is uniformly continuous. Consequently, there is a sequence of $t$'s, $a=t_0 < t_1 < \cdots < t_{k-1} < t_k = b$, such that $|z_0(t)-z_0(t_i)| < \delta/2$ for $t_{i-1} \leqq t \leqq t_i$, $i=1, \cdots, k$.

Let $R_i$ be the set of points, $z$, determined by the inequality $|z-z_0(t_i)| < \delta$, $i=1, \cdots, k$. Then $R_i \subset G$. Define $\mathfrak{C}_n^i$ to be the curve $z=z_n(t)$, $t_{i-1} \leqq t \leqq t_i$; $i=1, \cdots, k$; $n=1, 2, 3, \cdots$. (The use of the German letter is justified because this curve is rectifiable.)

The curve $\mathfrak{A}_{np}^i: z=z_n(t_i)+[z_{n+p}(t_i)-z_n(t_i)]t$, $0 \leqq t \leqq 1$; $n=1, 2, 3, \cdots$; $p>0$; $i=0, \cdots, k$, is sometimes called the directed segment from $z_n(t_i)$ to $z_{n+p}(t_i)$. It is certainly rectifiable but need not lie in $G$.

On the other hand, since $z_n(t) \rightrightarrows z_0(t)$, $a \leqq t \leqq b$, there is an $n_0$ such that for $n > n_0$,

$$|z_n(t) - z_0(t)| < \delta/2, \qquad a \leqq t \leqq b.$$

Hence for $n > n_0$,

$$\mathfrak{C}_n^i \subset R_i, \qquad i = 1, \cdots, k, \text{ and}$$
$$\mathfrak{A}_{np}^{i-1} \subset R_i \supset \mathfrak{A}_{np}^i \qquad i = 1, \cdots, k.$$

But $f(z)$ is analytic in $R_i$, a simply connected region, for $i=1, \cdots, k$; hence the Cauchy Theorem implies that there is an analytic function $F_i(z)$, in $R_i$, $i=1, \cdots, k$, such that if $\mathfrak{C}$: $z=z(t)$, $c \leqq t \leqq d$ is a rectifiable curve in $R_i$, then $I(\mathfrak{C})=F_i[z(d)]-F_i[z(c)]$.

Therefore, $I(\mathfrak{C}_n^i)+I(\mathfrak{A}_{np}^i)-I(\mathfrak{C}_{n+p}^i)-I(\mathfrak{A}_{np}^{i-1}) = F_i[z_n(t_i)] - F_i[z_n(t_{i-1})] + F_i[z_{n+p}(t_i)] - F_i[z_n(t_i)] - F_i[z_{n+p}(t_i)] + F_i[z_{n+p}(t_{i-1})] - F_i[z_{n+p}(t_{i-1})] + F_i[z_n(t_{i-1})] = 0$.

(This device is employed because the symbol $(\mathfrak{C}_n^i + \mathfrak{A}_{np}^i - \mathfrak{C}_{n+p}^i - \mathfrak{A}_{np}^{i-1})$ has not been defined. Without using an equivalence relation it is not possible to define such a formal sum as a closed *curve* and use the Cauchy Theorem directly. An alternative device is to define the above symbol as a *chain* and consider integration over chains. In this connection the reader may consult the elegant development of Artin [1].)

To return to the argument, $\sum_{i=1}^{k} I(\mathfrak{C}_n^i) - \sum_{i=1}^{k} I(\mathfrak{C}_{n+p}^i) = I(\mathfrak{A}_{np}^0) - I(\mathfrak{A}_{np}^k)$; but $\sum_{i=1}^{k} I(\mathfrak{C}_n^i) = I(\mathfrak{C}_n)$, for any $n$, hence

$$\begin{aligned} | I(\mathfrak{C}_n) - I(\mathfrak{C}_{n+p}) | \leqq | I(\mathfrak{A}_{np}^0) | + | I(\mathfrak{A}_{np}^k) | \leqq | F_1[z_n(a)] - F_1[z_{n+p}(a)] | \\ + | F_k[z_n(b)] - F_k[z_{n+p}(b)] |. \end{aligned}$$

The continuity of $F_1$ and $F_k$ together with the fact that $\lim z_n(t) = z(t)$ for $t = a$ and $b$, guarantees that, for any $\epsilon > 0$ there is an $n_1 > n_0$ such that if $n > n_1$ then

$$| I(\mathfrak{C}_n) - I(\mathfrak{C}_{n+p}) | < \epsilon \qquad \text{for } p > 0.$$

Hence $\lim I(\mathfrak{C}_n)$ exists.

THEOREM 2. $\mathcal{C}$ *implies* $\mathcal{Z}$.

*Proof.* Since $\mathfrak{C}_n \to \zeta$ it is clear that the sequence $\mathfrak{C}_1, \zeta, \mathfrak{C}_2, \zeta, \cdots \to \zeta$. Hence the sequence $I(\mathfrak{C}_1), I(\zeta), I(\mathfrak{C}_2), I(\zeta), \cdots$ converges. But $I(\zeta) = 0$, therefore the limit must be zero, showing that $\lim I(\mathfrak{C}_n) = 0$.

THEOREM 3. $\mathcal{Z}$ *implies* $\mathcal{B}$.

This is quite obvious.

THEOREM 4. $\mathcal{B}$ *implies* $\mathcal{A}$.

*Proof.* If $f(z)$ is not analytic in $G$, then it fails to be analytic at some point $\zeta$ in $G$. Suppose that the distance from $\zeta$ to the boundary of $G$ is greater than $\delta > 0$. Let $R_n$ be the set of $z$'s determined by the unequality $|z - \zeta| < \delta/n$, $n = 1, 2, 3, \cdots$. For each $n$, $R_n \subset G$ and is simply connected. If $I(\mathfrak{C}) = 0$ for every *closed* curve $\mathfrak{C} \subset R_n$ then by the Morera Theorem, $f(z)$ is analytic in $R_n$. Hence there must be a closed curve $\mathfrak{C}_n$: $z = z_n(t)$, $a_n \leqq t \leqq b_n$ such that $I(\mathfrak{C}_n) \neq 0$, $n = 1, 2, 3, \cdots$. For each $n$ there is an integer $k_n > 0$ such that $k_n \cdot |I(\mathfrak{C}_n)| > n$. Let $l_n^i(t)$ be the unique linear mapping such that $l_n^i[(i-1)/k_n] = a_n$ and $l_n^i(i/k_n) = b_n$; $n = 1, 2, 3, \cdots$; $i = 1, \cdots, k_n$. Consider the curve $\mathfrak{C}_n^i$: $z = z_n^i(t) = z_n[l_n^i(t)]$, $(i-1)/k_n \leqq t \leqq i/k_n$, it is easy to see that the integral of $f(z)$ along $\mathfrak{C}_n$ is equal to the integral of $f(z)$ along $\mathfrak{C}_n^i$, $n = 1, 2, 3, \cdots$; $i = 1, \cdots, k_n$.

For each $n$, let $\bar{z}_n(t)$ be the function defined on $0 \leqq t \leqq 1$ as equal to $z_n^i(t)$ for $(i-1)/k_n \leqq t \leqq i/k_n$, $i = 1, \cdots, k_n$. Then, as each $z_n(t)$ is closed, $\bar{z}_n(t)$ is a continuous function on $0 \leqq t \leqq 1$, and so is a curve $\overline{\mathfrak{C}}_n$. Moreover,

$$| I(\overline{\mathfrak{C}}_n) | = \left| \sum_{i=1}^{k_n} I(\mathfrak{C}_n^i) \right| = k_n | I(\mathfrak{C}_n) | > n, \quad n = 1, 2, 3, \cdots.$$

On the other hand $\overline{\mathfrak{C}}_n \subset R_n$ and hence $|\bar{z}_n(t) - \zeta| < \delta/n$, $0 \leqq t \leqq 1$. Consequently $\overline{\mathfrak{C}}_n \to \zeta$, and by hypothesis there is a number $B$ such that $|I(\overline{\mathfrak{C}}_n)| < B$, $n = 1, 2, 3, \cdots$.

This contradiction shows that $f(z)$ must be analytic in $G$.

**4. General comments.** The first theorem makes it possible to define a complex number $I^*(C,f)$ for any curve, $C$, and a function, $f(z)$, analytic in a region, $G$, containing $C$. In fact, let $\{\mathfrak{C}_n\}$ be a sequence of rectifiable curves such that $\mathfrak{C}_n \to C$. Such sequences certainly exist: in particular each $\mathfrak{C}_n$ may be taken to be a polygon inscribed in $C$. Moreover, it may be assumed that $\mathfrak{C}_n \subset G$, $n=1, 2, 3, \cdots$. Let $I^*(C,f) = \lim I(\mathfrak{C}_n, f)$, which is known to exist by Theorem 1. In the event $\overline{\mathfrak{C}}_n \to C$, then the sequence $\mathfrak{C}_1, \overline{\mathfrak{C}}_1, \mathfrak{C}_2, \overline{\mathfrak{C}}_2, \cdots \to C$, from which it follows that $\lim I(\mathfrak{C}_n, f) = \lim I(\overline{\mathfrak{C}}_n, f)$. In other words $I^*(C,f)$ depends only on $C$ and $f$. This number may be called *a continuous extension* of the ordinary integral, since it is but a remark to show that it agrees with the ordinary integral in the event $C$ is rectifiable. Moreover, it enjoys all the common properties of the ordinary integral; in particular it is a continuous functional of the curve of integration. Indeed, it may be spoken of as *the continuous extended integral*, since any extension of the ordinary integral which is a continuous functional of the curve of integration must agree with this particular extension. In view of these comments it is possible to state—

THEOREM 5. *A function $f(z)$ which is continuous in a region $G$ is analytic in $G$ if and only if the continuous extended integral of $f(z)$ exists along every curve $C$ in $G$.*

It should be pointed out that, if $f(z)$ is analytic in a simply connected region $R$, then the continuous extended integral of $f(z)$ along a curve $C$ in $R$ may be defined as a trivial consequence of classical results. For, if $f(z)$ is analytic in $R$ then, as has been noted in Theorem 1, there is an analytic function $F(z)$ in $R$ such that, if $\mathfrak{C}$: $z=z(t)$, $a \leqq t \leqq b$ is in $R$, then $I(\mathfrak{C}, f) = F[z(b)] - F[z(a)]$. Consequently, if $C$: $z = \bar{z}(t)$, $\bar{a} \leqq t \leqq \bar{b}$ is a general curve, then the continuous extended integral $I^*(C,f) = F[\bar{z}(\bar{b})] - F[\bar{z}(\bar{a})]$ and may be so defined, *a priori*. For a region $G$ which is not simply connected this artifice will not always work, as in general no such function $F(z)$ exists. The method outlined in this paper is, in a sense, simply one device which extends this trivial remark to general regions.

A question of some interest is concerned with the following observation. Suppose $G$ is a region and $\mathfrak{F}$ the totality of functions each of which is defined and analytic on $G$. If $C$ is a curve lying in $G$, then what properties of $C$ are necessary and sufficient for the usual process (noted in the introduction) to yield an integral of $f(z)$ along $C$ for every $f(z)$ in $\mathfrak{F}$? (The condition that $\mathfrak{C}$ is rectifiable is sufficient for the purpose, but it is not necessary.) The question has been completely answered by Zorn [**2**].

Let $C$: $z=z(t)$, $a \leqq t \leqq b$ be any curve in $G$ and $\pi$: $t_0, \tau_1, t_1, \cdots, t_{k-1}, \tau_k, t_k$ be a partition of the interval $a \leqq t \leqq b$. The symbol $\pi(C)$ is defined to be $\sum_{i=1}^{k} |z(t_i) - z(t_{i-1})|^2$. The symbols $|\pi|$ and $J[\pi, z(t), f]$ have been defined in the introduction. For temporary convenience, the curve $C$ will be called *quasi-rectifiable* if $\lim_{|\pi| \to 0} \pi(C) = 0$.

The result of Zorn is that $\lim J[\pi, z(t), f]$ *exists if and only if $C$ is quasi-rectifiable.* There are curves which are quasi-rectifiable without being rectifiable; on the other hand, not every curve is quasi-rectifiable.

In conclusion, it is noted that, if $f(z)$ is in $\mathfrak{F}$ and $C$ is a quasi-rectifiable curve in $G$, then $\lim J[\pi, z(t), f] = \dot{I}^*(C, f)$.

**References**

1. Artin, Emil. On the Theory of Complex Functions. Notre Dame Mathematical Lectures, No. 4, (1944) pp. 57–70.
2. Zorn, M. A. Approximating Sums. This MONTHLY, vol. 54, (1947) pp. 148–151.

## COLIN MACLAURIN

H. W. TURNBULL, St. Andrews University, Scotland

On June 14, 1946, honor was paid to the memory of Colin Maclaurin when members of the Edinburgh Mathematical Society laid a wreath beside his grave in Greyfriars Churchyard at Edinburgh on the occasion of the bicentenary of his death. At a special meeting of the Society in December, 1946, a vivid tribute was paid by Dr. Robert Schlapp, a former President of the Society, who described the life and work of Maclaurin. I was fortunate to be present to hear the address, and am much indebted to what I heard in what follows.

Colin Maclaurin was born at Kilmodan, Argyll, in February, 1698. He was one of three sons. John, the eldest, following in his father's steps, became a minister; he was a public spirited man of profound learning, and corresponded with Jonathan Edwards, the American metaphysician. The second son, Daniel, died young after having given signs of extraordinary genius. Colin was the youngest. His father died when Colin was six weeks old, and nine years later he lost his mother. The care of the children thereafter devolved entirely upon their uncle. At the age of eleven Colin followed his brother to the University of Glasgow where he came under the influence of Robert Simson, Professor of Mathematics, from whom he gained a lifelong interest in Euclid and ancient geometry. He graduated at the age of fifteen when he wrote and publicly defended a thesis, *On the power of gravity*. After spending a year reading Divinity he left Glasgow to live with his uncle in their highland home beside Loch Fyne, reading mathematics, philosophy, and the classics, wandering on the hills, actively searching out the scientific secrets of their stones and plants. Unfinished scraps in his notebooks reveal the sensitivity of his nature as he would sometimes break out into poetic rhapsody on the beauties of the scene and the perfections of its Author. Such was the background of the startlingly mature work upon the organic description of plane curves which he completed in 1719, at the age of twenty-one.

Maclaurin was the third and greatest of three scholars who, more than any others, made a critical and understanding study of the *Principia* of Newton, that had been published at Halley's prompting in 1687. David Gregory, astronomer and mathematician, was the first; Roger Cotes, the second, died in 1716 at the age of 34. Cotes helped Newton prepare the second edition of the *Prin-*

*cipia*, and Newton once wrote, "Had Cotes lived we might have known something." Cotes left a confused heap of manuscripts from which one particular geometrical theorem was extracted by his cousin Robert Smith (who later became Master of Trinity College, Cambridge) and sent to Maclaurin. It led to his second great work which has proved to be the foundation stone of the theory of plane cubic curves.

At the age of nineteen Maclaurin was appointed to the Chair of Mathematics in the Marischal College, Aberdeen, where honor was shown to his memory at a special lecture given in the College on February 4, 1947. He first met Sir Isaac Newton the next year in London, where he left his *Geometria Organica* to be printed, and was admitted a Fellow of the Royal Society. In 1722 he undertook a Continental journey as the tutor to the son of Lord Polwarth, and actually extended his absence from his Chair to three years. During the tour he was mathematically active, and for a thesis upon the percussion of bodies he was awarded a prize by the French Academy. Other awards, the same year, were made to Leonard Euler and Daniel Bernoulli. Maclaurin returned to Aberdeen after the sudden death of his pupil in France, and expressed his regret at his long absence without leave! In the following January his Chair was declared vacant, but meantime he had moved in November to Edinburgh where he deputized for the ageing Professor James Gregory, the nephew of his more famous namesake. Eventually and on the recommendation of Newton he succeeded to the Edinburgh Chair, which he occupied with great distinction. His classes were well attended, and his books on algebra and fluxions were models of lucid, careful elaboration from the very beginnings to the latest and deepest discoveries. Maclaurin laid the actuarial foundations that have borne fruit in the well known insurance societies of Edinburgh. He was the prime mover in the formation of a scientific gathering that sprang from the Medical Society, and soon grew to be the Royal Society of Edinburgh. In 1733, he married Anne, the daughter of Mr. Walter Stewart, Solicitor General for Scotland. Of their seven children, two sons, John and Colin, and three daughters, survived him.

Maclaurin was an able experimentalist. He proposed an observatory for Scotland, took an active part in improving the maps of Orkney and Shetland, and was prepared to embark on a voyage to discover a North Polar passage, should the opportunity occur. In 1745, when a Highland army marched on Edinburgh, Maclaurin took a leading part in rallying the townsfolk against the Jacobites and organizing the defenses of the city. Night and day he planned and supervised the hastily erected fortifications, a task of devotion which damaged his health. When the city was taken, he escaped and withdrew to England, reaching York where he became the guest of Thomas Herring, the Archbishop. "Here," wrote Maclaurin, "I live as happily as a man can do who is ignorant of the state of his family, and sees the ruin of his country." He returned the next year to Edinburgh, but the rigors of the journey broke his health, and in 1746 he died at the early age of forty-eight. Only a few hours before his death he was engaged in dictating a concluding passage for his work on the Philosophy of Newton; the

argument in favor of a future life contained in the last sentences is now well known; it proceeded from the lips of a dying man. After speaking of the beautiful scheme of nature and how imperfectly it can be apprehended in man's present state, "this," he says, "naturally leads us to consider our present state as only the dawn or beginning of our existence, and as a state of preparation or probation for further advancement."

The first mathematical work of Maclaurin was his *Geometria Organica sive Descriptio Linearum Curvarum Universalis* (London 1720), many of the properties therein contained being discovered while he was in his teens. The book not only bore the imprimatur of Sir Isaac Newton but took its rise from Newton's method of generating a conic or a cubic by rotating two angles of fixed sizes about their vertices $S$ and $C$ as fixed pivots.

If, as in Figure 1, $Q$, the intersection of two of the legs, lies on a fixed straight line $AE$, then $P$, the intersection of the other two legs, describes a conic which passes through $S$ and $C$. Maclaurin proved this by expressing $z$, the ordinate of $Q$,

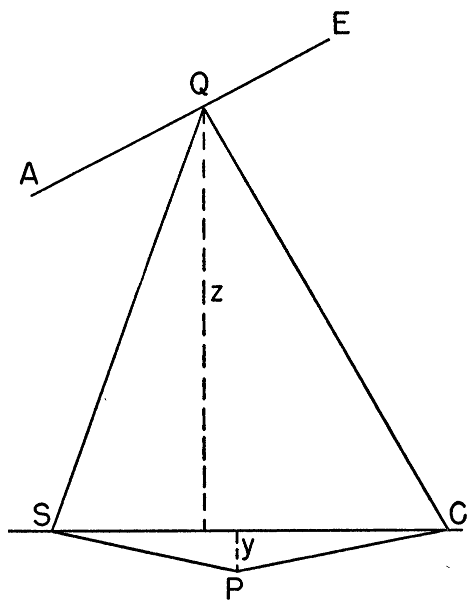



FIG. 1

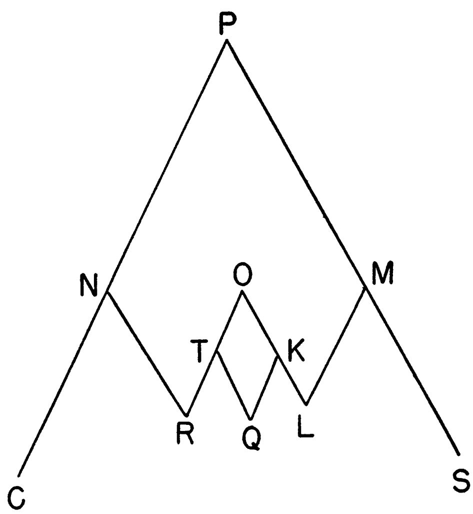



FIG. 2

bilinearly in terms of $x$ and $y$, where $C$ is the origin and $CS$ is the axis of $x$, and again where $S$ is the origin and $SC$ the axis of $x$. The two values of $z$ gave him a second degree equation in $x$ and $y$. The free use of several origins within one geometrical figure, and the persistent use of the same parameters $x$ and $y$ to express all the necessary magnitudes, is typical of his work throughout. He advances step by step, always using his two pivots $S$ and $C$ until he reaches highly elaborate properties such as those involved in the consideration of Figure 2.

In Figure 2, let $CNRTQ$ be a succession of given angles $CNR$, $NRT$, and so

on, whose vertices $N$, $R$, $T$ move on given lines (straight or curved) as guides, while $C$ is a fixed point. Let $SMLKQ$ be a similar succession, where $S$ is a fixed point. Let the first legs $CN$ and $SM$ reckoned from $C$ and $S$ meet at $P$, let the $r$th leg from $S$ and the $s$th leg from $C$ meet at $O$, while the $m$th from $S$ and the $n$th from $C$ meet at $Q$. (The figure, of course, requires $m=n=4$.) If $Q$ moves on a guiding curve, then the point $O$ describes a curve of order $lkitpuz\times(ms+nr+s+r)$ where the single letters denote the several orders of the curves described by $Q, N, R, T, M, L, K$. In particular, if all the guides are straight lines and $r=s=1$, the locus of $P$ is a curve of order $m+n+2$.

A vast number of properties are worked out, and all the rudiments of higher plane curves are dealt with, their nodes and cusps, intersections and freedoms, the number $\frac{1}{2}(n-1)(n-2)$ for the maximum number of nodes on an $n$-ic, and the $mn$ intersections of general curves of orders $m$ and $n$. He printed a proof of this last important result in a sequel a few years later, but apparently did not publish it.

In 1722, Maclaurin learned from Robert Simson the beautiful property of Pappus, his *porism*, that if each of the three sides of a variable triangle is made to pivot on a point while two of the vertices describe straight lines, then the third vertex also will describe a straight line, provided that all three pivots are collinear. This suggested to Maclaurin a sequel to the above, by relaxing the constancy of the moving angles but, instead, constraining each leg (not merely the first and the last) to pivot on a fixed point. He obtained a formula analogous to the above, for the locus of $O$ or $P$, and also found that the order of the locus of $P$ was halved when all the pivots were collinear. From the triangle and three non-collinear pivots he obtained a variant of Pascal's Theorem, which he proved while he was at sea on his memorable journey to the Continent in July, 1722.

His second great geometrical work, the *De Linearum Geometricarum Proprietatibus Generalibus*, published as an appendix to his treatise on Algebra, after his death, was inspired by Cotes' Theorem, and laid the foundation for the general theory of cubic curves. The work shows what can happen when two distinct ideas are perfectly blended. One idea came from Newton, and one from Cotes. From Newton came the property that the ratio of $PA\cdot PB\cdot PC$ to $Pa\cdot Pb\cdot Pc$ is constant when two lines through a variable point $P$ cut a curve of order $n$ in the points $A$, $B$, $C$ and $a$, $b$, $c$, respectively, while remaining parallel to two fixed lines. From Cotes came the property $\Sigma PA^{-1}=nPQ^{-1}$, where $Q$ is the point at which any line through $P$ cuts the polar line of $P$ with regard to the curve. The latter formula suggested logarithmic differentiation of the former, whence by a beautiful argument Maclaurin arrived at the property that if the tangents at the points $A, \cdots$, on the variable chord through $P$ cut a fixed chord at the points $K, \cdots$, and this chord cut the curve at $D, \cdots$, then

$$\Sigma PD^{-1} = \Sigma PK^{-1}.$$

By applying this result to the cubic, taking $P$ at an inflexion, or at another point of the curve, or again elsewhere, he discovered an immense number of properties,

many being reminiscent of the harmonic properties of conics. Typical examples are that four tangents can be drawn from a point of the curve, three only if the point is an inflexion, in which case the points of contact are collinear, and that a chord through two inflexions necessarily passes through a third.

*The Treatise of Fluxions* (Edinburgh, 1742) is a masterpiece of reasoning, in which Maclaurin gave a systematic account of Newton's theory, set out in both geometrical and analytical form, with a wealth of applications and many discoveries. It was written in answer to an attack which Bishop Berkeley made in *The Analyst* against the use of infinitesimals. In point of rigor it is a worthy link between the ancient method of exhaustions and the subsequent work of Cauchy and of Weierstrass. Maclaurin extended Newton's work on the gravitation of spheres to that of ellipsoids, and incidentally introduced the idea of confocal conics. The power of the work moved Clairaut to abandon analysis and attack the problem of the figure of the earth by pure geometry. Lagrange who, after Maclaurin, took the next great step forward in the theory, by developing the idea of potential, described this passage of Maclaurin as "le chef d'oevre de geometrie qu'on peut comparer a tout ce qu' Archimedes nous a laisse de plus beau et de plus ingenieux" (*Mem. de l'Acad. de Berlin*, 1773). *The Treatise of Fluxions* contains a theoretical and practical discussion of infinite series, including his well known integral test for convergence, the Maclaurin series, which he states to be a consequence of Brook Taylor's series (1715), and the summation formula discovered independently and nearly contemporaneously by Euler and Maclaurin. Perhaps the finest part of the whole work is that upon lines of swiftest descent and the isoperimetrical problems.

Maclaurin continued and developed the great work which Newton had begun, and today we are indebted to him for the discoveries and processes with which he enriched our mathematical heritage. Among his contemporaries in Scotland, Stirling alone was his equal in analytical power; and in geometry, Braikenridge alone contributed something new to his organic description of plane curves. The true significance of Maclaurin's geometrical power is seen in what it led to in the work of Poncelet, Steiner, Grassmann and Salmon.

**Published Works by Maclaurin**

1. Geometria Organica (dedicated in November, 1719, to Sir Isaac Newton), London, 1720.

2. Treatise of Fluxions, 2 vols., Edinburgh, 1742.

3. A Treatise of Algebra, with an Appendix De Linearum Geometricarum Proprietatibus Generalibus 1756, English translation of fifth edition, 1788. Published by Maclaurin's literary executors, Martin Folkes and Andrew Mitchell, M. P. of Abingdon, and John Hill, chaplain to Thomas Herring, Archbishop of York.

4. An account of Sir Isaac Newton's Philosophy, published by subscription by Patrick Murdoch for the benefit of Maclaurin's children, and prefaced by a memoir of Maclaurin. (First draft prepared by Maclaurin in 1728.)

5. Various communications mostly embodied in the above, and first published in the Philosophical Transactions, London, Nos. 356, 359, 394, 408, 439, 467, 469; A rule for finding the meridional parts of a spheroid, No. 461, and of the basis of the cells wherein the bees deposit their honey, No. 471.

## THE TEACHING OF COLLEGE MATHEMATICS TODAY*

M. S. KNEBELMAN, State College of Washington

**1. Introduction.** I believe that I should apologize for the title of my paper as it will be concerned primarily with a critique of the role the Association is playing in mathematical education. About two years ago the Board of Governors of the Association authorized its president to appoint a committee for the "co-ordination of studies in mathematical education." I have the honor to be a member of this committee under the highly capable and constructive chairmanship of Professor C. V. Newsom.† I must mention, and without any false modesty, that my own contribution to the progress of this committee's work has been minor indeed. I have, however, learned a few things about conditions and trends in present day mathematical education on the secondary and college levels so that my own ideas about these matters have become either modified or completely changed.

For these reasons I should make it clear that I speak for the committee only in part; most of what I have to say is an expression of my own discontent brought to an acute stage by the war and postwar conditions. This is not to say that we need look upon present day mathematical education with any great alarm; its main current flows in the right channel and the temporary fads and fashions can deflect it only temporarily. Nevertheless, we should attempt to remove all obstructions and impediments to its progress. Thus, as you know, during the war "acceleration" was all the rage and many of us "covered" the mathematical gamut from algebra to calculus in twelve weeks. As you also know, "acceleration" died a silent, but far from painless death; our chickens are coming home to roost now, and I think it is a regrettable fact, but a fact nevertheless, that most of that accelerated training was rather unsatisfactory. Now the height of style is "integration" and even if it persists, I do not believe it will leave a profound effect on mathematical education. My reason for this belief is that these integrated programs were in the main designed by humanists and educationists; mathematicians and other scientists were busy with war work at the time. I've attended a few meetings of such groups and I cannot say that their general ideas about mathematics are either very sound or very clear. It comes almost as a comic relief to read in a recent, widely accepted and publicized report of one of our foremost colleges, a comparison of the educational process with the three foci of an ellipse!

As I mentioned at the beginning it is not my purpose to tell you how to teach mathematics either on the secondary or on the college level. Neither will I presume to state what the most essential elements of any course are or, in my opinion, should be. Both of these things have recently been done in the pages of this MONTHLY, and any proposals I might make would lack the self-confidence

* Prepared for the Annual Meeting of the Association at Swarthmore, Pennsylvania, Dec. 26–27, 1946.

† Professor C. C. MacDuffee is now chairman.—EDITOR

that these papers seem to display. What I have in mind is an attempt to make mathematical instruction on the two above mentioned levels sufficiently thorough and useful to our students, and make a suggestion as to what the Association can do to improve the general mathematical atmosphere in which we live.

**2. Two main trends.** There are two main trends in present day mathematical education; the first is to reduce the content of a course to barest essentials. In many schools the approach is almost technological in nature. The justification for this was expressed to me by a colleague in the following terms: "If you want to teach a person how to drive a car, you do not try to teach him thermodynamics or the theory of internal combustion engines; you teach him how to push levers." And just as pushing levers will not design a better engine, so will the type of training indicated above be a very poor foundation for further study and development. It is the most difficult part of our job to make students grasp the full significance of a mathematical definition or concept; it is so much easier to learn a number of rules of procedure. I regard this type of training undesirable because the student neither obtains much useful information nor does he receive any mental stimulus or development from it.

The other trend is to relegate to lower levels instruction that traditionally was or is on a higher level. From my point of view this is a healthy trend that indicates progress in mathematical education. It is only a question of how rapidly this transition should be made, for experience and observation show that too rapid a transition is likely to prove injurious. There seems to be a desire in many quarters to start instruction in the calculus in the freshman year; such a development would relegate all preceding instruction to the high school. It seems to me that at least for the present this is not very realistic for the vast majority of high schools; also there is still the question of the maturity of the student. I don't know how desirable it would be to teach Fourier series or the Laplace Transform in the sophomore year. At any rate such reforms and progress must evidently be the joint effort of high school and college teachers of mathematics. Then, too, we receive our students from high school, and we have to build on the foundations laid down in the high school. That these foundations are often poor no one denies, but the college teacher, as a rule, does nothing to remedy these defects. We are too prone to criticize and condemn; in fact, quite recently, one of our eastern colleges held a symposium devoted to this pastime. Another group that teachers of mathematics are in the habit of damning is the group of professional educationists, the product of teachers' colleges. Quite naturally we resent the fact that there are state laws requiring teachers to "take" a certain number of educational courses in psychology and methodology; in some cases the ratio of the time spent in these courses to that spent on mathematics is 4 to 1. I know of no state law that requires a mathematics teacher to take a course in projective geometry or in number theory. It is a rare case indeed where a state board of education contains a mathematician or a person whose mathematical interest outweighs his general democratic educational interests. What

we refuse to admit is that all these groups are just as sincere in their efforts to do a good job, as we are, and are just as anxious to improve their work. In fact I am not at all sure that the college does a better job than the high school. This may sound like treason coming from a college teacher, but I know of more than one college where a student can graduate with a B.A. in mathematics without advancing any further than a first course in the calculus. I have seen their transcripts! At least the high school teachers have shown through the National Council of Teachers of Mathematics their sincere desire to improve secondary mathematical training. The Second Report of the Commission on Post War Plans which proposes "double-tracking" the high school curriculum so as to serve most effectively the various divisions of high school population is good evidence of this. And while I personally do not agree entirely either with their premises or their conclusions, nevertheless I believe that everyone of us should actively support their efforts. I am glad to say that through the efforts of Professor C. V. Newsom we are cooperating with the National Council closely and harmoniously not only in this matter but also in the matter of secondary school teacher training. Our committee is working on a program of teacher training which will improve mathematics teaching, but the adoption of the program will not come to pass unless the Association applies all the pressure it can in support of it.

**3. Need for cooperation with the other fields.** Another point, of the same nature, that I would like to bring to your attention is the fact that in our colleges by far the great majority of students who take courses in mathematics are primarily not students majoring in mathematics. They are usually engineers, physicists, chemists, and so forth. The mathematical part of their curriculum is almost never designed or controlled by departments of mathematics. Consider the case of E.C.P.D. (The Engineers' Council for Professional Development). They require engineering students to have had a course in solid geometry, either in high school or in college, and this quite frequently leads to a situation where a junior or a senior, after having taken a course in analytic geometry and calculus, is required to revert to his high school days. So far as I know they do not require any training in differential equations or Fourier series or any other mathematical discipline beyond a most rudimentary course in the calculus. Now while it is true that many colleges have introduced a more up-to-date mathematical program in their engineering curricula, in many cases efforts in this direction have met with failure and in some cases have led to a split so that we find in some colleges two departments of mathematics—one for engineers, the other for the rest of the college. The A.S.E.E. (The American Society for Engineering Education) has more to say about mathematical instruction for engineers than we have. The point I am trying to make is that cooperation with such an organization as E.C.P.D. or A.S.E.E. seems to offer the best chance for improvement and progress, and I believe that such cooperation must originate with the Association or it will not come about. Of course even all this will not insure more than better labels for our courses. It will not insure us against dull, poor, shallow

textbooks or dull, poor instruction. I've seen a book on high school geometry in which the only definition of a sphere I was able to find was the statement that a sphere looks like a baseball, and of course a picture of a baseball; notice the psychological approach! I have also seen a book on the calculus in which $dx$ is defined as a small piece of $x$. My harsh criticism of this was parried by the rejoinder that the definition was perfectly all right since the book was written primarily for engineers. It is my belief that the Association can exert an influence for better books either by not pulling punches in its reviews or by reviewing only books that are sound. I do not mean to say that we do not have many excellent books on mathematics, but these books are not the ones that have the widest adoption.

**4. Conclusion.** Finally it seems to me that if the Association, in collaboration with other groups interested in mathematical education, would work out a fairly general program of study for high schools and colleges and express its disapproval of any program that is far short of it, there would be many ways of introducing such programs in various schools, or at least express its approval of those schools that are maintaining reasonable standards of training. This may sound like the establishment of an accrediting agency for mathematics but it need not be that at all. Moreover, I have seen many improvements, in other fields, that were due to the existence of such an agency. One way of insuring decent standards in mathematical education is by means of cooperative tests in achievement. As a rule administrations do not oppose that; state colleges and universities may oppose entrance examinations and, in fact, they may be illegal, but examinations in course are accepted without resistance. I do not mean that the Association should engage in the business of designing and distributing tests; there is now a Cooperative Educational Testing Commission under the chairmanship of President J. B. Conant, its aim being the study and improvement of testing procedures in all areas of learning. What I do mean is that the Association should have a clear and rather loud voice in all matters pertaining to the teaching of mathematics. It should condemn or fail to approve any outmoded or archaic practices, and it should wield all its prestige and power necessary to insure progress and improvement in mathematical education. I know of no better organization to do this, and I know there are many schools where an official stand taken by the Association would be sufficient to enable the Department of Mathematics to introduce the necessary reforms.

# THE HERVEY POINT OF THE GENERAL n-LINE

R. GOORMAGHTIGH, Bruges, Belgium

**1. Introduction.** An arbitrary four-line determines four triangles and if, in each of these triangles, the perpendicular bisector of the segment joining the orthocenter to the circumcenter is drawn, then a well known theorem states that the four bisectors meet in a point which has been called the Hervey point [1] of the four-line. V. Thébault, however, has pointed out [2] that S. Kantor [3] mentioned the theorem two decades before Hervey announced it. In this MONTHLY, J. R. Musselman [4] has extended the Hervey theorem to the case of $n$ straight lines tangent to a parabola. Using the notation and methods of inversive geometry as developed by the Morleys [5], [6], we propose to obtain a generalization for an arbitrary $n$-line.

**2. Centric center of the n-line.** Given $n$ straight lines $\Delta_1, \Delta_2, \cdots, \Delta_n$ and, in a system of complex coördinates having $O$ as origin and $\Omega$ as base-point, let $x_1, x_2, \cdots, x_n$ be the coördinates of the images of $O$ in these lines. Denoting the conjugate of $c$ by $\bar{c}$, we note that $t_j = -\bar{x}_j/x_j$ is a turn. If now we consider the $k$ lines $\Delta_1, \Delta_2, \cdots, \Delta_k$, it is easily shown that the expression

$$\begin{aligned} a_{k,i} = {} & x_1 t_1^i/(t_1 - t_2)(t_1 - t_3) \cdots (t_1 - t_k) \\ & + x_2 t_2^i/(t_2 - t_1)(t_2 - t_3) \cdots (t_2 - t_k) \\ & + \cdots, \end{aligned}$$

satisfies the relations

$$a_{k,i} = a_{k+1,i+1} - t_{k+1} a_{k+1,i} \tag{1}$$

and

$$\bar{a}_{k,i} = (-1)^k \sigma_k a_{k,k-i-1}, \tag{2}$$

where $\sigma_k = t_1 t_2 \cdots t_k$.

Since the equation of the line $\Delta_j$ is

$$\bar{x} = t_j(x - x_j),$$

the point of intersection of lines $\Delta_1$ and $\Delta_2$ is given by the expression $a_{2,1}$, or, using (1), we have

$$z_2 = a_{2,1} = a_{3,2} - t_3 a_{3,1}.^*$$

Hence the intersection $a_{2,1}$ lies on the circle whose parametric equation is

$$x = a_{3,2} - \tau a_{3,1},$$

$\tau$ being a turn. Since $a_{3,2}$ and $a_{3,1}$ are symmetric with respect to the three lines

* To avoid confusion, it may be noted that generally $a_{j,i}$ is not the intersection of lines $\Delta_j$ and $\Delta_i$.

$\Delta_1$, $\Delta_2$, $\Delta_3$, it follows that this circle circumscribes the triangle formed by the three lines; its center is the point $a_{3,2}$.

Again, using (1), we may write

$$z_3 = a_{3,2} = a_{4,3} - t_4 a_{4,2},$$

and from the symmetry of the expressions $a_{4,3}$ and $a_{4,2}$ with respect to the four lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ we see that the circumcenters of the four triangles, formed by taking the four lines three at a time, lie on a circle, called the centric circle of the four-line. The center of the circle is the point $a_{4,3}$ and is called the centric center of the four-line.

Proceeding as before, we find that the centric centers of the five four-lines, obtained by omitting one of the five lines $\Delta_1, \Delta_2, \cdots, \Delta_5$, lie on a circle. This is the centric circle of the five-line and its center is the point $a_{5,4}$, which is called the centric center of the five-line. Continuing in this manner, we arrive at the theorem of de Longchamps [7], according to which the $n$ centric centers of the $(n-1)$-lines belonging to an arbitrary $n$-line are concyclic. This circle is the centric circle of the $n$-line and its center

$$z_n = a_{n,n-1}$$

is the centric center of the $n$-line.

**3. Centric focus of the n-line.** The parametric equation of the centric circle of the $(n-1)$-line $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$ is

$$x = a_{n-1,n-2} - \tau a_{n-1,n-3}, \tag{3}$$

where $\tau$ is a turn. Moreover, it follows from (1) and (2) that the ratio

$$-(t_n - a_{n,2}/a_{n,1}):(1 - t_n a_{n,n-3}/a_{n,n-2})$$

is a turn and for this value of $\tau$ the corresponding point on the circle (3) has the coördinate

$$\gamma_n = a_{n,n-1} - a_{n,2}a_{n,n-2}/a_{n,1}. \tag{4}$$

This expression being symmetric with respect to the coördinates $x_1, x_2, \cdots, x_n$, it follows that the $n$ centric circles of the $(n-1)$-lines belonging to the given $n$-line pass through the point $\gamma_n$.

In the case of the four-line, the point $\gamma_4$ is easily identified as the focus of the inscribed parabola; for this reason the point $\gamma_n$ is called the centric focus of the $n$-line. In the case of the four-line, the centric focus lies on the centric circle.

**4. Orthocenter of the n-line.** The orthocenter of the three-line may be thought of as the extremity of the resultant of the three vectors drawn from the circumcenter to the vertices of the triangle. Similarly, we define the orthocenter of the $n$-line to be the extremity of the resultant of the $n$ vectors drawn from the centric center of the $n$-line to the centric centers of the $(n-1)$-lines belonging to the given $n$-line.

The vector having $O$ for origin and equal to the vector joining the centric center of the $n$-line to the centric center of the $n-1$ lines $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$ is determined by

$$a_{n-1,n-2} - a_{n,n-1} = - t_n a_{n,n-2}.$$

Hence, it follows that the coördinate $h_n$ of the orthocenter of the $n$-line is given by

$$h_n = a_{n,n-1} - s_n a_{n,n-2}, \tag{5}$$

where

$$s_n = t_1 + t_2 + \cdots + t_n.$$

**5. The Hervey point of the n-line.** Let us now consider the distances $d$ and $\delta$ from the orthocenter of the $n$-line to the centric center and the orthocenter, respectively, of the $(n-1)$-line $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$.

The centric center of the $(n-1)$-line is given by

$$z_{n-1} = a_{n-1,n-2} = a_{n,n-1} - t_n a_{n,n-2},$$

and, using (5) and (1), we find for the orthocenter of the $(n-1)$-line

$$\begin{aligned} h_{n-1} &= a_{n-1,n-2} - s_{n-1} a_{n-1,n-3} \\ &= a_{n,n-1} - t_n a_{n,n-2} - s_{n-1}(a_{n,n-2} - t_n a_{n,n-3}) \\ &= a_{n,n-1} - s_n a_{n,n-2} + t_n s_{n-1} a_{n,n-3}. \end{aligned}$$

Combining these expressions with (5) and using (2), we get

$$\begin{aligned} d^2 &= (h_n - z_{n-1})(\bar{h}_n - \bar{z}_{n-1}) \\ &= (-1)^n \sigma_n s_{n-1} \bar{s}_{n-1} a_{n,1} a_{n,n-2}, \end{aligned}$$

and

$$\begin{aligned} \delta^2 &= (h_n - h_{n-1})(\bar{h}_n - \bar{h}_{n-1}) \\ &= (-1)^n \sigma_n s_{n-1} \bar{s}_{n-1} a_{n,2} a_{n,n-3}, \end{aligned}$$

and, therefore, their ratio is found to be

$$\lambda_n^2 = d^2/\delta^2 = a_{n,1} a_{n,n-2}/a_{n,2} a_{n,n-3},$$

which is symmetric with respect to the $n$ coördinates $x_1, x_2, \cdots, x_n$.

Moreover, in the given $n$-line, the square of the radius of the centric circle is

$$a_{n,n-2} \bar{a}_{n,n-2} = (-1)^n \sigma_n a_{n,1} a_{n,n-2}, \tag{6}$$

while the square of the distance from the centric center to the centric focus is

$$a_{n,2} a_{n,n-2} \bar{a}_{n,2} \bar{a}_{n,n-2}/a_{n,1} \bar{a}_{n,1} = (-1)^n \sigma_n a_{n,2} a_{n,n-3} \tag{7}$$

and the ratio of these two squares is precisely $\lambda_n^2$. Thus we have the result:

THEOREM. *The distances from the orthocenter of an n-line to the centric center and the orthocenter of an arbitrary* $(n-1)$*-line, obtained by omitting one of the given lines, are in a constant ratio. This ratio equals that of the radius of the centric circle of the n-line to the length of the segment joining the centric center to the centric focus.*

In particular, when $n=4$, the centric focus lies on the centric circle and $\lambda_4=1$. Thus our result reduces to Hervey's theorem, the orthocenter of the four-line being the Hervey point.

**6. Tangents to a parabola.** Let the $n$ given lines be tangent to a parabola. If the focus is taken as origin $O$ and if the base-point $\Omega$ is the projection of $O$ on the directrix, then the equation of the directrix is

$$x + \bar{x} = 2$$

and, since the image of $O$ in the tangent $\Delta_j$ lies on the directrix, it follows that

$$x_j = 2/(1 - t_j).$$

Introducing the notation

$$\pi_k = (1 - t_1)(1 - t_2) \cdots (1 - t_k),$$

we can easily show that

$$z_2 = a_{2,1} = 2/\pi_2.$$

Again, since the parametric equation of the centric circle of the three-line is

$$x = a_{3,2} - \tau a_{3,1} = 2(1 - \tau)/\pi_3,$$

we find

$$a_{3,2} = a_{3,1} = 2/\pi_3.$$

In a similar way, we get

$$a_{4,3} = a_{4,2} = 2/\pi_4.$$

Also, from (1), it follows that

$$a_{3,1} = a_{4,2} - t_4 a_{4,1}$$

and, solving for $a_{4,1}$, we have

$$a_{4,3} = a_{4,2} = a_{4,1} = 2/\pi_4.$$

Continuing in this manner, we finally obtain

$$a_{n,n-1} = a_{n,n-2} = \cdots = a_{n,1} = 2/\pi_n.$$

From (4) it now follows that $\gamma_n = 0$, which means that the centric focus of the $n$-line is the focus of the parabola; moreover, it is on the centric circle and, therefore, $\lambda_n = 1$. Thus we have the theorem:

THEOREM. *If $n$ straight lines are tangent to a parabola, then the segments joining the centric centers to the orthocenters of the $(n-1)$-lines, formed by omitting one of the lines, are such that the perpendiculars erected at their mid-points meet in a point* [4].

This point is the orthocenter of the $n$-line and, from (5), its coördinate is

$$h_n = 2(1 - s_n)/\pi_n.$$

**7. Tangents to a deltoid.** Given $n$ lines tangent to a deltoid. If the in-circle is taken as base-circle and if $\Omega$ is the symmetric image with respect to $O$ of an apse of the curve, then the line $\Delta_j$ is given by the equations

$$x - \bar{x}/t_j = t_j - 1/t_j^2$$

and

$$x_j = t_j - 1/t_j^2,$$

where $t_j$ is the mid-point of the segment cut off on the line by the deltoid.

By direct computation and using (1), we find

$$a_{2,1} = t_1 + t_2 + 1/t_1t_2 = s_2 + 1/\sigma_2,$$

$$a_{3,2} = t_1 + t_2 + t_3 = s_3, \qquad a_{3,1} = 1 - 1/t_1t_2t_3 = 1 - 1/\sigma_3,$$

$$a_{4,3} = s_4, \qquad a_{4,2} = 1, \qquad a_{4,1} = 1/\sigma_4.$$

Continuing in this manner, we are able to prove that, when $n \geqq 5$,

$$a_{n,n-1} = s_n, \qquad a_{n,n-2} = 1, \qquad a_{n,n-3} = 0.$$

These results combined with (5) lead to the theorem:

THEOREM. *Four or more tangents to a deltoid form an $n$-line, whose orthocenter coincides with the center $O$ of the deltoid.*

#### References

1. F. R. J. Hervey, Educational Times, Reprint, 1891, p. 37.
2. V. Thébault, Bulletin de la Société royale des Sciences de Liége, 1944, p. 293.
3. S. Kantor, Bulletin des Sciences mathématiques et astronomiques, 1870, p. 138.
4. J. R. Musselman, On four lines and their associated parabola, this MONTHLY, vol. 44, 1937, p. 513.
5. Frank Morley, Metric geometry of the plane $n$-line, Transactions of the American Mathematical Society, vol. 1, 1900, p. 97.
6. Frank Morley and F. V. Morley, Inversive Geometry, p. 160.
7. G. de Longchamps, Nouvelles correspondances de mathématiques, vol. 3. 1887, pp. 306–312, 340–347.

# MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

## SOME REMARKS ON EULER'S TOTIENT

V. L. KLEE, JR., University of Virginia

Ratat [1] and Goormaghtigh [2] have recorded values of $n$ for which $\phi(n)=\phi(n+1)$. A list of all such values of $n<3000$, found from Glaisher's table [3] follows.

$\phi(n)=\phi(n+1)$ for $n=1$, 3, 15, 104, 164, 194, 255, 495, 584, 975, 2204, 2625, and 2834.

The last five entries are new. It is noteworthy that for all the stated values except 1 and 3, whichever of $n$ and $n+1$ is odd is divisible by 15.

Ratat [1] also noted that for $n<125$, $\phi(2n\pm 1)\geqq\phi(2n)$. We list here all values of $n<1500$ for which this inequality fails to hold:

$\phi(2n-1)<\phi(2n)$ for $n=263$, 293, 368, 578, 683, 743, 788, 878, 893, 908, 998, 1073, 1103, 1208, 1238, 1268, 1403, and 1418.

$\phi(2n+1)<\phi(2n)$ for $n=157$, 262, 367, 412, 577, 682, 787, 877, 892, 907, 997, 1072, 1207, 1237, 1312, and 1402.

Now if $p$ is an odd prime and $2p-1$ is prime, then $\phi(n)=\phi(n+2)$ for $n=2(2p-1)$. There are also the following values of $n<3000$ for which $\phi(n)=\phi(n+2)$ although $n$ and $n+2$ are not related in the manner just indicated: 4, 7, 8, 32, 70, 308, 512, 572, 635, 728, 2170, and 2695.

Finally, we note that if $n<3000$ and $\phi(n)+2=\phi(n+2)$, then either $n$ and $n+2$ are prime or $n$ has the form $4p$, where $p$ and $2p+1$ are prime.

### References

1. R. Ratat, L'intermedaire des math., 24 (1917) 101–102.
2. R. Goormaghtigh, ibid., 25 (1918) 42–44.
3. J. W. L. Glaisher, Number-Divisor Tables, Cambridge, 1940.

## GAUSSIAN MEANS

WILLIAM GUSTIN, University of California, Los Angeles

Let

$$A = (a_1, \cdots, a_n)$$

be a sequence of $n$ positive real numbers, and let

$$Q = (q_1, \cdots, q_n)$$

be another sequence of $n$ positive real numbers, called weights, such that

$$\sum_{i=1}^{n} q_i = 1.$$

For any real number $t$ the mean of order $t$ of the sequence $A$ weighted by the sequence $Q$ is defined as

$$\mathfrak{M}(t, A, Q) = \phi_t^{-1}\left[\sum_{i=1}^{n} q_i\phi_t(a_i)\right],$$

where the function $\phi_t$ with inverse $\phi_t^{-1}$ is given by

$$\phi_t(a) = \begin{cases} \log a, & t = 0, \\ a^t, & t \neq 0, \end{cases}$$

for every positive real argument $a$. It is known that $\mathfrak{M}(t, A, Q)$ is a nondecreasing continuous function of $t$ whose g.l.b. is min $A$ and whose l.u.b. is max $A$ [1].

We now generalize this mean of order $t$ to a mean of order $T$, where

$$T = (t_1, \cdots, t_n)$$

is any sequence of $n$ real numbers. Let the sequences

$$\overset{k}{A} = (\overset{k}{a_1}, \cdots, \overset{k}{a_n}), \qquad (k = 0, 1, 2, \cdots),$$

be defined recursively as follows. Put

$$A^0 = A.$$

Then determine the elements of the sequence $A^{k+1}$ from the sequence $A^k$ by the formula

$$\overset{k+1}{a_i} = \mathfrak{M}(t_i, \overset{k}{A}, Q), \qquad (i = 1, \cdots, n;\ k \geqq 0).$$

We shall prove that for a fixed index $i$ the infinite sequence

$$A_i = (\overset{1}{a_i}, \overset{2}{a_i}, \cdots)$$

has a limit independent of $i$. We call this common limit the mean of order $T$ of the sequence $A$ weighted by $Q$, and denote it by

$$\mathfrak{M}(T, A, Q).$$

It is our purpose here merely to establish the existence of this mean and not to investigate its properties.

The particular mean

$$\mathfrak{M}[(0, 1), (a_1, a_2), (\tfrac{1}{2}, \tfrac{1}{2})]$$

is known as the arithmetico-geometric mean of $a_1$ and $a_2$; it was first investigated by Gauss in connection with a problem in potential theory [2]. The special methods which have been employed to prove the convergence of the infinite sequences $A_1$ and $A_2$ to the arithmetico-geometric mean of $a_1$ and $a_2$ do not carry over to the more general mean discussed here.

For convenience we permute the indices of each of the sequences $T$, $A$, $Q$

so that $T$ is nondecreasing. Thus, in particular,

$$t_1 \leqq t_i \leqq t_n.$$

Since the mean of order $t$ is a nondecreasing function of $t$, we see that

$$\overset{k}{a_1} \leqq \overset{k}{a_i} \leqq \overset{k}{a_n}, \qquad (k \geqq 1). \tag{1}$$

Therefore, the sequence $A^k$ is bounded below by $\overset{k}{a_1}$ and is bounded above by $\overset{k}{a_n}$, so that

$$\overset{k}{a_1} \leqq \mathfrak{M}(t_1, \overset{k}{A}, Q) = \overset{k+1}{a_1} \leqq \overset{k+1}{a_n} = \mathfrak{M}(t_n, \overset{k}{A}, Q) \leqq \overset{k}{a_n}, \qquad (k \geqq 1).$$

The bounded nondecreasing infinite sequence $A_1$ then has a limit $\alpha_1$, and similarly the bounded nonincreasing infinite sequence $A_n$ has a limit $\alpha_n$. Taking inferior and superior limits of (1), we have

$$\alpha_1 = \underline{\lim}_{k\to\infty} \overset{k}{a_1} \leqq \underline{\lim}_{k\to\infty} \overset{k}{a_i} \leqq \overline{\lim}_{k\to\infty} \overset{k}{a_i} \leqq \overline{\lim}_{k\to\infty} \overset{k}{a_n} = \alpha_n. \tag{2}$$

From (2) and the fact that the function $\phi_{t_1}(a)$, written below as $\phi(a)$, is a continuous strictly increasing function of $a$ provided $t_1 \geqq 0$, we obtain the following extended inequality:

$$\begin{aligned}
\phi(\alpha_1) &= \phi(\underline{\lim}_{k\to\infty} \overset{k+1}{a_1}) \\
&= \underline{\lim}_{k\to\infty} \phi(\overset{k+1}{a_1}) \\
&= \underline{\lim}_{k\to\infty} \sum_{i=1}^{n} q_i\phi(\overset{k}{a_i}) \\
&\geqq \sum_{i=1}^{n} q_i\phi(\underline{\lim}_{k\to\infty} \overset{k}{a_i}) \\
&\geqq \sum_{i=1}^{n-1} q_i\phi(\alpha_1) + q_n\phi(\alpha_n) \\
&= \sum_{i=1}^{n} q_i\phi(\alpha_1) + q_n[\phi(\alpha_n) - \phi(\alpha_1)] \\
&= \phi(\alpha_1) + q_n[\phi(\alpha_n) - \phi(\alpha_1)] \\
&\geqq \phi(\alpha_1).
\end{aligned}$$

Subtracting $\phi(\alpha_1)$ from this inequality we get

$$0 \geqq q_n[\phi(\alpha_n) - \phi(\alpha_1)] \geqq 0,$$

whence $\alpha_1 = \alpha_n$, if $t_1 \geqq 0$. If $t_1 < 0$, the function $\phi(a)$ is a continuous strictly decreasing function of $a$, and a similar argument, in which we use superior limits and reverse the inequality signs, also shows that $\alpha_1 = \alpha_n$. Finally we see from (2) that this common limit $\alpha = \alpha_1 = \alpha_n$ is the limit of each of the infinite sequences

$A_i$, so that

$$\alpha = \mathfrak{M}(T, A, Q).$$

**References**


1. Hardy, Littlewood, Pólya, Inequalities, Cambridge, England, 1934, Chapter II.
2. O. D. Kellogg, Foundations of Potential Theory, Berlin, 1929, pp. 60–62.


# CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics.*

## OBJECTIVES IN CALCULUS

C. C. MACDUFFEE, University of Wisconsin

What should be the objectives in a beginning course in the calculus? That is a question which many college teachers ask themselves, and to which it is difficult to frame an answer. Calculus is the course for which the student has long been preparing through college algebra, trigonometry and analytics, and for many a student it is the last mathematics course which he will ever take. The amount of interesting and valuable material which is at that point open to him is large and beyond the capacity of the time available for its complete presentation. What gems shall be presented and which omitted is a problem which we all have to face.

Our problem is complicated by the fact that no two students have exactly the same backgrounds, interests, personalities or plans for the future. Education should be a very personal matter between Professor A and Student X. The ideal college would give to each student a tailor-made course fitted to his exact needs and capacities, allowing him to proceed as rapidly and deeply as his abilities allow. But such methods under competent teachers are too expensive for any modern college so that the student must be fitted as best he may with a ready-made suit of clothes. The problem is to design the suit so that it will fit the largest number of students reasonably well.

The first objective in a course in calculus has to be the basic techniques of differentiation and integration. Just as the fundamentals of spelling and grammar have to be learned before one can compose literature, so these techniques have to be acquired before one can use the calculus. Any student who can qualify to enter a class in calculus can with patience master these techniques. The pity of it is that so many students get not much else for their labors.

Beyond the fundamental techniques, there seems to be a difference of opinion as to the best procedure. There seems to be one school of thought which would have the student solve large numbers of problems in mechanics and physics without much attempt at incisive reasoning or rigor, apparently with the idea that after enough experience the student's subconscious mind will take over and set up the problem for him. This is known as "standard engineering practice."

There is another school which tends to minimize the applications and to teach the calculus as a pure and lofty discipline of the mind. Most of the class time is spent on real variable theory, and a few students may even learn to throw $\epsilon$'s and $\delta$'s around with what the teacher believes (until the day of the final examination) to be a fair degree of intelligence. This method is in great favor with young Ph.D.'s. It is not in favor among deans.

The ideal method obviously lies in neither of these directions. The mathematician cannot afford to forget that the calculus was developed for the purpose of solving problems in mechanics and physics, and that its greatest glory even now is in connection with the applications. The engineer too must remember that Newton and Laplace and the Bernoullis were deep and incisive thinkers whose intuitions were merely the manifestations of careful and rigorous thinking.

If we grant that the education of a scientist consists in the development of the power to do, we must admit that our proper goal in the calculus is to develop the student's ability to interpret the physical world in mathematical terminology. This presupposes of course that he shall be able to speak the language of mathematics, that he shall have a command of the techniques and also of the theory behind these techniques. But over and beyond this fluency, he must have achieved an intuitive feeling for the elementary concepts of mechanics and physics, and his thinking along these lines must be in the language of mathematics. This is a large order for a first course in calculus, and obviously incapable of complete achievement. But I believe it is a measuring rod upon which the success of such a course must be judged.

There is a pedagogical sequence for the presentation of ideas which seems to be inherent in the human animal, and which seems to be quite unrelated to what we consider the logical order. Successful teaching respects this pedagogical order. Thus the modern foundations of the calculus, which make the concept of limit a purely static concept without appeal to the notions of time or motion, is magnificent. Every graduate student should be required to master it. But it should not be the calculus teacher's one and only god.

One of the weaknesses of American universities is their intense departmentalization. Nature does not recognize the fine distinctions between what belongs to mathematics, what to physics, and what to chemistry. Neither did the great universities of Europe to the extent that we do. We sometimes teach calculus with no applications to physics, we frequently try to teach physics without using mathematics. What is perhaps even more demoralizing to the students, our mathematics teachers sometimes demonstrate an incompetence in physics and vice versa.

There have been attempts to coördinate the teaching of calculus and mechanics to the extent of preparing textbooks in the two courses which can be used in the two courses simultaneously, the calculus being available by the time it is needed in mechanics, and the problems in mechanics ready to illustrate the theorems of the calculus. Perhaps by the time our elective system has been modified so that the average student does not have an irregular program, this scheme will be more widely tried.

In these days of educational experimentation, various combinations of courses are being tried which have never been tried before. What could be more natural than a combination course of basic physics and calculus? This course would probably have to be spread over two years if it were to contain a complete course in both physics and calculus. It would have to be given by a man who is competent in and sympathetic toward both courses. He could not be a physicist who teaches a little mathematics as a "tool," nor a mathematician who "runs in a few illustrations from physics." But he would be able to develop physicists to whom mathematics is a mother tongue. Can you think of a better background for scientists of the present age?

Regardless of the framework in which it is taught, the first course in calculus must be handled with a fine sense of balance. It should be rigorous up to the capacity of the student to appreciate rigor, and this rigorous treatment must be extended to the problems, not merely confined to the proof of the existence of the definite integral. But the fundamental and basic problem is to develop the student's intuitions so that mathematics is to him a spoken language. Then and only then is he in a position to appreciate the meaning of rigor. For is rigor anything else than clarity?

## FORMULA FOR THE AREA OF A TRIANGLE

M. K. FORT, JR., University of Virginia

We prove in Theorem 1 that a certain determinant is an invariant. Theorem 1 is then used to prove the well known formula

$$A = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

for the area of a triangle.

THEOREM 1. *If $P_1$, $P_2$, $P_3$ are points in a plane, and these points have coördinates $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ and $(x_1', y_1')$, $(x_2', y_2')$, $(x_3', y_3')$ respectively in rectangular coördinate systems $C$ and $C'$ (which we shall assume to have the orientation commonly used in analytic geometry texts); then*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

The transformation from the $C'$ system to the $C$ system is given by equations of the form

$$x = mx' - ny' + h$$
$$y = nx' + my' + k$$

where $m^2+n^2=1$. Therefore

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} mx_1' - ny_1' + h & nx_1' + my_1' + k & 1 \\ mx_2' - ny_2' + h & nx_2' + my_2' + k & 1 \\ mx_3' - ny_3' + h & nx_3' + my_3' + k & 1 \end{vmatrix}.$$

The determinant on the right side of the above equation can be simplified by subtracting the proper multiples of the last column from the first two columns. If we do this we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} mx_1' - ny_1' & nx_1' + my_1' & 1 \\ mx_2' - ny_2' & nx_2' + my_2' & 1 \\ mx_3' - ny_3' & nx_3' + my_3' & 1 \end{vmatrix}.$$

The determinant on the right is equal to

$$(m^2 + n^2) \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

Since $m^2+n^2=1$, we see that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

THEOREM 2. *If $P_1$, $P_2$, $P_3$ are the vertices of a triangle and the cyclic order $P_1P_2P_3P_1$ induces a counter-clockwise orientation on the boundary of the triangle, then the area $A$ of the triangle is given by*

$$A = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Choose a coördinate system $C'$ so that $P_1$ is at the origin and $P_2$ is on the positive $x'$-axis. It follows from the fact that $P_1P_2P_3P_1$ induces a counter-clockwise orientation, that $P_3$ must be in either the first or second quadrant. For this choice of $C'$ we now see that $x_1' = y_1' = y_2' = 0$, that $x_2'$ is the length of the side $P_1P_2$, and that $y_3'$ is the length of the altitude perpendicular to this side. Thus the area

of the triangle satisfies $2A = x_2'y_3'$. We now apply Theorem 1 and obtain

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x_2' & 0 & 1 \\ x_3' & y_3' & 1 \end{vmatrix} = x_2' y_3' = 2A.$$

Therefore

$$A = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In a similar fashion we can prove:

THEOREM 3. *If $P_1$, $P_2$, $P_3$ are the vertices of a triangle and the cyclic order $P_1P_2P_3P_1$ induces a clockwise orientation on the boundary of the triangle, then the area $A$ of the triangle is given by*

$$A = -1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three morths after publication of problems.*

### PROBLEMS FOR SOLUTION

E 776. *Proposed by L. R. Ford, Illinois Institute of Technology*

"Are those your children that I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on; "they prefer baseball but there are not enough children to make two teams. Curiously enough," he mused, "the product of the numbers in the four groups is my house number, which you saw when you came in."

"I am something of a mathematician," said the visitor, "let me see whether I can find the numbers of children in the various families." After figuring for a

time he said, "I need more information. Does your cousin's family consist of a single child?" The host answered his question, whereupon the visitor said, "Knowing your house number and knowing the answer to my question, I can now deduce the exact number of children in each family."

How many children were there in each of the four families?

E 777. *Proposed by C. R. Perisho, McCook Junior College*

Find the number of permutations of $n$ objects with the restriction that in no arrangement may an object be adjacent to either of its neighbors in the original order.

E 778. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $ABCD$, find the point $P$ in space such that the shortest paths separating $P$ from each of the vertices $A$, $B$, $C$, $D$, after having touched the opposite faces $BCD$, $CDA$, $DAB$, $ABC$, are equal to each other.

E 779. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Let $x$, $y$, $z$ be three positive integers and set

$$a = x + y, \qquad b = x + z, \qquad c = x + y + z.$$

Show that for any prime exponent $p > 2$,

$$(ab)^p - (cx)^p - (yz)^p$$

is divisible by the product $pabcxyz$.

E 780. *Proposed by G. Pólya, Stanford University*

A lampshade has the shape of a frustum of a right circular cone. Its perimeter is $P$ at the bottom, $p$ at the top, and its slant height is $s$. Show that such a lampshade can be cut out in one piece from a rectangular sheet of paper with dimensions

$$P \quad \text{and} \quad s + p(P - p)/8s.$$

You can even save paper for a flap to glue the two ends together, except in the limiting case where $P = p$, when not a bit of paper is wasted.

## SOLUTIONS

### Bounds for Finite Harmonic Series

E 746 [1946, 591]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that for all positive integers $r$

$$r\{1 - (n + 1)^{-1/r}\} < 1/1 + 1/2 + \cdots + 1/n \leqq r(n^{1/r} - 1) + 1.$$

I. *Solution by J. H. Simester, University of Louisville.* We shall use the function

$$f(n) = 1 + 1/2 + \cdots + 1/n - \ln(1 + n),$$

(whose limit when $n\to\infty$ is Euler's constant). A property of $f(n)$ is that

$$0 < f(n) < 1 - 1/(1+n),$$

from which it follows that, for $n>1$,

$$f(n-1) + 1/n < 1 - 1/n + 1/n = 1.$$

Now, from

$$\int_1^{1+n} t^{-1-1/r}dt < \int_1^{1+n} t^{-1}dt = \ln(1+n),$$

it follows that

$$r\{1-(1+n)^{-1/r}\} < 1 + 1/2 + \cdots + 1/n \tag{1}$$

for any positive integral values of $r$ and $n$. Also, for $n>1$,

$$\begin{aligned} 1 + 1/2 + \cdots + 1/n &= \ln n + \{f(n-1) + 1/n\} < \ln n + 1 \\ &= \int_1^n t^{-1}dt + 1 < \int_1^n t^{-1+1/r}dt + 1 = r(n^{1/r} - 1) + 1, \end{aligned}$$

whence, for all positive integers $r$ and $n$

$$1 + 1/2 + \cdots + 1/n \leqq r(n^{1/r} - 1) + 1. \tag{2}$$

Results (1) and (2) are the desired inequalities.

II. *Solution by the Proposer.* The identity

$$\frac{(m+1) - m}{(m+1)^{1/r} - m^{1/r}} = \sum_{i=1}^{r} (m+1)^{1-i/r} m^{(i-1)/r}$$

gives

$$\frac{(m+1) - m}{(m+1)^{1/r} - m^{1/r}} \leqq r(m+1)^{1-1/r} \leqq r(m+1), \qquad m \geqq 0.$$

Similarly

$$\frac{\dfrac{1}{m} - \dfrac{1}{m+1}}{\left(\dfrac{1}{m}\right)^{1/r} - \left(\dfrac{1}{m+1}\right)^{1/r}} = \sum_{i=1}^{r} \left(\frac{1}{m}\right)^{1-i/r} \left(\frac{1}{m+1}\right)^{(i-1)/r}$$

gives

$$\frac{\dfrac{1}{m} - \dfrac{1}{m+1}}{\left(\dfrac{1}{m}\right)^{1/r} - \left(\dfrac{1}{m+1}\right)^{1/r}} > \frac{r}{(m+1)^{1-1/r}} > \frac{r}{m+1}, \qquad m > 0.$$

Hence, for $m \geqq 0$,

$$r\{(m+1)^{-1/r}-(m+2)^{-1/r}\} < 1/(m+1) \leqq r\{(m+1)^{1/r}-m^{1/r}\}.$$

Summation with respect to $m$ now leads to the required result.

III. *Solution by Barney Bissinger, Fitchburg, Massachusetts.* The left hand member of the inequalities is equal to the area under the curve $y=(x+1)^{-1-1/r}$, above the $x$-axis, and between the ordinates at $x=1$ and $x=n$. This area is obviously less than $\sum_1^n 1/i$.

Similarly, the right hand member of the inequalities is equal to 1 plus the area under the curve $y=x^{-1+1/r}$, above the $x$-axis, and between the ordinates at $x=1$ and $x=n$. This area is clearly greater than $1+\sum_2^n 1/i$; equality occurs only when $r=n=1$.

Also solved by Paul Brock, Ragnar Dybvik, and Norman Miller.

### Diophantine Vectors

E 747 [1946, 591]. *Proposed by H. W. Becker, Omaha, Nebraska*

Can we decompose an integer force $F$ into $n$ integer forces $F_1, \cdots, F_n$, such that the sum of any number of the components is also an integer force, where $n>2$?

*Editorial Note.* The answer is *yes*. For consider a tetrahedron $ABCD$ in which $AC=AD=BA=BC=BD=3$, and $CD=4$. Consider the three vectors $F_1\equiv\overrightarrow{AB}$, $F_2\equiv\overrightarrow{BC}$, $F_3\equiv\overrightarrow{CD}$. Then it is easily seen that $F_1+F_2+F_3$, $F_2+F_3$, $F_3+F_1$, $F_1+F_2$ all have integral magnitudes. Other solutions are of course possible.

Becker has also proposed the problem of determining the existence or non-existence of $n>2$ mutually orthogonal integer vectors such that the sum of any number of them will also be an integer vector. This problem seems much more difficult and possesses a considerable associated literature.

### Lateral Area of an Oblique Cone of Revolution

E 748 [1946, 591]. *Proposed by George Pólya, Stanford University*

Let $\alpha$ be the angle between the axis and any element of a right circular cone. If the cone is cut through by a plane (not necessarily perpendicular to the axis) show that the lateral area of the part remaining between the plane and the vertex is given by $\pi AG \sin\alpha$, where $A$ and $G$ are respectively the arithmetic and the geometric mean of the longest and the shortest remaining elements.

*Solution by G. K. Klausner, Cooper Union Institute of Technology.* The required area $S$ is easily shown to be $E \csc\alpha$, where $E$ is the projection of $S$ on a plane perpendicular to the axis of the cone. From geometry the semi-axes of the ellipse $E$ are found to be $\frac{1}{2}(a+b)\sin\alpha$ and $\sqrt{ab}\sin\alpha$, where $a$ and $b$ are the lengths of the longest and shortest elements. Since $A=\frac{1}{2}(a+b)$ and $G=\sqrt{ab}$, $E=\pi AG\sin^2\alpha$ and $S=\pi AG\sin\alpha$.

Also solved by F. Ballantine, D. H. Browne, H. E. Fettis, Free Jamison, D. W. Matlock, R. K. Morley, D. K. Pease, C. F. Pinzka, P. W. A. Raine, J. H. Simester, A. Sisk, P. D. Thomas, R. H. Urbano, and the proposer. Thomas noted that this problem occurs as example 13, p. 380 of Bowser's *Differential and Integral Calculus* (1910).

### A Cone Inscribed in a Sphere

E 749 [1946, 591]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a given sphere inscribe a right circular cone whose lateral area is equal to the area of the zone beneath its base. Show that the total area of the cone is equal to the area of the zone in which it is inscribed.

*Solution by the Proposer.* Let $S$ be the vertex of the cone, $SH=x$, its altitude, piercing the sphere again at $D$, and $SA$ any arbitrary element of the cone. Let $d$ be the diameter of the sphere. The lateral area of the cone is given by

$$\pi(HA)(SA) = \pi\sqrt{x(d-x)dx}$$

and that of the zone beneath the cone by

$$\pi(DA)^2 = \pi d(d-x).$$

The relation $\pi(HA)(SA) = \pi(DA)^2$ gives the equation

$$x\sqrt{d(d-x)} = d(d-x),$$

which reduces to

$$x^2 - d(d-x) = 0.$$

Without the need of solving, this last equation shows that *the point $H$ divides the diameter $SD$ of the sphere in mean and extreme ratio.* There is thus a classical geometrical construction of the point $H$, and consequently a construction of the required cone.

The second part of the problem follows from the relations

$$\pi(HA)(SA) + \pi(HA)^2 = \pi dx = \pi(SA)^2.$$

The solution of this problem implicitly contains the solutions of the following problems:

1. To inscribe in a given sphere a right circular cone such that the lateral area and the total area have a sum equal to the area of the sphere.

2. To cut a sphere by a plane such that the area of the section will be equal to the difference of the areas of the zones determined by the plane.

3. To cut a sphere by two parallel planes such that the area of each section will be equal to that of the zone intercepted by the planes.

4. The pole $P$, with respect to the sphere, of the base of the cone is such that $PH=2d$, and the distance $PD=d(\sqrt{5}+1)/2$ is equal to twice the side of a regu-

lar 3/10 star inscribed in a great circle of the given sphere.

Also solved by W. E. Buker, G. Y. Cherlin, Ragnar Dybvik, D. H. Erkiletian, Jr., R. B. Herrera, B. R. Leeds, D. K. Pease, C. F. Pinzka, P. W. A. Raine, C. D. Smith, and P. D. Thomas.

*Remarks by C. D. Smith, Mississippi State College.* The problem of finding the inscribed right circular cone with a volume equal to that of the spherical segment below leads to the cubic

$$2h^3 + Rh^2 - 4R^2h + R^3 = 0,$$

where $R$ is the radius of the sphere and $h$ is the distance from the center of the sphere to the base of the cone. Removing the undesired root $h=R$, we find $h=R(\sqrt{17}-3)/4$. The similar distance $h$ for the given problem is $R(\sqrt{5}-2)$. Using these values of $h$ to calculate the radii of the bases of the cones for the two problems we find that these radii differ from each other by only (approximately) $0.012R$.

More surprising is the case of a spheroid with axis of rotation $2R$. If we take the inscribed cone with altitude $R+h$ along the axis $2R$, and insist that the volume of the cone be equal to that of the segment beneath the base, we find the same cubic as for the sphere. This equation is independent of the eccentricity of the generating ellipse, and hence we have cones of equal altitudes for all spheroids having axis of rotation equal to $2R$.

*Editorial Note.* It might be apropos to mention here a curiosity sent some time ago to this department by T. A. Bickerstaff of the University of Mississippi: *The vertex angle of the maximum inscribed cone in a sphere equals a base angle of the minimum circumscribed cone about the sphere.* The proof is very straightforward.

**Interior Diagonal Points**

E 750 [1946, 591]. *Proposed by Paul Erdös, Syracuse University*

Find the number of intersections of the diagonals of a convex polygon of $n$ sides.

I. *Solution by Norbert Kaufman and R. H. Koch, Chicago, Illinois.* Consider a convex polygon of $n \geqq 4$ sides. Every combination of the $n$ vertices taken four at a time determines a quadrilateral which has two intersecting diagonals. Also, every two intersecting diagonals of the polygon determine a quadrilateral. Therefore the required number of intersections is $\binom{n}{4}$. These intersections, of course, may not all be distinct.

II. *Solution by Arthur Rosenthal, University of New Mexico.* If $k$ vertices lie on one side of a diagonal $d$, then $n-k-2$ vertices lie on the other side, and hence $d$ intersects $k(n-k-2)$ other diagonals. Taking into account that each diagonal is determined by two vertices and each intersection by two diagonals, one finds

that the number of intersections of the diagonals of a simple convex polygon of $n$ sides is

$$(n/4)\sum_{k=1}^{n-3} k(n-k-2) = \binom{n}{4},$$

not counting the $n$ vertices of the polygon.

A much more difficult problem is to find the number of distinct points of intersection of the diagonals of a simple regular polygon of $n$ sides.

Also solved by D. W. Alling, Fred Ballantyne, H. W. Becker, D. H. Browne, F. A. Butter, Jr., G. Y. Cherlin, Monte Dernham (two ways), Harley Flanders, William Gustin, J. B. Kelly, D. W. Matlock, W. E. Patten, C. F. Pinzka, and the proposer.

Several solvers also found the number of exterior diagonal points. Dernham pointed out the interesting fact that, as $n$ increases, the decreasing ratio of interior to exterior intersections,

$$(n-1)(n-2)/2(n-4)(n-5),$$

approaches the limit 1/2. He also suggested the problem of finding the *minimum* possible number of distinct interior diagonal points that may be possessed by a convex $n$-gon. Patten pointed out that the proposed problem occurs as ex. 7, p. 34, vol. II (2nd ed.) of Chrystal's *Text Book of Algebra*. As an allied problem he suggested proving that all interior diagonal points of a regular convex odd-sided polygon are simple intersections. As a still more difficult problem he proposed that of determining the existence or non-existence of $n$-gons possessing prescribed numbers of diagonal points of given multiplicity. If $I_j$ designates the number of interior diagonal points of multiplicity $j$, then $(I_2, I_3, \cdots, I_m)$ may be called the *index* of the $n$-gon. For $n=7$ Patten stated that $I_m=0$ for $m>3$, and that 7-gons exist for $(I_2, I_3)=(35, 0)$, $(32, 1)$, $(29, 2)$, and $(26, 3)$.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4254. *Proposed by Paul Erdös, Syracuse University*

We have seven points in the plane. Prove that we can always select three

which do not form an isosceles triangle. For six points this does not necessarily hold. (If $A$, $B$, $C$ are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

4255. *Proposed by G. Pólya, Stanford University*

A sequence $\{x_n\}$ is defined recursively, in terms of two numbers $x_0$ and $x_1$, by the formula

$$x_n = \frac{(n-1)g}{1+(n-1)g}x_{n-1} + \frac{1}{1+(n-1)g}x_{n-2},$$

where $g$ is a given positive quantity. Find an expression for the limit of $x_n$ as $n \to \infty$. (This generalizes problem E 694 (1945, 516) which corresponds to the special case $g=1$.)

4256. *Proposed by N. A. Court, University of Oklahoma*

Given a sphere orthogonal to two circles lying in two distinct planes. If the center of the sphere is conjugate, with respect to one of the circles, to the point in which the plane of that circle cuts the axis of the other circle, the same is true of the center of the sphere, if the roles of the two circles are interchanged.

Note. A circle is orthogonal to a sphere if the plane of the circle cuts the sphere along a great circle orthogonal to the given circle (see, for instance, the Proposer's *Modern Pure Solid Geometry*, p. 138, art. 416).

4257. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In any arithmetic progression show that the difference (sum) of the product of $n$ consecutive terms and the product of $n$ other consecutive terms is always divisible, provided $n$ is even (odd), by the sum of the greatest and least of the terms. Prove also the corollary: If $a$ and $b$ are positive integers and if $a+b+1=p$ is prime, then

$$a!b! \pm 1 \equiv 0 \pmod{a+b+1},$$

the sign being $+$ or $-$ according as $a$ and $b$ are even or odd.

4258. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the necessary and sufficient condition that four non-collinear points are such that each is the orthocenter of the other three, is

$$\pm\, 34\cdot 42\cdot 23 \pm 41\cdot 13\cdot 34 \pm 12\cdot 24\cdot 41 \pm 23\cdot 31\cdot 12 = 0,$$

where $rs$ denotes the distance between the $r$th and $s$th points, and three of the signs differ from the fourth.

## SOLUTIONS

NOTE. 4183 (1947–235) was proposed by P. M. Hummel, University of Alabama, rather than Cezar Coşniţa as stated in error.

**Binomial Coefficients**

4189 [1946, 103]. *Proposed by Albert Wilansky, Brown University*

Prove that

$$\sum_{r=0}^{m}(-1)^r\binom{m+k+\alpha}{r+k+\alpha}\binom{r+k}{k}=\binom{m+\alpha-1}{m}, \qquad \alpha \geqq 0.$$

*Solution by S. F. Lee, Yenching University, Peiping, China.* Recalling the expansion of $(1-x)^\mu$, we have

$$(1-x)^{m+k+\alpha}=\sum_{n=0}^{\infty}(-1)^n\binom{m+k+\alpha}{n}x^n=\sum_{n=0}^{\infty}(-1)^n\binom{m+k+\alpha}{m-n+k+\alpha}x^n,$$

$$(1-x)^{-k-1}=\sum_{n=0}^{\infty}(-1)^n\binom{-k-1}{n}x^n=\sum_{n=0}^{\infty}\binom{k+n}{k}x^n,$$

and their product

$$(1-x)^{m+\alpha-1}=\sum_{n=0}^{\infty}(-1)^n\binom{m+\alpha-1}{n}x^n.$$

Equating the coefficients of $x^m$, we obtain

$$\sum_{r=0}^{m}(-1)^{m-r}\binom{m+k+\alpha}{r+k+\alpha}\binom{r+k}{k}=(-1)^m\binom{m+\alpha-1}{m},$$

which is equivalent to the proposed result. We note that the restriction $\alpha \geqq 0$ has not been used in the proof.

Solved also by D. W. Alling, W. J. Combellack, H. S. Grant, J. R. Kinney, M. S. Knebelman, H. L. Krall, A. S. Peters, James Singer, N. Wyman, and the Proposer.

**Summation**

4191 [1946, 103]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1}{\cosh(\pi/2)}-\frac{1}{3\cosh(3\pi/2)}+\frac{1}{5\cosh(5\pi/2)}-\cdots=\frac{\pi}{8}.$$

*Solution by H. E. Fettis, Dayton, Ohio.* The Fourier series for $\sinh(bx)$, obtained by the standard procedure, is

$$\sinh(bx)=\frac{2\sinh(b\pi)}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n\sin(nx)}{b^2+n^2}.$$

Therefore

$$\frac{\pi}{2}\frac{\sinh(bx)}{\sinh(b\pi)} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n\sin(nx)}{b^2+n^2}.$$

Since the series is valid for $-\pi \leqq x \leqq \pi$, we may replace $x$ by $\pi - x$, and obtain

$$\frac{\pi}{2}\frac{\sinh(b\pi - bx)}{\sinh(b\pi)} = \sum_{n=1}^{\infty}\frac{n\sin(nx)}{b^2+n^2}.$$

Adding the two series, and noting that all terms with $n$ even cancel, we obtain the result

$$\frac{\pi}{4}\frac{\cosh\frac{1}{2}b(\pi-2x)}{\cosh\frac{1}{2}b\pi} = \sum_{n=0}^{\infty}\frac{(2n+1)\sin(2n+1)x}{b^2+(2n+1)^2},$$

and, in particular if $x = \pi/2$

$$\operatorname{sech}\tfrac{1}{2}b\pi = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{b^2+(2n+1)^2}.$$

The series under consideration may therefore be written

$$\begin{aligned}
\sum_{m=0}^{\infty}(-1)^m\frac{\operatorname{sech}\frac{1}{2}(2m+1)\pi}{(2m+1)}
&= \frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2n+1)}{(2m+1)[(2n+1)^2+(2m+1)^2]}\\
&= \frac{4}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{2n+1}\left[\frac{1}{2m+1}-\frac{2m+1}{(2n+1)^2+(2m+1)^2}\right]\\
&= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[\frac{\pi}{4}-\frac{\pi}{4}\operatorname{sech}\tfrac{1}{2}(2n+1)\pi\right]\\
&= \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2n+1}-\frac{1}{2n+1}\operatorname{sech}\tfrac{1}{2}(2n+1)\pi\right]\\
&= \frac{\pi}{4}-\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\operatorname{sech}\tfrac{1}{2}(2n+1)\pi.
\end{aligned}$$

Since the two summations are identical, the desired result follows immediately.

*Note by C. D. Olds, San Jose College, California.* The stated result follows immediately from a theorem given by Ramanujan in his famous letter to G. H. Hardy in 1913, namely:

$$\int_0^{\infty}\frac{\sin 2nx}{x(\cosh\pi x+\cos\pi x)}dx = \frac{\pi}{4}-2\left(\frac{e^{-n}\cos n}{\cosh\pi/2}-\frac{e^{-3n}\cos 3n}{3\cosh 3\pi/2}+\cdots\right).$$

The present theorem is the special case when $n=0$.

Ramanujan's theorem was first proved by C. T. Preece, using contour integration (*Journal London Math. Society*, vol. 3, 1928, p. 215).

Solved also by H. S. Grant, H. Y. Hsü and S. F. Lee (jointly), J. P. McCarthy, C. D. Olds, and the Proposer.

### Concurrent Lines in a Pentagon

4193 [1946, 160]. *Proposed by Hüseyin Demir, Columbia University*

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2}\|A_iA_{i+1}$, and $B_iA_{i+3}\|A_iA_{i+4}$, then the lines $A_iB_i$ concur in a point $C$.

*Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania.* Take the triangle $A_1A_2A_4$ as the triangle of reference for a system of homogeneous trilinear coördinates. Let $A_1$ be (1, 0, 0), $A_2$ (0, 1, 0,), $A_3$ $(d, e, f)$, $A_4$ (0, 0, 1), $A_5$ $(k, l, m)$.

Then the equations of the line through $A_3$ parallel to $A_1A_2$ and of the line through $A_4$ parallel to $A_1A_5$ are

$$afx + bfy - (ad + be)z = 0, \qquad alx + (bl + cm)y = 0.$$

Thus $B_1$ has the coördinates

$$(bl + cm)(ad + be), \qquad - al(ad + be), \qquad acfm;$$

and the equation of $A_1B_1$ is

$$cfmy + l(ad + be)z = 0.$$

In the same way the equations of $A_2B_2$ and $A_4B_4$ are found to be

$$cfmx + d(ak + bl)z = 0, \qquad l(ad + be)x - d(ak + bl)y = 0.$$

These three lines are easily seen to be concurrent in a point $C$ which has the coördinates

$$d(ak + bl), \qquad l(ad + be), \qquad - cfm.$$

Using the triangles $A_1A_3A_4$ and $A_2A_4A_5$ we can prove in the same way that $A_3B_3$ and $A_5B_5$ also pass through the point $C$.

Solved also by the Proposer.

*Editorial Note.* Clawson gave a second proof using the converse of Ceva's Theorem. The Proposer employed the pencils of lines $A_3B_3$, $A_4B_4$ formed when the side $A_2A_3$ rotates about $A_3$, other sides remaining fixed; since the correspondence between $A_3B_3$ and $A_4B_4$ is homographic, the locus of the intersection $C$ of the rays $A_3B_3$, $A_4B_4$ is a conic; this conic decomposes into $A_1B_1$ and $A_3A_4$, thus giving the proof.

### A Quadrilateral Similar to a Cyclic Quadrilateral

4194 [1946, 160]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

In each of the triangles formed by three of the vertices of a cyclic quadrilat-

eral, we consider the projection of the orthocenter on the circumdiameter parallel to the Simson line of the fourth vertex of the quadrilateral with respect to the triangle. The four projections form a quadrilateral inversely similar to the one given and are on a circle concentric to the circumcircle of that quadrilateral.

I. *Solution by J. H. Butchart and Richard Meyer, Arizona State College at Flagstaff.* Let the vertices of the cyclic quadrilateral be $A_1$, $A_2$, $A_3$, $A_4$ and the orthocenter of the triangle formed by the omission of $A_i$ be $H_i$. Let the projection of $H_i$ of the line through the circumcenter $O$ parallel to the Simson line of $A_i$ be $A_i'$. We wish to show (1) that $\measuredangle A_i'OA_j' = -\measuredangle A_iOA_j$, and (2) that $OA_1' = OA_2' = OA_3' = OA_4'$.

It is well known that the Simson line of $A_1$ is parallel to $A_2K$, where $K$ is the intersection with the circumcircle of the line through $A_1$ perpendicular to $A_3A_4$, and similarly the Simson line of $A_2$ is parallel to $A_1L$, where $A_2L$ is perpendicular to $A_3A_4$ and $L$ is on the circumcircle. Since $A_1K$ and $A_2L$ are parallel, the angle between $A_2K$ and $A_1L$ equals $\measuredangle A_1OA_2$. For the proof of (2), note that $OH_1$ is the vector sum of $OA_2$, $OA_3$, $OA_4$ and that $A_2K$ makes the same angles with these segments as $A_1L$ makes with $OA_1$, $OA_4$, $OA_3$. The projections $OA_1'$, $OA_2'$ of $OH_1$, $OH_2$ on parallels to $A_2K$, $A_1L$ through $O$ are therefore equal. It is clear that the arc $A_1A_2$ is opposite in direction to the arc $KL$ and thus that $\measuredangle A_1'OA_2' = -\measuredangle A_1OA_2$.

II. *Solution by Ou Li, Yenching University, Peiping, China.* In a system of complex coördinates having the given circumcircle as its base circle, the equation of the circumdiameter parallel to the Simson line of $A_1$ with respect to the triangle $A_2A_3A_4$ is

$$t_1x - t_2t_3t_4\bar{x} = 0,*$$

where $t_i$ are the coördinates of the vertices $A_i$ $(i=1, 2, 3, 4)$. Then the coördinates of the projection $H_1'$ of the orthocenter $H_1$ of the triangle $A_2A_3A_4$ on this line are easily found to be

$$x_1 = \tfrac{1}{2}\sigma_2\bar{t}_1 = \tfrac{1}{2}|\sigma_2|T\bar{t}_1$$

where $|T| = 1$ and $\sigma_2 = \sum t_it_j$. Analogous results evidently hold for the other three cases. Hence the four projections lie on a circle which is concentric to the base circle and whose radius is $\frac{1}{2}|\sigma_2|$, the radius of the base circle being taken as unity.

Furthermore, since

$$\frac{|x_1 - x_2|}{|t_1 - t_2|} = \frac{|\sigma_2|}{2},$$

the quadrilateral $H_i'$ is similar to the given quadrilateral $A_i$. As $x_it_i$ is a constant, the two quadrilaterals are situated in an inverse order. Also, from the relation

$$\frac{x_i - x_j}{\bar{x}_i - \bar{x}_j} = -t_st_k = \frac{t_s - t_k}{\bar{t}_s - \bar{t}_k} \qquad (i \neq j \neq s \neq k),$$

* R. Goormaghtigh, Analytic Treatment of Some Orthopole Theorems, this MONTHLY, vol. 46 1939 p. 266.

it follows that $H_i'H_j'$ is parallel to $A_iA_j$.

Solved also by J. W. Clawson and the Proposer.

**Upper Bound for an Integral**

4198 [1946, 225]. *Proposed by C. D. Olds, San Jose State College*

Prove that

$$\frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt < \left(\frac{2}{e}\right)^n,$$

where $t$ is real, $n$ is a positive integer, and $w(t)=(t-1)(t-2)\cdots(t-n+1)$.

*Solution by Philip Davis, Harvard University, and A. M. Peiser, Rutgers University.* Let

$$H_{n-1} = \frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt$$

and write $I_n=H_ne^{n+1}$. While a shorter proof of the stated inequality is possible, it seems desirable to establish the improved estimate

$$H_{n-1} < (e-1)^{-n}.$$

This is the best possible estimate. We shall show, in fact, that

(2) $$H_{n-1} \sim (e-1)^{-n}, \qquad n\to\infty.$$

A simple change of variable yields

$$I_n = \frac{1}{n!}\int_0^\infty (t+1)(t+2)\cdots(t+n)e^{-t}dt.$$

Now,

$$\begin{aligned} I_{n+1} - I_n &= \frac{1}{(n+1)!}\int_0^\infty t(t+1)\cdots(t+n)e^{-t}dt \\ &= K_n + e^{-1}I_{n+1}, \end{aligned}$$

where

(3) $$K_n = \frac{1}{(n+1)!}\int_0^1 t(t+1)\cdots(t+n)e^{-t}dt.$$

Thus

(4) $$(1-e^{-1})I_{n+1} - I_n = K_n$$

and, in particular,

$$(1-e^{-1})I_{n+1} > I_n,$$

so that $J_n=I_n(1-e^{-1})^{n+1}$ is a monotonic increasing sequence. Using (4), we have

$$J_{n+1} - J_n = K_n(1 - e^{-1})^{n+1},$$

so that

$$J_m - J_1 = \sum_{n=1}^{m-1} (J_{n+1} - J_n) = \sum_{n=1}^{m-1} K_n(1 - e^{-1})^{n+1}.$$

It follows from (3) that $K_n < 1$, $n = 1, 2, \cdots$, so that we may write

$$\lim_{n\to\infty} J_n = J_1 + \sum_{n=1}^{\infty} K_n(1 - e^{-1})^{n+1}. \tag{5}$$

Now for $t \geqq 0$,

$$e^t = [1 - (1 - e^{-1})]^{-t} = 1 + t(1 - e^{-1}) + \frac{t(t+1)(1 - e^{-1})^2}{2!} + \cdots,$$

or

$$1 = e^{-t} + te^{-t}(1 - e^{-1}) + \frac{t(t+1)e^{-t}(1 - e^{-1})^2}{2!} + \cdots.$$

Integrating over $0 \leqq t \leqq 1$, a clearly justified termwise integration yields

$$1 = \int_0^1 e^{-t}dt + (1 - e^{-1}) \int_0^1 te^{-t}dt + \sum_{n=1}^{\infty} K_n(1 - e^{-1})^{n+1},$$

and by (5)

$$\lim_{n\to\infty} J_n = 1 + J_1 - \int_0^1 e^{-t}dt - (1 - e^{-1}) \int_0^1 te^{-t}dt.$$

An easy computation shows that $J_1 = 2(1 - e^{-1})^2$, and that

$$\lim_{n\to\infty} J_n = 1.$$

This is the desired asymptotic result (2). The inequality (1) follows immediately from the monotonicity of $J_n$.

Also solved by D. W. Alling, Joshua Barlaz, P. T. Bateman and N. J. Fine, R. G. Blake, J. E. Brock and M. J. Gottlieb, Paul Brock, F. A. Butter, Jr., N. J. Fine, Fritz Herzog, P. M. Hummel, J. B. Kelly, S. F. Lee, A. S. Peters, M. W. Powell, J. H. Simester, J. G. Wendel, and the Proposer.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*Mathematics of Finance.* By F. S. Harper. Scranton, Pa., International Textbook Co., 1946. 9+327 pages. $3.25.

In the author's words, "It is the purpose of this book to present the subject so that anyone with a background of college algebra may obtain in a one semester, or better a two-semester, course a working knowledge of the fundamentals of the subject which will enable him to understand and apply them to a wide range of practical problems." The book attains this goal in an admirable fashion. It is well written, carefully edited, and quite readable. In the development of the theory and the solutions of problems, emphasis is placed on the use of line diagrams, the equation of value principle, and a clear understanding of the origin and meaning of the tabulated functions. Considerable space is devoted to solutions of illustrative examples in which convenient methods of computing are described. The use of logarithms is encouraged throughout.

The book differs in many respects from others on the same subject, the most striking difference being in the order of the topics. The table of contents will indicate the unusual order. I. Interest rates. II. Ordinary annuities. III. Discount rates. IV. More general annuities. V. Bonds. VI. Miscellaneous problems. VII. Life annuities. VIII. Life-insurance premiums. IX. Terminal reserves.

Compound interest is introduced at the outset, simple interest being introduced later as an approximation to compound interest. The discussion of simple discount is postponed to Chapter III where it is defined as a convenient approximation to compound discount. This order results in the desirable repetition of certain fundamental ideas, as well as the early introduction of ordinary annuities.

The general annuity is treated in a simple manner by modifying the payment period to make it coincide with the interest-conversion period. Amortization, sinking-funds, capitalized cost, and depreciation are discussed adequately in Chapter VI. The treatment of life annuities and life insurance is particularly good. A feature of the text is the use of the 1937 Standard Annuity Table as well as the American Experience Table. Over fifty sets of exercises furnish numerous problems for drill. "Since numerical answers rarely give any clue to the method of solution to be employed, each exercise and problem is followed conveniently by its answer."

Tables at the end of the book have the standard functions tabulated on the left-hand pages and the corresponding logarithms on the right. The compound interest functions are given to seven places of decimals, the logarithms to six places. The tables are printed by offset from typescript, detracting somewhat

from the appearance of the book.

This reviewer found practically no misprints, the only one of importance being in the table on page 209. Here the formulas are numbered incorrectly, and $x+n$ in the fourth line should read $x+m$. In problem 20, page 87, the discount rate $d$ is used before it is defined. If $i$ is the rate per coupon period, then $t$ should be replaced by $t/t'$ in formulas 37–38 on page 140. In order that the statement at the top of page 141 be correct, $P$ should be replaced by $P_0$ and the phrase "equation (37)" should be deleted. The author's definition of the "and accrued interest price" for bonds is not the customary definition, and no mention is made of the "professional practice" in connection with bonds. No distinction is made between a nominal rate *compounded* $p$ times a year and a nominal rate *payable* $p$ times a year.

To sum up, this book is a refreshing addition to the texts on the mathematics of finance. It should prove to be an effective teaching instrument.

H. D. LARSEN

## NEW BOOKS RECEIVED

*A First Course in Mathematical Statistics.* By C. E. Weatherburn. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 15+271 pages. $3.50.

*Essentials of College Algebra and Mathematics of Investment.* By W. L. Hart. Boston, D. C. Heath and Co., 1946. 10+304+126 pages. $4.75.

*A Locus with* 25920 *Linear Self-transformations.* (Cambridge Tracts in Mathematics and Mathematical Physics, No. 39.) By H. F. Baker. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 11+107 pages. $2.00.

*Mathematical Recreations and Essays.* By W. W. R. Ball. Revised by H. S. M. Coxeter. New York, The Macmillan Company, 1947. 16+418 pages. $2.95.

*Methods of Mathematical Physics.* By Harold Jeffreys and Bertha Jeffreys. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 9+679 pages. $15.00.

*The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coördinates.* By E. A. Maxwell. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 19+230 pages. $2.75.

*The Cambridge Four-figure Mathematical Tables.* New Edition. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 32 pages. $0.45.

*Plane Trigonometry.* Revised Edition. By W. K. Morrill. New York, Rinehart and Co., 1946. 10+245 pages. $2.50.

*The Theory of Functions of Real Variables.* By L. M. Graves. New York and London, McGraw-Hill Book Co., 1946. 10+300 pages. $4.00.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### KAPPA MU EPSILON CONVENTION

The sixth national convention of *Kappa Mu Epsilon* was held at Illinois State Normal University at Normal, Illinois during April 10–12, 1947. One hundred seventy-one members of the fraternity representing twenty-six chapters were registered at the meetings.

The general theme, as announced by the chairman of the convention, Miss Elinor B. Flagg of *Illinois Alpha*, the host chapter, was Keep Mathematics Effective. This theme was evident at council meetings as well as at general sessions where the emphasis was placed on greater participation in mathematics by undergraduates. The National Council approved plans whereby *The Pentagon*, the official publication of *Kappa Mu Epsilon*, would publish more student papers. To encourage the writing of papers by undergraduates, a prize is to be given to the student author of the best paper published in *The Pentagon* in each biennium. This prize is to be an all-expense trip to the next national convention.

A committee was appointed to investigate and compile reports on opportunities for mathematicians in the various industries and professions together with the type of training needed for each. These reports are to be published periodically in the official journal.

The advisability of dividing *Kappa Mu Epsilon* into districts with separate district officers to make the work of *Kappa Mu Epsilon* more effective was also discussed. Definite action was postponed until the present growth of the fraternity became stabilized.

The convention approved applications for charters of five new chapters. These are:

William Jewell College at Liberty, Missouri
Texas State College for Women at Denton, Texas
Texas Christian University at Fort Worth, Texas
Baldwin-Wallace College at Berea, Ohio
Mount Mary College at Milwaukee, Wisconsin.

These chapters will receive their charters before the end of the semester. Other colleges and universities have indicated their interest in the fraternity but further action on these was postponed until September, 1947.

The principal address was given by Professor C. V. Newsom of Oberlin College, Oberlin, Ohio, Past-President of *Kappa Mu Epsilon*. The title of his paper was *The Mathematical Method*. Other faculty members who addressed the convention were:

President R. W. Fairchild of Illinois State Normal University
Professor E. R. Sleight, *Michigan Alpha*, National President of *Kappa Mu*

*Epsilon* and presiding officer at all meetings

Professor H. Van Egen of *Iowa Alpha*, President-Elect of *Kappa Mu Epsilon*

Miss Ruth Yates, Illinois State Normal University

Professor E. H. Taylor, *Illinois Beta*

Miss E. Marie Hove, *New York Alpha*, National Secretary of *Kappa Mu Epsilon*

Professor H. D. Larsen, *New Mexico Alpha*, Editor of *The Pentagon*

Sister Helen Sullivan, *Kansas Gamma*, National Historian of *Kappa Mu Epsilon*

Professor L. F. Ollmann, *New York Alpha*, National Treasurer of *Kappa Mu Epsilon*

Professor C. N. Mills of *Illinois Alpha*.

The following undergraduate papers were presented and will be published in *The Pentagon*:

*Pattern forms of divisibility* by Robert J. Weeks of *Illinois Alpha*

*Computation of firing data for field artillery* by Thomas Selby of *Michigan Beta*

*A plea for non-isolationism in mathematics* by Victoria Fritton, of *Kansas Gamma*

*Mathematics in Scotland from* 1717–1838 by Shirley Searls of *Michigan Alpha* and read by Audrey R. Schuett of *Michigan Alpha*.

Specially composed *Kappa Mu Epsilon* songs were sung by Victoria Fritton of *Kansas Gamma*, Miss B. Rohr of *New York Alpha* and Doris Wyatt of *Texas Beta*.

The officers elected for the biennium 1947–49 are:

President, Professor H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa.

Vice-President, Professor H. R. Mathias, Bowling Green State University, Bowling Green, Ohio.

Secretary, Miss E. Marie Hove, Hofstra College, Hempstead, New York.

Treasurer, Professor L. F. Ollmann, Hofstra College, Hempstead, New York.

Historian, Professor C. C. Richtmeyer, Central Michigan College of Education, Mount Pleasant, Michigan.

Immediate Past-president, Professor E. R. Sleight, Albion College, Albion, Michigan.

Editor of *The Pentagon*, Professor H. D. Larsen, University of New Mexico, Albuquerque, New Mexico.

### Metropolitan Intercollegiate Mathematics Convention

The second annual Metropolitan Intercollegiate Mathematics Convention was sponsored by the *Pi Mu Epsilon* chapters of Brooklyn College, Hunter College, and New York University, on April 19, 1947 and held at Brooklyn College, Brooklyn, New York. The program included the following addresses:

Introductory address by George Shapiro of Brooklyn College

*Topics in Topology* by Professor Paul A. Smith of Columbia University

*Applications of Probability* by Edward C. Molina of Newark College of Engineering.

The committee members in charge of the convention were: George Shapiro, Brooklyn College; Gerard Washnitzer, Brooklyn College; Lillian Kaleko, Brooklyn College, Secretary; Cecile Salwen, Hunter College; Annette Drucker, Hunter College; Joyce Marrits, Hunter College; Elaine Margolin, New York University; and Bernice Goldberg, New York University.

## CLUB REPORTS

### Mathematics Club, University of Richmond

The first meeting of the current session was devoted to the initiation and orientation of new members. Subsequently, two meetings were held at which the topics listed below were presented. Refreshments were served at each meeting.

*An introduction to the theory of relativity* by Dr. B. C. Holtzclaw, Professor of Philosophy, University of Richmond

*Additions to the periodic table* by Dr. W. E. Trout, Jr., Professor of Chemistry, University of Richmond.

### Kappa Mu Epsilon, Wayne University

The Wayne University *Mathematics Club* was formally installed as the *Michigan Gamma* chapter of *Kappa Mu Epsilon* on May 10, 1946. The ceremony was conducted in the Rackham Building while the installation banquet, honoring the twenty-seven charter members, was held at the Book-Cadillac Hotel. Professor E. R. Sleight of Albion College, National President of *Kappa Mu Epsilon*, officiated at the installation of the new chapter and gave the banquet address.

Topics discussed during subsequent meetings include:

*Two mathematical puzzles*, by Ted Slaby

*Comprehension and retention*, by Nadine Zelinek

*Curves of constant breadth*, by Professor John W. Baldwin.

The officers include: President, Ted Slaby; Corresponding Secretary, Professor D. C. Morrow.

### Kappa Mu Epsilon, Illinois State Normal University

*Illinois Alpha* chapter of *Kappa Mu Epsilon* held eleven meetings during 1945–46, of which two were special meetings. Six new members were added to give a total membership of thirty-one for the year. Some of the activities were: a news letter was sent to all *Illinois Alpha* alumni, a home-coming breakfast for the alumni, and a Spring picnic.

At the annual Spring banquet, at which forty-three members were present, Miss Eunice Blackburn, missionary in Yucatan, spoke on *Education in Yucatan*.

Other outstanding programs include:

*Exhibit and discussion of old mathematics books* by Dr. C. N. Mills

*Significant figures* by Dr. C. T. McCormick

*Introduction to the mathematics of factor analysis* by Dr. B. R. Ullsvick.

Officers for 1945–46 were: President, Maxine Sponsler; Vice-President, Mary Donnell; Secretary, Martha Lewis; Treasurer, LaVerne Wenzelman; Historian, Janice Posey; Social Director, Alice McCorkle; Corresponding Secretary, Dr. C. N. Mills; Sponsor, Miss Elinor B. Flagg.

## NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.*

### SUMMER COURSES

*The University of Notre Dame* announces the following advanced courses for its summer session from June 18 to August 13: by Professor Ross, introduction to modern algebra; by Professor Vandiver (University of Texas), advanced topics in algebra and number theory; by an instructor yet to be announced, introduction to topology.

*Syracuse University* announces the following advanced courses from June 16 to September 7: by Professor Bers, differential geometry; by Dr. Bruns, vector analysis; by Dr. Dubisch, higher algebra; by Professor Gelbart, functions of a real variable; by Dr. Gilbert, introduction to topology; by Professor Harwood, differential equations; by Professor Loewner, fluid dynamics; and by Professor Welch, probability. From July 7 to August 16 the following courses will be offered: by Dr. Bernstein, history of mathematics; by Professor Cairns, fundamentals of analysis; and by Professor Stokes, introduction to modern mathematics.

### DUKE UNIVERSITY INSTITUTE FOR TEACHERS

The Duke University Institute for Teachers of Mathematics will be held from August 5 to August 15. This conference, under the direction of W. W. Rankin, is the seventh annual session of the Mathematics Institute for Teachers. In the past six summers more than 450 teachers from twenty states have attended the Institute. The theme of this year's Institute will be "Mathematics at Work."

The purpose of the Institute is to bring together high school and college teachers of mathematics, to study intensively the problems of common interest and to learn new uses of mathematics. The work of the Institute centers around the Mathematics Laboratory located in the West Duke Building on the East Campus. The Mathematics Laboratory makes available in one place a wide range of materials relating mathematics to science, industry, engineering, education, and commerce.

There will be a registration fee for the Institute of $3.00. This fee includes admission to the recreation facilities, tennis, swimming, *etc.*, of the Duke University Summer Session. Room and board will be furnished by Duke University at the rate of $3.00 for a double room and $3.50 per day for a single room. Room reservations should be made early. No fee or deposit need be made before arriving. Further information may be obtained by writing to Professor Rankin.

## AID TO DEVASTATED LIBRARIES

The Committee on Aid to Devastated Libraries has prepared for shipment abroad a number of packages containing periodicals and a few books. These packages are being sent to libraries in Belgium, China, Czechoslovakia, Finland, Germany, Greece, Hungary, India, Italy, and The Netherlands, through the assistance of the American Book Center. The Committee wishes to thank all those members of the Association who have made the sending of these packages possible by their response to the appeal which was published in the MONTHLY for October, 1946. It also wishes to express its indebtedness to the American Book Center (Library of Congress, Washington 25, D. C.), which continues throughout the present year the important activity of collecting and forwarding all material that can be used in the intellectual reconstruction abroad.

The amount and the character of the material that the Committee has received thus far has fallen far short, however, of what is needed to meet the needs that have been made known. There are needed many books published during the war years, as well as reprints of important papers and journals. That is the reason for the present appeal. Help can be given as follows:
(1) By contributions of money to be sent to the New York office of the Society. Checks should be made out to the American Mathematical Society (with an indication FOR FOREIGN LIBRARIES). Money so contributed will be used solely for the purchase of books and journals not published by either the Society or the Association, and not available to the Committee from other sources. The cost of preparing the material for shipment, and of sending it abroad is defrayed by a small appropriation made for this purpose by the Trustees of the Society and by the cooperation of the American Book Center.
(2) By sending books and journals, in prepaid packages, to the American Mathematical Society, Butler Library, Columbia University, 531 West 116th Street, New York 27, N. Y., with the notation FOR FOREIGN LIBRARIES. In order to give an idea of the character of the books that are needed, an indication is given below of the specific requests that have been made. These requests have come from libraries of universities and scientific societies in Austria, Belgium, China, Czechoslovakia, England, Finland, France, Germany, Greece, Hungary, India, Italy, Japan, the Netherlands, Poland, Roumania, Yugoslavia.

There are several requests for journals published during the war years. There is particular need for the *Annals*, the *Duke Journal*, the *Journal of Mathematics and Physics*, as well as for the various journals published or supported by the

Society. Among the books asked for, the various numbers of the *Colloquium Series*, and the two numbers of *Mathematical Surveys* appear repeatedly; so do the volumes in the *Princeton Mathematical Series* and the *Annals of Mathematics Studies*. The following individual titles are taken from long lists of requests:

Albert, *Introduction to Algebraic Theories*,
Artin, *Galois Theory*
Bennett and Bayliss, *Formal Logic*
Birkhoff and MacLane, *Survey of Modern Algebra*
Churchill, *Fourier Series*
Davis, *Principles of Econometrics*
Dickson, *History of the Theory of Numbers*
Eisenhart, *Non-Riemannian Geometry*
Eisenhart, *Differential Geometry*
James, *Mathematics Dictionary*
Lane, *Treatise on Projective Differential Geometry*
Sternberg-Smith, *Theory of the Potential*
Tarski, *Introduction to Logic*
Taylor, *Vector Analysis*
Uspensky-Heaslet, *Elementary Number Theory*
Veblen and Young, *Projective Geometry*
Watson, *Bessel Functions*
Widder, *The Laplace Transform*
Wilks, *Mathematical Statistics*

LEO ZIPPIN
PAUL A. SMITH
ARNOLD DRESDEN, *Chairman*

**PERSONAL ITEMS**

Professor H. J. Ettlinger of the University of Texas represented the Mathematical Association at the installation of William V. Houston as President of Rice Institute on April 10, 1947.

Professor J. W. Alexander of the Institute for Advanced Study has been awarded an honorary doctorate of science by Princeton University.

Dr. A. S. Householder of the Monsanto Chemical Company has received a Bureau of Ordnance Development award for work with project N-111; Applied Psychology Panel.

Professor J. R. Kline of the University of Pennsylvania has been elected to membership in the Society of Science and Letters at Warsaw.

K. A. Bush of Mohawk College has been promoted to an associate professorship.

F. M. Carpenter of the State University of Iowa has been appointed to an assistant professorship at Missouri School of Mines, Rolla, Missouri.

Associate Professor H. S. M. Coxeter of the University of Toronto has been appointed to a visiting professorship at the University of Notre Dame.

Assistant Professor M. J. Gottlieb of Washington University is on leave of absence this term to work at the Institute for Advanced Study.

Drs. G. H. A. Grosheide and J. Haantjes of the Free University of Amster-

dam have been promoted to professorships. This is a correction of an item in the February issue of the MONTHLY.

Dr. F. C. Jonah of Chance Vought Aircraft Division of the United Aircraft Corporation has been promoted to the position of Staff Project Engineer.

C. W. Jordan, Jr., has been appointed to an assistant professorship at Williams College.

Dr. H. D. Kloosterman of the University of Leiden has been promoted to a professorship.

R. R. Kuebler of Dickinson College has been promoted to an assistant professorship.

Assistant Professor J. K. L. MacDonald of Cooper Union has been appointed to a professorship of graduate mathematical physics at New York University.

Assistant Professor M. G. Moore of Bradley University, Peoria, Illinois, has been promoted to an associate professorship.

Dr. Z. I. Mosesson of the Prudential Insurance Company of America has been promoted to the position of Assistant Mathematician.

Dr. W. D. Munro of the University of Minnesota has been promoted to an assistant professorship in mathematics and mechanics.

Dr. P. F. Nemenyi has been appointed physicist with the Naval Ordnance Laboratory, White Oak, Maryland.

Dr. J. Popken has been appointed to a professorship at the University of Utrecht.

Professor E. B. Wilson of the School of Public Health, Harvard University, has retired with the title emeritus.

The following appointments to instructorships are announced:

New York Institute of Optics: L. D. Levine

United States Naval Academy: J. R. Gorman

Word has been received of the death of Professor Ettore Bortolotti of the University of Bologna on February 17, 1947.

Professor Emeritus B. L. Remick of Kansas State College died March 18, 1947.

Professor W. T. Short of Oklahoma Baptist University died February 19, 1947.

# THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## ANNUAL MEETING OF THE ILLINOIS SECTION

The twenty-fifth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois State Normal University, Normal, Illinois, on Friday and Saturday, May 10–11, 1946.

There were forty-six persons in attendance, including the following twenty-

one members of the Association: Edith I. Atkin, D. R. Bey, S. F. Bibb, W. H. Coulter, Elinor B. Flagg, L. R. Ford, A. E. Gault, Mildred Hunt, E. C. Kiefer, W. C. Krathwohl, Luise Lange, C. T. McCormick, W. C. McDaniel, C. N. Mills, M. G. Moore, E. J. Moulton, E. W. Ploenges, J. M. Sachs, M. Anice Seybold, E. H. Taylor, B. R. Ullsvick.

At the business meeting the following officers were elected for the coming year: Chairman, C. N. Mills, Illinois State Normal University; Vice-Chairman, M. G. Moore, Bradley Polytechnic Institute; Secretary, E. C. Kiefer, James Millikin University. It was decided to hold the next meeting on May 10–11, 1947 at Bradley Polytechnic Institute in Peoria, Illinois.

The following papers were presented:

1. *A note on the history of the concept of infinity in mathematics*, by Dr. E. H. Taylor, Eastern Illinois State Teachers College.

The speaker noted the central role of the concept of infinity in the development of mathematics. He discussed the place of the concept in the growth of modern geometry, in the development of the calculus, and in modern attempts to attain rigor in mathematics. Attention was called to the conflict between the school of Hilbert and that of Brouwer growing out of difficulties with infinite processes and infinite classes.

2. *On a graphical interpretation of the criteria for conic sections*, by Dr. Luise Lange, Chicago City Junior Colleges, Woodrow Wilson Branch.

3. *Forum*: *What college mathematics for the returning service man?* by Professor C. N. Mills, Illinois Normal University, Professor E. C. Kiefer, James Millikin University, and Professor S. F. Bibb, Illinois Institute of Technology.

The various speakers told of efforts made in their own schools to assist returned service men.

4. *Two theorems on Brocard points derived from group theory*, by Sister Mary Phillip, Rosary College.

A geometric representation of the dihedral group $G_6$ in the real inversive plane shows that certain sets of points with special significance from the group point of view play an important part in the geometry of the configuration. The triangles of a certain poristic system under $G_6$ have the same Brocard points, namely, the transforms of their common circumcenter under the invariant subgroup of $G_6$. Moreover, the Brocard angles of these triangles are equal.

The Brocard angle of this system of triangles under $G_6$ is equal to the Brocard angle which determines the circumcircle of the system as a Neuberg circle on a segment of the Desargues' axis of related pairs of triangles of the system with limiting points the common isodynamic points of the triangles of the system.

5. *Application of mathematics to war problems*, by Professor E. J. Moulton, Northwestern University.

Professor Moulton talked briefly of the extent to which professional mathe-

maticians participated in the war effort. Dividing the field of mathematics into nine sections ranging from arithmetic to topology, he indicated where each was used in connection with the recent conflict. He spoke of such diverse problems as the design of clothing, weather forecasting, and the atomic bomb. Numerous other applications were mentioned.

E. C. KIEFER, *Secretary*

## NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 30, 1946. Professor P. A. Caris, Chairman of the Section, presided at the morning and afternoon sessions.

There were fifty-one present, including the following thirty-five members of the Association: C. B. Allendoerfer, Joshua Barlaz, P. T. Bateman, A. L. Billig, T. A. Botts, H. W. Brinkmann, L. H. Bunyan, W. B. Campbell, P. A. Caris, J. A. Clarkson, J. W. Clawson, L. J. Deck, F. L. Dennis, Arnold Dresden, W. H. Gottschalk, J. F. Heyda, J. R. Kline, F. L. Manning, Clifford Marburger, D. L. McDonough, S. S. McNeary, A. E. Meder, Jr., Martin Moliver, W. R. Murray, C. A. Nelson, J. C. Oxtoby, C. J. Rees, I. J. Schoenberg, Benjamin Slepin, L. L. Smail, E. P. Starke, G. L. Walker, R. M. Walter, G. C. Webber, Anna Pell Wheeler.

At the business meeting the following officers were elected for the coming year: Chairman, C. J. Rees, University of Delaware; Secretary, W. H. Gottschalk, University of Pennsylvania. The Program Committee for the next meeting will be: I. J. Schoenberg (Chairman), University of Pennsylvania, C. B. Allendoerfer, Haverford College, and A. E. Pitcher, Lehigh University. The next meeting will be held on Saturday, November 29, 1947.

The program consisted of the following papers:

1. *Convex sets*, by Professor T. A. Botts, University of Delaware.

The elementary properties of convex sets in the euclidean plane were discussed from the intuitive geometric viewpoint. The plane-of-support property of convex sets was established in two (well known) ways, each of them avoiding the type of limiting process usually employed. It was pointed out that the geometric arguments used could be phrased analytically so as to be valid for the corresponding theorems in $n$-dimensional euclidean space.

2. *Slope in solid analytic geometry*, by Professor C. B. Allendoerfer, Haverford College.

This paper has been published in this MONTHLY, vol. 53 (1946), pp. 241–247.

3. *Generalizations of the Weierstrass approximation theorem*, by Professor Edwin Hewitt, Bryn Mawr College, introduced by Professor W. R. Murray.

The celebrated approximation theorem of Weierstrass asserts that, given a closed interval $[a, b]$ in the real number system, a real-valued continuous function $\phi(x)$ defined on $[a, b]$, and any positive number $\epsilon$, there exists a polynomial

$p(x)$ such that $|p(x)-\phi(x)|<\epsilon$ for all $x$ in $[a, b]$. Many different proofs have been given for this theorem, some depending upon the existence of integrals, others using uniform continuity. M. H. Stone has proved (*Trans. Amer. Math. Soc.*, vol. 41, 1937, p. 466, Theorem 82) a striking generalization of this theorem, as follows. Let $X$ be any bicompact Hausdorff space, and $\mathcal{J}$ any set of real-valued continuous functions defined on $X$ such that for every $p$, $q \epsilon X$ which are distinct points, there is a function $f \epsilon \mathcal{J}$ such that $f(p) \neq f(q)$. Then any continuous real-valued function defined on $X$ can be approximated uniformly to an arbitrary degree of accuracy of means of polynomials $p(f_1, \cdots, f_n)$, where $f_1, \cdots, f_n$ are in $\mathcal{J}$. This theorem is proved as a special case of a more general approximation theorem valid in arbitrary completely regular spaces.

W. H. GOTTSCHALK, *Secretary*

## FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University on Saturday, December 7, 1946, with a morning session, luncheon, and an afternoon session. Professor W. K. Morrill, Chairman of the Section, presided at both sessions.

The attendance was one hundred and one including the following thirty-seven members: N. H. Ball, Archie Blake, S. G. Bourne, H. H. Campaigne, Abraham Cohen, G. F. Cramer, J. A. Duerksen, E. J. Finan, M. K. Fort, Jr., C. H. Frick, Michael Goldberg, J. R. Hammond, G. A. Hedlund, Sister Mary Cordia Karl, L. M. Kells, Katharine B. Keppler, V. L. Klee, Jr., W. D. Lambert, M. H. Martin, E. S. Mayer, Carol V. McCamman, Mary E. Meade, Emanuel Mehr, Joseph Milkman, T. W. Moore, F. D. Murnaghan, W. H. Norris, Jr., Grace S. Quinn, C. E. Rhodes, R. E. Root, E. D. Schell, H. R. Smith, S. Helen Taylor, Marian M. Torrey, W. R. Utz, A. L. Whiteman, G. T. Whyburn.

The first five of the following papers were read at the morning session. Professor Murnaghan's paper was read at the afternoon session at the invitation of the Section.

1. *Needed changes in curriculum materials and in methods as suggested by war-time experience*, by Dr. S. Helen Taylor, State Teachers College, Frostburg, Maryland.

As a result of her war-time experience, Dr. Taylor recommended several changes in the curriculum material now generally used in the teaching of mathematics at the college level.

2. *On systems of constructible number theory*, by Dr. David Nelson, George Washington University, introduced by the Secretary.

A formal system of classical number theory $C$ may be extended by the addition of further logical primitives for implication, alternation, and the existential quantifier and further rules of inference to give an intuitionistic system $I$. This system may be further extended by the addition of a new primitive for negation to give a system $N$. Each of these extensions may be characterized by a depend-

ence among the logical symbols in $C$ and in $I$. If the logical symbols of $N$ are independent, an interpretation of the logical connectives which is divergent from the three represented by these systems is suggested; if not, a characterization of $N$ is provided.

3. *A construction of the extended real number system*, by M. K. Fort, Jr., University of Virginia.

The terms "closed interval of rational numbers" and "directed set of closed intervals" were first defined. An "extended real number" was then defined to be a maximal directed set. The collection of all such extended real numbers contains in addition to the usual real numbers two elements which are called $-\infty$ and $\infty$. Finally, multiplication and addition were defined and some properties of the extended real number system were discussed.

4. *On the equation* $\phi(x)=n$, by V. L. Klee, Jr., University of Virginia.

A brief survey of previous results on the problem of obtaining solutions of $\phi(x)=n$ was followed by an outline of the proof and an application of the following theorem due to the author: If $m$ is an odd integer greater than 1, expressible in the form

$$m = \prod (2^{j_i} + 1)^{a_i} \prod d_i(2^{k_i}d_i + 1)^{b_i}$$

where the integers $2^{j_i}+1$ and $2^{k_i}d_i+1$ are prime, if the $a_i$'s and the $b_i$'s are non-negative integers, and if

$$1 \leqq \sum j_i + \sum k_i \leqq n,$$

then

$$x = 2^{n+1} - \sum j_i - \sum k_i \prod (2^{j_i} + 1)^{a_i+1} \prod (2^{k_i}d_i + 1)^{b_i+1}$$

is a solution of the equation $\phi(x)=2^n m$. And if $\sum j_i+\sum k_i=n$, then $x/2$ is also a solution. Furthermore, all solutions are obtainable in this manner from expressions of $m$ in the above form.

5. *On the decomposition of meromorphic functions*, by W. R. Utz, University of Virginia.

In a recent paper L. H. Loomis has given a necessary and sufficient condition for the decomposition of a meromorphic function, defined on the interior of the unit circle, into a regular, bounded, univalent function followed by a rational function. This condition is given in terms of the behavior of the function near the boundary of the region of its definition. In the present paper the author employs extensions to the boundary of mappings from the unit circle upon an arbitrary bounded simply connected region with locally connected boundary to secure an analogous theorem for a function meromorphic on a bounded, simply connected region with locally connected boundary.

6. *The teaching of mathematics below the college level*, by Professor E. D. Murnaghan, Johns Hopkins University.

This paper will be published in the *Mathematics Teacher*.

E. J. FINAN, *Secretary*

## NEW MEMBERS

Professor W. B. Carver, Secretary-Treasurer, announces that the following one hundred persons have been elected to membership on applications duly certified:

A. T. ANDERSON, A.M. (Michigan) Instr., Cooper Union, New York, N. Y.
PETER ANDRIS, B.S. in E.E. (Illinois) Teacher, Harrison Tech. High School, Chicago, Ill.
W. E. F. APPUHN, A.M. (Columbia) Adj. Prof., Poly. Inst. of Brooklyn, Brooklyn, N. Y.
J. D. ARMSTRONG, B.S.E. (Florida) Instr., Aeronautical Univ., Chicago, Ill.
A. W. ASHBURN, Ph.D. (Virginia) Asso. Prof., Texas State Coll. for Women, Denton, Tex.
GLADYS F. BADGER, A.M. (Northwestern) Teacher, Roosevelt High School, Chicago, Ill.
INA M. BRAMBLETT, A.M. (Texas) Asst. Prof., Texas Christian Univ., Fort Worth, Tex.
FRANCES M. BRENEMAN, M.S. (Kansas St. T. C., Emporia) Instr., Washburn Univ., Topeka, Kans.
J. P. BREWSTER, A.M. (Duke) Asst. Prof., Clemson Agric. Coll., Clemson, S. C.
A. C. BURDETTE, Ph.D. (Illinois) Asst. Prof., Univ. of California, Coll. of Agric., Davis, Calif.
R. L. CASKEY, A.M. (Oklahoma) Asst. Prof., Oklahoma A. and M. Coll., Stillwater, Okla.
ELIZABETH C. CATHEY, M.S. (Louisiana State) Instr., Univ. of Alabama, University, Ala.
D. G. CHAPMAN, A.M. (California) Asst. Prof., Univ. of British Columbia, Vancouver, B. C., Canada
JOSEPH CLARE, M.Eng. (Liverpool, England) Asso. Prof., Knox Coll., Galesburg, Ill.
C. E. DENNY, B.S. (U. S. Naval Acad.) Instr., Central Coll., Fayette, Mo.
E. G. DOUGLAS, A.M. (Mercer Univ.) Instr., Univ. of South Carolina, Columbia, S. C.
E. J. DOWNIE, A.B. (Colgate) Instr., Colgate Univ., Hamilton, N. Y.
GENEVA E. DURHAM, A.M. (Northwestern) Instr., Atlantic Union Coll., South Lancaster, Mass.
JEANETTE R. DURST (Mrs. T. N.), A.M. (Tennessee) Instr., Univ. of South Carolina, Columbia, S. C.
W. F. EBERLEIN, Ph.D. (Harvard) Instr., Univ. of Michigan, Ann Arbor, Mich.
L. P. EDWARDS, M.A. (Acadia) Asst. Prof., Univ. of New Brunswick, Fredericton, N. B., Canada
N. A. EISEN, B.S.E. (Missouri) Chemist, Mid Continent Petroleum Corp., Tulsa, Okla.
WALTER FLEMING, A.M. (Minnesota) Lecturer, Univ. of Manitoba, Fort Garry Site, Winnipeg, Man., Canada
R. W. FRANKEL, Student, Univ. of Michigan, Ann Arbor, Mich.
W. C. G. FRASER, Ph.D. (Toronto) Instr., Dartmouth Coll., Hanover, N. H.
(MR.) EMILE FRÉCHETTE, Lic. ès Sci. Math. (Montreal) Teacher, Seminary of Philosophy, 3880 Côte-des-Neiges Road, Montreal, P. Q., Canada
N. S. FREE, M.A. (Univ. of British Columbia) Lecturer, Univ. of British Columbia, Vancouver, B. C., Canada
MARIANO GARCIA, JR., Ph.D. (Virginia) Asso. Prof., Coll. of Agriculture, Univ. of Puerto Rico, Mayaguez, Puerto Rico
S. A. GHANI, M.A. (Muslim Univ., India) Lecturer, Nizam Coll., Hyderabad, Deccan, India
ESTHER F. GIBNEY. Chm. of Dept., Roosevelt High School, Chicago, Ill.
A. L. GILMORE, JR., B.S. (Miss. Southern Coll.) Teacher, High School, Picayune, Miss.
F. MONICA GOEN, M.S. (Iowa) Instr., Mississippi State Coll., State College, Miss.
I. L. GOLDMAN, A.B. (Columbia) Part-time Instr., Columbia Univ., New York, N. Y.
CASSIE C. GREER, A.M. (Chicago) Teacher, Englewood High School, Chicago, Ill.
A. C. GRIMES, A.M. (Mississippi) Instr., Mississippi State Coll., State College, Miss.
FRANK HARARY, A.M. (Brooklyn) Teaching Asst., Univ. of California, Berkeley, Calif.
L. J. HARRIS, B.E.E. (Rensselaer Poly. Inst.) Elec. Engr., Aluminum Co. of America, Alcoa Reduction Plant, Alcoa, Tenn.
R. H. HOPKINS, A.M. (Denver) Instr., Mississippi State Coll., State College, Miss.
J. M. HURT, A.M. (Texas) Instr., Math. and Astr., Univ. of Texas, Austin, Tex.

C. W. JENKE, B.S. (St. Mary's Univ.) Chemist, San Antonio Brewing Association, San Antonio, Tex.

WALTER JENNINGS, A.M. (Ohio State) Asst. Prof., Virginia Poly. Inst., Blacksburg, Va.

M. L. JOHNSON, A.M. (Western Reserve) Asst. Prof., Kent State Univ., Kent, Ohio

L. O. JONES, A.M. (Peabody) Prof., Chm. of Dept., William Jewell Coll., Liberty, Mo.

G. A. KEYES, M.S. (Syracuse) Lt. (j.g.), Asst. Instr., U. S. Merchant Marine Cadet School, San Mateo, Calif.

SISTER CONSTANTIA KLOS, B.S. (St. Joseph's Coll., Md.) St. Joseph's Coll., Emmitsburg, Md.

W. E. KRUSE, A.M. (Columbia) Instr., St. Peter's Coll., Jersey City, N. J.

W. H. KRUSKAL, A.M. (Harvard) Vice President, Kruskal and Kruskal, Inc., New York, N. Y.

J. W. LAGRONE, A.M. (Vanderbilt) Asso. Prof. Clemson Agric. Coll., Clemson, S. C.

GEORGE LAUSH, B.S. (Pittsburgh) Grad. Asst., Cornell Univ., Ithaca, N. Y.

M. E. LEVENSON, M.S. (New York Univ.) Instr., Cooper Union, Cooper Square, New York, N. Y.

S. L. LEVY, B.S.E.E. (Ill. Inst. of Tech.) Grad. Asst., Illinois Inst. of Tech., Chicago 16, Ill.

EUNICE LEWIS, A.M. (Oklahoma) Teacher, Central High School, Tulsa, Okla.

P. E. LEWIS, Ph.D. (Illinois) Asso. Prof., Oklahoma A. and M. Coll., Stillwater, Okla.

F. W. LIGHT, JR., M.D. (Johns Hopkins) Asst. Prof., Johns Hopkins Univ., Baltimore 18, Md.

W. S. LOUD, Ph.D. (Mass. Inst. of Tech.) Instr., Massachusetts Inst. of Tech., Cambridge 39, Mass.

R. B. LOWE, Student, Poly. Inst. of Brooklyn, 85 Livingston St., Brooklyn, N. Y.

SISTER MARY BENEDICTA, A.B. (Good Counsel Coll.) Good Counsel Coll., White Plains, N. Y.

HAZEL L. MASON, A.M. (New Mexico) Head of Dept., Math. and Physics, Mary Hardin-Baylor Coll., Belton, Tex.

BERNARD MASON, A.M. (Columbia) Instr., Physics, Hofstra Coll., Hempstead, N. Y.

KENNETH MAY, Ph.D. (California) Asst. Prof., Carleton Coll., Northfield, Minn.

R. D. MCDOLE, M.S. (Okla. A. and M.) Oklahoma A. and M. Coll., Stillwater, Okla.

D. F. MELA, A.B. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

M. H. A. MILLER. Student, Univ. of South Dakota, Vermillion, S. D.

I. H. MONELL, A.M. (Loyola Univ., Ill.) Teacher, Austin High School, Chicago, Ill.

H. W. MORROW, JR., B.M.E. (Minnesota) Teaching Fellow, Math. and Physics, Univ. of South Dakota, Vermillion, S. D.

K. W. MORROW, A.B. (South Dakota) Grad. Student, Univ. of South Dakota, Vermillion, S. D.

A. F. NICKEL, A.B. (Queens Coll.) Lt. (j.g.), Instr., U. S. Merchant Marine Cadet School, San Mateo, Calif.

T. H. O'BEIRNE, M.A. (Glasgow) Head of Math. Section, Torpedo Experimental Establishment, Royal Naval Torpedo Factory, Greenock, Renfrewshire, Scotland

C. S. OGILVY, A.M. (Columbia) Instr., Trinity Coll., Hartford 6, Conn.

R. C. OSBORN, A.M. (Texas) Instr., Math. and Astr., Univ. of Texas, Austin, Tex.

C. J. PIPES, B.S. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla.

FLORENCE M. POHLEY, B.S. (Chicago) Teacher, High School, Chicago, Ill.

J. W. POPOW, B.S. (Washington and Jefferson) Instr., U. S. Naval Acad., Annapolis, Md.

EDGAR REICH, Student, Poly. Inst. of Brooklyn, Brooklyn, N. Y.

E. S. ROBBINS, A.B. (Wichita) Teaching Fellow, Univ. of Wichita, Wichita, Kans.

G. J. ROSS, M.S. in Educ. (C.C.N.Y.) Teacher, Erasmus Hall High School, Brooklyn, N. Y.

J. J. ROWLAND, A.M. (Oregon) Instr., State Coll. of Washington, Pullman, Wash.

L. W. RUTLAND, JR., M.S. (E. Tex. St. T. C.) Instr., Eng. Math., Univ. of Colorado, Boulder, Colo.

K. S. B. SASTRI, M.A. (Madras Univ.) Prof., D. G. N. Coll., Hyderabad, Sind, India

KLAUS SCHOCKEN, Ph.D. (Berlin) Instr., Xavier Univ., Cincinnati 7, Ohio

D. C. SHELDON, Ph.D. (California) Head of Dept., Clemson Agric. Coll., Clemson, S. C.

R. W. SHEPHARD, Ph.D. (California) Asst. Prof., New York Univ., New York, N. Y.

C. M. SHEPHERD, M.S.Ch.E. (Case School)

Research Electrochemist, Naval Research Lab., Washington, D. C.

W. L. SHEPHERD, M.S. (Okla. A. and M.) Asst. Prof., Oklahoma A. and M. Coll., Stillwater, Okla.

A. A. SMITH, A.M. (N. Tex. State T. C.) Chm. of Dept., San Angelo Coll., San Angelo, Tex.

MRS. DOROTHY B. SMITH, A.B. (Phillips Univ.) Special Instr., Univ. of Oklahoma, Norman, Okla.

W. K. SPEARS, A.B. (Dartmouth) 1st Lt., AUS. 318th Base Unit, Lockbourne AAB, Columbus, Ohio

W. M. STONE, A.M. (Oregon State Coll.) Instr., Iowa State Coll., Ames, Iowa

MILDRED M. SULLIVAN, Ph.D. (Radcliffe) Asst. Prof., Queens Coll., 65-30 Kissena Blvd., Flushing, N. Y.

M. FRANCES SUTER, A.M. (Illinois) Instr., Madison Coll., Harrisonburg, Va.

M. M. TEMPLE, M.S. (Mississippi) Mississippi State Coll., State College, Miss.

BETTY THOMAS, A.M. (Alabama) Asst. Prof., Wesleyan Coll., Macon, Ga.

J. B. TYSVER, A.B. (State Coll. of Washington) Grad. Student, Part-time Instr., State Coll. of Washington, Pullman, Wash.

WILLIAM WALLACE, B.S. (Northeastern) Instr., Northeastern Univ., 360 Huntington Ave., Boston, Mass.

F. C. WARNER, A.B. (Coll. of Wooster) Instr., Univ. of Buffalo, Buffalo, N. Y.

INA W. WELMERS (Mrs. E. T.), A.M. (Michigan) Instr., Univ. of Buffalo, Buffalo, N. Y.

J. E. WILKINS, JR., Ph.D. (Chicago) Mathematician, American Optical Co., Box A, Buffalo 15, N. Y.

G. M. WING, A.B. (Rochester) Instr., Univ. of Rochester, Rochester, N. Y.

B. L. WOLFE, B.S. in Educ. (St. T. C., Indiana, Pa.) Instr., San Angelo Coll., San Angelo, Tex.

SISTER FRANCIS XAVIER, Ph.D. (Fordham Univ.) Head of Dept., St. Joseph's Coll. for Women, 232 Clinton Ave., Brooklyn 5, N. Y.

## CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Connecticut, September 1–2, 1947.

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA, Fairfield, April 1, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 7


PART 1

CONTENTS




AUGUST-SEPTEMBER 1947


**The Association is in need of copies of the February and March, 1947, issues of the MONTHLY, and will pay $1.00 each for copies of these issues until 75 copies of the February issue and 50 copies of the March issue have been received. They should be mailed to Mathematical Association of America, McGraw Hall, Cornell University, Ithaca, New York, with the name and address of the sender clearly marked on the package.**

# The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)





EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions and advertising in the MONTHLY, should be sent to the Secretary-Treasurer, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 20th of the month preceding that in which the change becomes effective.


---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---


Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# RANDOM WALK AND THE THEORY OF BROWNIAN MOTION*

MARK KAC,† Cornell University

1. **Introduction.** In 1827 an English botanist, Robert Brown, noticed that small particles suspended in fluids perform peculiarly erratic movements. This phenomenon, which can also be observed in gases, is referred to as Brownian motion. Although it soon became clear that Brownian motion is an outward manifestation of the molecular motion postulated by the kinetic theory of matter, it was not until 1905 that Albert Einstein first advanced a satisfactory theory.

The theory was then considerably generalized and extended by the Polish physicist Marjan Smoluchowski, and further important contributions were made by Fokker, Planck, Burger, Fürth, Ornstein, Uhlenbeck, Chandrasekhar, Kramers and others [1]. On the purely mathematical side various aspects of the theory were analyzed by Wiener, Kolomgoroff, Feller, Lévy, Doob, and Fortet [2]. Einstein considered the case of the *free* particle, that is, a particle on which no forces other than those due to the molecules of the surrounding medium are acting. His results can be briefly summarized as follows.

Consider the motion of the projection of the free particle‡ on a straight line which we shall call the $x$-axis. What one wants is the probability

$$\int_{x_1}^{x_2} P(x_0 \mid x; t)dx$$

that at time $t$ the particle will be between $x_1$ and $x_2$ if it was at $x_0$ at time $t=0$. Einstein was then able to show that the "probability density" $P(x_0 \mid x; t)$§ must satisfy the partial differential equation

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}, \tag{1}$$

where $D$ is a certain physical constant. The conditions imposed on $P$ are

$$\begin{aligned} &(a) && P \geqq 0 \\ (2) \quad &(b) && \int_{-\infty}^{\infty} P(x_0 \mid x; t)dx = 1 \\ &(c) && \lim_{t\to 0} P(x_0 \mid x; t) = 0, \qquad \text{for } x \neq x_0. \end{aligned}$$

---

* This is an extended version of an address delivered at the annual meeting of the Association at Swarthmore, Pennsylvania, December 26–27, 1946.

† John Simon Guggenheim Memorial Fellow.

‡ In what follows we shall identify this projection with the particle itself and hence consider the so-called one-dimensional Brownian motion.

§ The notation $P(x_0 \mid x; t)$ and $P(n \mid m; s)$ for conditional probabilities is that used by Wang and Uhlenbeck [1]. It does not conform with the notation adopted in the statistical literature. Had we adopted the latter notation we would write $P(x; t \mid x_0)$ and $P(m; s \mid n)$.

Conditions (*a*) and (*b*) are the usual ones imposed upon a probability density and condition (*c*) expresses the *certainty* that at $t=0$ the particle was at $x_0$. It is well known that (1) and (2) imply that

$$P(x_0 \mid x;\, t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-(x-x_0)^2/4Dt} \tag{3}$$

and that the solution (3) is unique.

The greatness of Einstein's contribution was not, however, solely due to the derivation of (1), and hence (3). From the point of view of physical applications it was equally, or perhaps even more, important that he was able to show that

$$D = \frac{2RT}{Nf}, \tag{4}$$

where $R$ is the universal gas constant, $T$ the absolute temperature, $N$ the Avogadro number, and $f$ the friction coefficient. The friction coefficient $f$, in the case the medium is a liquid or a gas at ordinary pressure, can in turn be expressed in terms of viscosity and the size of the particle [3].

It was relation (4) that made possible the determination of the Avogadro number from Brownian motion experiments, an achievement for which Perrin was awarded the Nobel prize in 1926. However, the derivation of (4) belongs to physics proper, and presents no particular mathematical interest; we shall therefore not be concerned with it in the sequel.

As soon as the theory for the free particle was established, a natural question arose as to how it should be modified in order to take into account outside forces as, for example, gravity. Assuming that the outside force acts in the direction of the $x$-axis and is given by an expression $F(x)$, Smoluchowski has shown that (1) should in this case be replaced by

$$\frac{\partial P}{\partial t} = -\frac{1}{f}\frac{\partial}{\partial x}(PF) + D\frac{\partial^2 P}{\partial x^2}. \tag{5}$$

Two cases of special interest and importance are:

$F(x) = -a$; field of constant force (for example, gravity).

$F(x) = -bx$; elastically bound particle (for example, small pendulum).

At this point it must be strongly emphasized that theories based on (1) and (5) are only approximate. They are valid only for relatively large $t$ and, in the case of an elastically bound particle, only in the overdamped case, that is, when the friction coefficient is sufficiently large. These limitations of the theory were already recognized by Einstein and Smoluchowski but are often disregarded by writers who stress that in Brownian motion the velocity of the particle is infinite. This paradoxical conclusion is a result of stretching the theory beyond the bounds of its applicability. An improved theory (known as "exact") was advanced by Uhlenbeck and Ornstein and by Kramers. The Uhlenbeck-Orn-

stein approach was further elaborated by Chandrasekhar and Doob.

In what follows we shall be concerned with a discrete approach to the Einstein-Smoluchowski (approximate) theory. This approach was first suggested by Smoluchowski himself; it consists in treating Brownian motion as a discrete random walk. Smoluchowski used this approach only in connection with a free particle but we shall also treat other classical cases. Moreover, a re-interpretation of one of the discrete models will allow us to discuss the important question of recurrence and irreversibility in thermodynamics.

The main advantages of a discrete approach are pedagogical, inasmuch as one is able to circumvent various conceptual difficulties inherent to the continuous approach. It is also not without a purely scientific interest and it is hoped that it may suggest various generalizations which will contribute to the development of the Calculus of Probability.

2. **The free particle.** Imagine a particle which moves along the $x$-axis in such a way that in each step it can move either $\Delta$ to the right or $\Delta$ to the left, the duration of each step being $\tau$. The fact that we are dealing with a free particle is interpreted by assuming that the probabilities of moving to the right or to the left are equal, and hence each equal $\frac{1}{2}$. Instead of $P(x_0|x;\, t)$ we now consider $P(n\Delta|m\Delta;\, s\tau)=P(n|m;\, s)$ which is the probability that the particle is at $m\Delta$ at time $s\tau$, if at the beginning it was at $n\Delta$. Noticing that $P(n|m;\, s)$ is also the probability that after $s$ games of "heads or tails" the gain of a player is $\nu=m-n$, we can write

$$(6)\quad P(n\,|\,m;s) = \begin{cases} \dfrac{1}{2^s}\,\dfrac{s!}{\left(\dfrac{s+|\nu|}{2}\right)!\left(\dfrac{s-|\nu|}{2}\right)!} & \text{if } |\nu| \leqq s \text{ and } |\nu|+s \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that $\Delta$ and $\tau$ approach 0 in such a way that

$$(7)\qquad \frac{\Delta^2}{2\tau} = D, \qquad n\Delta \to x_0, \qquad s\tau = t.$$

It then follows from the classical Laplace-De Moivre theorem [4] that

$$(8)\qquad \lim \sum_{x_1<m\Delta<x_2} P(n\,|\,m;s) = \frac{1}{2\sqrt{\pi D t}}\int_{x_1}^{x_2} e^{-(x-x_0)^2/4Dt}dx,$$

and hence the fundamental result of Einstein emerges as a consequence of what in probability theory we call a "limit theorem."

It is both important and instructive to point out a striking formal connection between the discrete (random walk) and the continuous (Einstein) approaches. We notice that $P(n|m;\, s)$ satisfies the difference equation

$$(9)\qquad P(n\,|\,m;s+1) = \tfrac{1}{2}P(n\,|\,m-1;s) + \tfrac{1}{2}P(n\,|\,m+1;s),$$

which we write in the equivalent form

$$\begin{aligned} &\frac{P(n\Delta \mid m\Delta;\,(s+1)\tau) - P(n\Delta \mid m\Delta;\,s\tau)}{\tau} \\ (10)\qquad &= \frac{\Delta^2}{2\tau}\left\{\frac{P(n\Delta \mid (m+1)\Delta;\,s\tau) - 2P(n\Delta \mid m\Delta;\,s\tau) + P(n\Delta \mid (m-1)\Delta;\,s\tau)}{\Delta^2}\right\}. \end{aligned}$$

In the limit (7) this difference equation goes over formally into the differential equation

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}, \tag{11}$$

which as noted before was the basis of Einstein's theory. This formal connection between the two approaches can be made rigorous, but we shall not go into this. However, we shall use it as a guiding heuristic principle in constructing models of Brownian motion when outside forces are taken into account.

Finally, let us mention that it is the relation

$$\frac{\Delta^2}{2\tau} = D,$$

which is responsible for the conclusion that the velocity of a Brownian particle is infinite. In fact, in our model, the ratio $\Delta/\tau$ plays the role of the instantaneous velocity and it obviously approaches infinity as $\Delta \to 0$.

3. **Particle in a field of constant force and in the presence of a reflecting barrier.** We again consider random walk along the $x$-axis in which a particle can move $\Delta$ to the right or $\Delta$ to the left, the duration of each step being $\tau$. We now introduce the following new assumptions:

(a) The probability of a move to the right is $q = \frac{1}{2} - \beta\Delta$, and consequently the probability of a move to the left is $p = \frac{1}{2} + \beta\Delta$. Here $\beta$ is a certain physical constant, and $\Delta$ must be chosen sufficiently small so that $q > 0$.

(b) When the particle reaches the point $x = 0$ (*reflecting barrier*) it must, in the next step, move $\Delta$ to the right.

Without the assumption (b) the problem would be quite simple and of no great physical interest. In actual experiments with heavy Brownian particles, like those of Perrin, the bottom of the container acts as a reflecting barrier and the elucidation of its influence on the Brownian motion is of considerable theoretical interest.

This problem has been solved by Smoluchowski, on the basis of his equation (5) but we shall show that one can solve the discrete problem and obtain Smoluchowski's result by passing to a limit.

Assuming that the particle starts from $n\Delta \geqq 0$ ($n$ an integer) we seek an explicit expression for $P(n \mid m;\, s)$. We first notice that $P(n \mid m;\, s)$ satisfies, for

$m \geqq 2$, the difference equation

$$P(n \mid m; s+1) = qP(n \mid m-1; s) + pP(n \mid m+1; s), \tag{12}$$

and that for $m=1$ and $m=0$ we have

$$P(n \mid 1; s+1) = P(n \mid 0; s) + pP(n \mid 2; s), \tag{12a}$$

$$P(n \mid 0; s+1) = pP(n \mid 1; s). \tag{12b}$$

We also have the initial condition

$$P(n \mid m; 0) = \delta(m, n), \tag{13}$$

where $\delta(m, n)$ denotes, as usual, the Kronecker delta.

The difference equation (12) when rewritten in the form analogous to (10) can be shown to go over formally (in the limit $\Delta \to 0$, $\tau \to 0$, $\Delta^2/2\tau = D$, $n\Delta \to x_0$, $m\Delta \to x$, $s\tau = t$) into the differential equation

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} + 4\beta D \frac{\partial P}{\partial x}, \tag{14}$$

which is of the form (5) with $F(x) = -a = -4\beta Df$.

To find $P(n \mid m; s)$ we use a method which is basic in the study of the so-called Markoff chains, of which our problem is but a particular example, and which in its essentials goes back to Poincaré [5]. Let $(p)_s$ be the (infinite) vector

$$(p)_s = \begin{bmatrix} P(n \mid 0; s) \\ P(n \mid 1; s) \\ P(n \mid 2; s) \\ \vdots \end{bmatrix} \tag{15}$$

and $A$ the infinite matrix

$$A = \begin{Vmatrix} 0 & p & 0 & 0 & 0 \cdots \\ 1 & 0 & p & 0 & 0 \cdots \\ 0 & q & 0 & p & 0 \cdots \\ 0 & 0 & q & 0 & p \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{Vmatrix}. \tag{16}$$

Then, the difference equation (12), (12$a$) and (12$b$), can be written in the matrix form as

$$(p)_{s+1} = A(p)_s. \tag{17}$$

Thus it follows immediately that

$$(p)_s = A^s(p)_0, \tag{18}$$

where $(p)_0$ is the vector

$$(p)_0 = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix},$$

1 being the $n$th component, the components being numbered from zero on. Interpreting (18), we see that

$$P(n \mid m; s) = \text{the } (m, n) \text{ element of } A^s. \tag{19}$$

To make use of (19), we notice that if $R > n+s$ and we consider the finite matrix $A_R$, which is the upper left $R$ by $R$ submatrix of $A$, then for $m < R$

$$\text{the } (m, n) \text{ element of } A^s = \text{the } (m, n) \text{ element of } A_R^s, \tag{20}$$

or equivalently,

$$\text{the } (m, n) \text{ element of } A^s = \lim_{R \to \infty} \text{of the } (m, n) \text{ element of } A_R^s. \tag{21}$$

For each $R$ there exist matrices $P_R$ and $Q_R$ such that

$$P_R Q_R = I \tag{22}$$

and

$$A_R = P_R \begin{bmatrix} \lambda_0(R) & & & 0 \\ & \lambda_1(R) & & \\ & & \ddots & \\ 0 & & & \lambda_{R-1}(R) \end{bmatrix} Q_R, \tag{23}$$

$\lambda_0(R), \lambda_1(R), \cdots, \lambda_{R-1}(R)$ being the eigenvalues of the matrix $A_R$. To simplify the notation we write $\lambda_j$ for $\lambda_j(R)$.

Multiplying the matrix $A_R$ $s$ times by itself and making use of (22), we obtain

$$A_R^s = P_R \begin{bmatrix} \lambda_0^s & & & 0 \\ & \lambda_1^s & & \\ & & \ddots & \\ 0 & & & \lambda_{R-1}^s \end{bmatrix} Q_R \tag{24}$$

and one can calculate the $(m, n)$ element of $A_R^s$ explicitly provided the diag-

onalization (23) can be performed explicitly. This indeed is the case.

Let $(x)_0, (x)_1, \cdots, (x)_{R-1}$ be the "right" and $(y)_0, (y)_1, \cdots, (y)_{R-1}$ the "left" eigenvectors belonging respectively to the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_{R-1}$. In other words, for $k=0, 1, \cdots, R-1$, we have

$$A_R(x)_k = \lambda_k(x)_k,$$
$$A_R'(y)_k = \lambda_k(y)_k,$$

where $A_R'$ is the transpose of $A_R$.

Suppose furthermore that the vectors can be so normalized that

$$(x)_k \cdot (y)_k = 1, \qquad k = 0, 1, \cdots, R-1, \tag{25}$$

where $(x)_k \cdot (y)_k$ is the inner (dot) product of the vectors. Since it is well known that

$$(x)_j \cdot (y)_k = 0 \qquad \text{for } \lambda_j \neq \lambda_k$$

we see that, in the case when all the eigenvalues are distinct, we can take as $P_R$ the matrix whose columns are the vectors $(x)_k$ and for $Q_R$ the matrix whose rows are the vectors $(y)_k$.

In order to determine the eigenvalues and the right eigenvectors we consider the system of linear equations

$$\begin{aligned} px_1 &= \lambda x_0 \\ x_0 + px_2 &= \lambda x_1 \\ qx_1 + px_3 &= \lambda x_2 \\ \cdots&\cdots \\ qx_{R-2} \qquad &= \lambda x_{R-1}, \end{aligned} \tag{26}$$

and the extended infinite system

$$\begin{aligned} px_1 &= \lambda x_0 \\ x_0 + px_2 &= \lambda x_1 \\ \cdots&\cdots \\ qx_{R-1} + px_{R+1} &= \lambda x_R \\ \cdots&\cdots . \end{aligned} \tag{27}$$

If we can find non-trivial solutions of (27), for which

$$x_R = 0, \tag{28}$$

we will have found solutions of (26).

It turns out that (28) will yield an equation in $\lambda$ which has $R$ distinct roots and thus our procedure gives us all eigenvalues, and consequently all right eigenvectors. Multiplying the members of the equations of (27) by 1,

$z, z^2, \ldots$, and adding, we obtain formally

$$x_0 z + q\sum_1^\infty x_k z^{k+1} + p\sum_1^\infty x_k z^{k-1} = \lambda \sum_0^\infty x_k z^k$$

or, upon introducing the abbreviation,

$$f(z) = \sum_0^\infty x_k z^k, \tag{29}$$

we have

$$x_0 z + qz[f(z) - x_0] + \frac{p}{z}[f(z) - x_0] = \lambda f(z). \tag{30}$$

From (30) we obtain

$$f(z) = \frac{p}{q} x_0 \left\{ -1 + \frac{1 - \lambda z}{qz^2 - \lambda z + p} \right\}, \tag{31}$$

and since this function is analytic in the neighborhood of zero the formal procedure used above can be justified.

Let $\rho_1$ and $\rho_2$ be the *reciprocals* of the roots of

$$qz^2 - \lambda z + p = 0. \tag{32}$$

We have then

$$f(z) = \frac{p}{q} x_0 \left\{ -1 + \frac{1 - \lambda z}{p(1 - \rho_1 z)(1 - \rho_2 z)} \right\}, \tag{33}$$

and introducing partial fractions,

$$\frac{1 - \lambda z}{p(1 - \rho_1 z)(1 - \rho_2 z)} = \frac{1}{p}\frac{\lambda - \rho_1}{\rho_2 - \rho_1}\frac{1}{1 - \rho_1 z} + \frac{1}{p}\frac{\rho_2 - \lambda}{\rho_2 - \rho_1}\frac{1}{1 - \rho_2 z}. \tag{34}$$

Thus

$$x_k = \frac{x_0}{q}\left( \frac{\lambda - \rho_1}{\rho_2 - \rho_1}\rho_1^k + \frac{\rho_2 - \lambda}{\rho_2 - \rho_1}\rho_2^k \right) \qquad \text{for } k \geqq 1, \tag{35}$$

and, in particular, the equation $x_R = 0$ assumes the form

$$\frac{\lambda - \rho_1}{\rho_2 - \rho_1}\rho_1^R + \frac{\rho_2 - \lambda}{\rho_2 - \rho_1}\rho_2^R = 0. \tag{36}$$

Equation (36) must now be solved for $\lambda$. Assuming $R$ to be even, and seeking solutions in the form

$$\lambda = 2\sqrt{pq}\cos\Theta, \qquad 0 \leqq \Theta \leqq \pi,$$

we are led to the equation

$$\frac{\tan R\Theta}{\tan \Theta} = \frac{1}{2p - 1}.$$

For $R > (2p-1)^{-1}$ this equation is seen to have $R-2$ distinct roots, $\Theta_1, \Theta_2, \cdots, \Theta_{R-2}$, which lie in the subintervals

$$\left(\frac{j\pi}{R} - \frac{\pi}{2R}, \frac{j\pi}{R} + \frac{\pi}{2R}\right),$$

where $j$ ranges through the integers from 1 to $R-1$ with the exception of $j = R/2$. Corresponding to $\Theta_1, \Theta_2, \cdots, \Theta_{R-2}$ we have $R-2$ distinct eigenvalues,

$$\lambda_k = 2\sqrt{pq}\cos\Theta_k, \qquad k = 1, 2, \cdots, R-2,$$

and the components of the right eigenvector belonging to $\lambda_k$ can be written in the form

$$x_k^{(m)} = a_k\left(\frac{q}{p}\right)_*^{m/2}\left(\cos m\Theta_k - 2\beta\Delta\frac{\sin m\Theta_k}{\sin\Theta_k}\right),$$

where

$$\left(\frac{q}{p}\right)_*^{\mu} = \begin{cases} \left(\dfrac{q}{p}\right)^{\mu} & \text{if } \mu > 0, \\ q & \text{if } \mu = 0, \end{cases}$$

and $a_k$ is a normalizing constant which will be fixed later. For sufficiently large $R$ the remaining eigenvalues $\lambda_0$ and $\lambda_{R-1}$ can be shown to be given by the formulas

$$\lambda_0 = 2\sqrt{pq}\cosh\theta_0, \qquad \lambda_{R-1} = -\lambda_0,$$

where $\theta_0$ is the only positive root of the equation

$$\frac{\tanh R\theta}{\tanh\theta} = \frac{1}{2p-1}.$$

The components of the corresponding right eigenvectors are given by the expressions

$$x_0^{(m)} = a_0\left(\frac{q}{p}\right)_*^{m/2}\left(\cosh m\theta_0 - 2\beta\Delta\frac{\sinh m\theta_0}{\sinh\theta_0}\right)$$

$$x_{R-1}^{(m)} = a_{R-1}(-1)^m\left(\frac{q}{p}\right)_*^{m/2}\left(\cosh m\theta_0 - 2\beta\Delta\frac{\sinh m\theta_0}{\sinh\theta_0}\right).$$

It remains now to find the left eigenvectors. This can be accomplished in exactly the same manner and we merely quote the results. We obtain

$$y_k^{(m)} = b_k\left(\frac{p}{q}\right)^{m/2}\left(\cos m\Theta_k - 2\beta\Delta\frac{\sin m\Theta_k}{\sin\Theta_k}\right)$$

for $m=0, 1,\ldots, R-1$; $k=1, 2, \cdots, R-2$, and

$$y_0^{(m)} = b_0\left(\frac{p}{q}\right)^{m/2}\left(\cosh m\theta_0 - 2\beta\Delta\frac{\sinh m\theta_0}{\sinh\theta_0}\right)$$

$$y_{R-1}^{(m)} = b_{R-1}(-1)^m\left(\frac{p}{q}\right)^{m/2}\left(\cosh m\theta_0 - 2\beta\Delta\frac{\sinh m\theta_0}{\sinh\theta_0}\right).$$

To satisfy the normalization conditions (25) we must have

$$a_kb_k\left(q + \sum_{m=1}^{R-1} f_m^2(\Theta_k)\right) = 1, \qquad k = 1, 2, \cdots, R-2, \tag{37}$$

$$a_kb_k\left(q + \sum_{m=1}^{R-1} F_m^2(\theta_0)\right) = 1, \qquad k = 0, R-1, \tag{38}$$

where

$$f_m(\Theta) = \cos m\Theta - 2\beta\Delta\frac{\sin m\Theta}{\sin\Theta}$$

and

$$F_m(\theta) = \cosh m\theta - 2\beta\Delta\frac{\sinh m\theta}{\sinh\theta}\cdot$$

We can, of course, put $a_0=a_1=\cdots=a_{R-1}=1$, and determine the $b$'s from (37) and (38). Referring back to (19), (20), and (24), and recalling that columns of $P_R$ are the right eigenvectors $(x)_k$, and the rows of $Q_R$ are the left eigenvectors $(y)_k$, we obtain

$$P(n\,|\,m; s) = \sum_{k=0}^{R-1} \lambda_k^s x_k^{(m)} y_k^{(n)}, \tag{39}$$

or, more explicitly,

$$\begin{aligned} P(n\,|\,m; s) = {} & b_0(2\sqrt{pq}\cosh\theta_0)^s\left(\frac{p}{q}\right)^{n/2}\left(\frac{q}{p}\right)_*^{m/2} F_m(\theta_0)F_n(\theta_0)\left[1 + (-1)^{m+n+s}\right] \\ & + \left(\frac{p}{q}\right)^{n/2}\left(\frac{q}{p}\right)_*^{m/2}(2\sqrt{pq})^s\sum_{k=1}^{R-2} b_k\cos^s\Theta_k f_m(\Theta_k)f_n(\Theta_k). \end{aligned} \tag{40}$$

Making use of (21), we can achieve considerable simplification by letting $R\to\infty$. In fact, it can be shown that

$$P(n\,|\,m; s) = \frac{p-q}{2pq}\left(\frac{q}{p}\right)_*^m\left[1 + (-1)^{m+n+s}\right] \tag{41}$$

$$+\frac{2}{\pi}\left(\frac{p}{q}\right)^{n/2}\left(\frac{q}{p}\right)_*^{m/2}(2\sqrt{pq})^s\int_0^\pi \cos^s\Theta\frac{\tan^2\Theta}{(p-q)^2+\tan^2\Theta}f_n(\Theta)f_m(\Theta)d\Theta.$$

Although in various places we have tacitly assumed that $p$ and $q$ are different from $\frac{1}{2}$, the final formula (41) can easily be seen to be valid also for the case $p=q=\frac{1}{2}$. In this case (free particle in the presence of a reflecting barrier) the formula assumes the remarkably simple form

$$(42)\qquad P(n\mid m;\, s) = \frac{2}{\pi}\,(1)_*^{m/2}\int_0^\pi \cos^s\Theta\cos m\Theta\cos n\Theta d\Theta,$$

and the right member can be expressed in terms of binomial coefficients. This formula can also be derived in a much simpler way using, for instance, the classical method of images.

In the limit

$$\Delta\to 0,\qquad \tau\to 0,\qquad \frac{\Delta^2}{2\tau}=D,\qquad n\Delta\to x_0,\qquad s\tau=t,$$

one can show that

$$\lim\sum_{x_1<m\Delta<x_2} P(n\mid m;\, s) = \int_{x1}^{x2} P(x_0\mid x;\, t)dx,$$

where

$$(43)\quad P(x_0\mid x;\, t) = 4\beta e^{-4\beta x} + e^{-2\beta(x-x_0)}e^{-4\beta^2Dt}\,\frac{2}{\pi}\int_0^\infty e^{-Dy^2t}\frac{y^2}{4\beta^2+y^2}g(x,y)g(x_0,y)dy,$$

and

$$g(x,\, y) = \cos xy - 2\beta(\sin xy)/y.$$

The proof of this theorem is not elementary inasmuch as it utilizes the so called "continuity theorem for Fourier-Stieltjes transforms" [6]. Formula (43) can be shown to be equivalent with Smoluchowski's formula given in [1].

4. **An elastically bound particle.** Again the particle can move either $\Delta$ to the right or $\Delta$ to the left, and the duration of each step is $\tau$. However, the probability of moving in either direction depends on the position of the particle. More precisely, if the particle is at $k\Delta$ the probabilities of moving right or left are

$$\frac{1}{2}\left(1-\frac{k}{R}\right)\quad\text{or}\quad\frac{1}{2}\left(1+\frac{k}{R}\right),$$

respectively. $R$ is a certain integer, and possible positions of the particle are limited by the condition $-R\leqq k\leqq R$. The basic probabilities $P(n\mid m;\, s)$ now satisfy the difference equation

$$(44)\quad P(n \mid m; s+1) = \frac{R+m+1}{2R} P(n \mid m+1; s) + \frac{R-m+1}{2R} P(n \mid m-1; s),$$

which must be solved with the initial condition

$$(45)\qquad P(n \mid m; 0) = \delta(m, n).$$

In the limit

$$\Delta \to 0, \qquad \tau \to 0, \qquad R \to \infty, \qquad \frac{\Delta^2}{2\tau} = D, \qquad \frac{1}{R\tau} \to \gamma,$$

$$s\tau = t, \qquad n\Delta \to x_0, \qquad m\Delta \to x,$$

the difference equation (44) is seen to go over formally into the differential equation

$$(46)\qquad \frac{\partial P}{\partial t} = \gamma \frac{\partial(xP)}{\partial x} + D \frac{\partial^2 P}{\partial x^2}$$

which is Smoluchowski's equation (5) with $F(x) = -x/\gamma f$.

The discrete problem in a different form and in a different connection was first proposed and discussed by P. and T. Ehrenfest in 1907 [7]. In the next section we shall discuss their original formulation. A fairly detailed treatment was given by Schrödinger and Kohlrausch in 1926 [8] and a brief exposition can be found in the review article of Wang and Uhlenbeck [1]. It seems that Schrödinger and Kohlrausch were the first to point out the connection between the Ehrenfest model and Brownian motion of an elastically bound particle. However, an explicit solution of (44) with the initial condition (45) was apparently not known. I have recently found such a solution using the matrix method described in Section 3 [9]. Instead of the infinite matrix of that section we must now consider the finite matrix

$$(47)\qquad B = \begin{pmatrix} 0 & \frac{1}{2R} & 0 & 0 & 0 \cdots 0 \\ 1 & 0 & \frac{2}{2R} & 0 & 0 \cdots 0 \\ 0 & 1 - \frac{1}{2R} & 0 & \frac{3}{2R} & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & & \frac{1}{2R} \quad 0 \end{pmatrix}$$

and the problem is again reduced to finding the eigenvalues $\lambda_{-R}, \lambda_{-R+1}, \cdots, \lambda_0, \cdots, \lambda_{R-1}, \lambda_R$ of $B$ and matrices $P$ and $Q$ such that

(48)
$$PQ = I$$

and

(49)
$$B = P \begin{pmatrix} \lambda_{-R} & & & & \\ & \lambda_{-R+1} & & 0 & \\ & & \ddots & & \\ & 0 & & \lambda_{R-1} & \\ & & & & \lambda_R \end{pmatrix} Q.$$

As before, $P(n|m; s)$ is the $(m, n)$ element of $B^s$, where

(50)
$$B^s = P \begin{pmatrix} \lambda_{-R}^s & & & & \\ & \lambda_{-R+1}^s & & 0 & \\ & & \ddots & & \\ & 0 & & \lambda_{R-1}^s & \\ & & & & \lambda_R^s \end{pmatrix} Q.$$

In order to perform the diagonalization (49) explicitly we start (following the procedure of Section 3) by trying to find the eigenvalues and the right eigenvectors of $B$. For this purpose we consider the system of linear equations

(51)
$$\begin{aligned} \frac{1}{2R}\, x_1 &= \lambda x_0 \\ x_0 + \frac{2}{2R}\, x_2 &= \lambda x_1 \\ \left(1 - \frac{1}{2R}\right) x_1 + \frac{3}{2R}\, x_3 &= \lambda x_2 \\ \cdots\cdots\cdots\cdots & \\ \frac{1}{2R}\, x_{2R-1} &= \lambda x_{2R}, \end{aligned}$$

and the auxiliary infinite system

(52)
$$\begin{aligned} \frac{1}{2R}\, x_1 &= \lambda x_0 \\ x_0 + \frac{2}{2R}\, x_2 &= \lambda x_1 \\ \left(1 - \frac{1}{2R}\right) x_1 + \frac{3}{2R}\, x_3 &= \lambda x_2 \\ \cdots\cdots\cdots\cdots & \end{aligned}$$

$$\frac{1}{2R}x_{2R-1}+\frac{2R+1}{2R}x_{2R+1} = \lambda x_{2R}$$

$$\frac{2R+2}{2R}x_{2R+2} = \lambda x_{2R+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

If we can find non-trivial solutions of (52) for which

$$x_{2R+1} = 0 \tag{53}$$

we will have found solutions of (51). It will turn out that this procedure will again yield all eigenvalues and right eigenvectors. Multiplying the members of the equations of (52) by $1, z, z^2, \cdots$, and adding, we obtain formally

$$\sum_{k=0}^{\infty}\left(1-\frac{k}{2R}\right)x_k z^{k+1}+\sum_{k=0}^{\infty}\frac{k}{2R}x_k z^{k-1} = \lambda\sum_{k=0}^{\infty}x_k z^k,$$

or, introducing the abbreviation

$$f(z) = \sum_{k=0}^{\infty}x_k z^k,$$

$$zf(z)-\frac{z^2}{2R}f'(z)+\frac{1}{2R}f'(z) = \lambda f(z).$$

We thus get the differential equation

$$f'(z) = 2R\frac{\lambda - z}{1-z^2}f(z), \tag{54}$$

whose solution satisfying $f(0)=x_0$ is easily found to be

$$f(z) = x_0(1-z)^{R(1-\lambda)}(1+z)^{R(1+\lambda)}. \tag{55}$$

Since $f(z)$ is analytic in the neighborhood of $z=0$ the formal procedure can be justified.

We now notice that if

$$\lambda = \frac{j}{R}, \qquad j = -R, -R+1, \cdots, 0, \cdots, R-1, R, \tag{56}$$

$f(z)$ is a polynomial of degree $2R$, and hence $x_{2R+1}=0$. The numbers (56) are thus seen to be eigenvalues of $B$ and, since there are $2R+1$ of them, we see that we have found *all* the eigenvalues. It also follows that the components of the right eigenvector belonging to the eigenvalue $\lambda_j=j/R$ can be taken as

$$C_0^{(j)} = 1,\ C_1^{(j)},\ C_2^{(j)}, \cdots, C_{2R}^{(j)},$$

where the $C$'s are defined by the identity

$$(1 - z)^{R-j}(1 + z)^{R+j} \equiv \sum_{k=0}^{2R} C_k^{(j)} z^k. \tag{57}$$

So far we have followed very closely the procedure described in Section 3. Surprisingly enough, we encounter unexpected difficulties in trying to carry out the analogy still further and determine by similar means the left eigenvectors.

To find the matrix $Q$ we resort to a different method. Let us first recall that $P$ can be taken as the matrix whose $j$th column (for convenience columns and rows are numbered from $-R$ to $R$) is

$$\begin{matrix} 1 \\ C_1^{(j)} \\ C_2^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ C_{2R}^{(j)}. \end{matrix}$$

Matrix $Q$ must satisfy the equation

$$P'Q' = I,$$

which is an immediate consequence of the equation $PQ=I$, and hence denoting by $\alpha_{-R}, \cdots, \alpha_0, \cdots, \alpha_R$, the consecutive elements of the $j$th column of $Q'$, we must have

$$\sum_{k=-R}^{R} C_{R+r}^{(k)}\alpha_k = \delta(j, r), \qquad r = -R, \cdots, R. \tag{58}$$

From (58) it follows that

$$z^{R+j} = \sum_{r=-R}^{R} \delta(j, r) z^{R+r} = \sum_{r=-R}^{R} z^{R+r} \sum_{k=-R}^{R} C_{R+r}^{(k)}\alpha_k = \sum_{k=-R}^{R} \alpha_k \sum_{r=-R}^{R} C_{R+r}^{(k)} z^{R+r}$$

$$= \sum_{k=-R}^{R} \alpha_k \sum_{s=0}^{2R} C_s^{(k)} z^s,$$

or, by virtue of (57),

$$z^{R+j} = \sum_{k=-R}^{R} \alpha_k(1 - z)^{R-k}(1 + z)^{R+k} = (1 - z)^{2R} \sum_{l=0}^{2R} \alpha_{l-R}\left(\frac{1 + z}{1 - z}\right)^l.$$

Thus

$$\frac{z^{R+j}}{(1 - z)^{2R}} = \sum_{l=0}^{2R} \alpha_{l-R}\left(\frac{1 + z}{1 - z}\right)^l. \tag{59}$$

Let

$$\zeta = \frac{1+z}{1-z},$$

so that

$$z = -\frac{1-\zeta}{1+\zeta} \quad \text{and} \quad 1 - z = \frac{2}{1+\zeta}\cdot$$

In terms of $\zeta$ (59) assumes the form

$$\frac{(-1)^{R+j}}{2^{2R}}(1-\zeta)^{R+j}(1+\zeta)^{R-j} = \sum_{l=0}^{2R} \alpha_{l-R}\zeta^l, \tag{60}$$

and since by (57)

$$(1-\zeta)^{R+j}(1+\zeta)^{R-j} = \sum_{l=0}^{2R} C_l^{(-j)}\zeta^l,$$

we obtain, by comparing coefficients of corresponding powers of $\zeta$,

$$\alpha_{l-R} = \frac{(-1)^{R+j}}{2^{2R}} C_l^{(-j)},$$

or finally,

$$\alpha_s = \frac{(-1)^{R+j}}{2^{2R}} C_{R+s}^{(-j)}. \tag{61}$$

Formula (61) determines explicitly the elements of $Q'$ (and hence of $Q$), and it is now possible to write an explicit expression for $P(n|m;s)$. In fact, making use of (50), we obtain

$$P(n \mid m; s) = \frac{(-1)^{R+n}}{2^{2R}} \sum_{j=-R}^{R} \left(\frac{j}{R}\right)^s C_{R+j}^{(-n)} C_{R+m}^{(j)}. \tag{62}$$

In the limit

$$\Delta \to 0, \qquad \tau \to 0, \qquad \frac{\Delta^2}{2\tau} = D, \qquad \frac{1}{R\tau} \to \gamma, \qquad s\tau = t, \qquad n\Delta \to x_0,$$

we have

$$\lim \sum_{x_1 < m\Delta < x_2} P(n \mid m; s) = \int_{x_1}^{x_2} P(x_0 \mid x; t)\,dx,$$

where

$$P(x_0 \mid x; t) = \frac{\sqrt{\gamma}}{\sqrt{2\pi D(1-e^{-2\gamma t})}} e^{-\gamma(x - x_0 e^{-\gamma t})^2/2\gamma(1-e^{-2\gamma t})}. \tag{63}$$

The proof is again made to depend on the continuity theorm for Fourier-Stieltjes transforms.

The frequency function (63) was first discovered by Lord Raleigh [10]. Its connection with Brownian motion of an elastically bound particle, in the strongly overdamped case, was established by Smoluchowski who arrived at it quite independently.

5. **The Ehrenfest model. Irreversibility and recurrence.** Imagine $2R$ balls numbered consecutively from 1 to $2R$, distributed in two boxes (I and II) so that at the beginning there are $R+n$, $-R\leqq n\leqq R$, balls in box I. We chose at random an integer between 1 and $2R$ (all these integers are assumed to be equiprobable) and move the ball, whose number has been drawn from the box in which it is, to the other box. This process is then repeated $s$ times and we ask for the probability $Q(R+n|R+m;\ s)$ that after $s$ drawings there should be $R+m$ balls in box I.

A moment's reflection will persuade one that this formulation (originally proposed by P. and T. Ehrenfest) [7] is equivalent to the random walk formulation of Section 4, if one interprets the excess over $R$ of balls in box I as the displacement of the particle $(\Delta=1)$. Thus

$$Q(R + n \mid R + m;\ ) = P(n \mid m; s),$$

where $P(n|m; s)$ has the meaning of Section 4.

In the present formulation we have a simple and convenient model of heat exchange between two isolated bodies of unequal temperatures. The temperatures are symbolized by the numbers of balls in the boxes and the heat exchange is not an orderly process, as in classical thermodynamics, but a random one like in the kinetic theory of matter. The realistic value of the model is greatly enhanced by the fact that the average excess over $R$ of the number of balls in box I, namely, the quantity

$$\sum_{m=-R}^{R} mP(n \mid m; s)$$

can easily be shown to be equal to

$$n\left(1 - \frac{1}{R}\right)^{s} \tag{64}$$

which in the limit $R\rightarrow\infty$, $1/R\tau\rightarrow\gamma$, $s\tau=t$, gives

$$ne^{-\gamma t},$$

or the Newton law of cooling.

There are several proofs of (64) [11]. The most straightforward one, which is not however the simplest, is based on formula (62).

The Ehrenfest model is also particularly suited for the discussion of a famous paradox which at the turn of this century nearly wrecked Boltzmann's inspired

efforts to explain thermodynamics on the basis of kinetic theory. In classical thermodynamics the process of heat exchange of two isolated bodies of unequal temperatures is irreversible. On the other hand, if the bodies are treated as a dynamical system the famed "Wiederkehrsatz" of Poincaré asserts that "almost every" state (except for a set of states which, when interpreted as points in phase space, form a set of Lebesgue measure 0) of the system will be, to an arbitrarily prescribed degree of accuracy, again approximately achieved. Thus, argued Zermelo, the irreversibility postulated in thermodynamics and the "recurrence" properties of dynamical systems are irreconcilable. Boltzmann then replied that the "Poincaré cycles" (time intervals after which states "nearly recur" for the first time,—the word "nearly" requiring further specification) are so long compared to time intervals involved in ordinary experiences that predictions based on classical thermodynamics can be fully trusted. This explanation, though correct in principle, was set forth in a manner which was not quite convincing and the controversy raged on. It was mainly through the efforts of Ehrenfest and Smoluchowski that the situation became completely clarified, and the irreversibility interpreted in a proper statistical manner.

It will now be easy to discuss this explanation by appealing to the Ehrenfest model. Let $P'(n|m;\, s)$ denote the probability that after $s$ drawings (the duration of each drawing is $\tau$) $R+m$ balls will be observed *for the first time* in box I if there were $R+n$ balls in that box at the beginning. In particular, $P'(n|n;\, s)$ is the probability that the recurrence time of the state "$n$" (defined by the presence of $R+n$ balls in box I) is $s\tau$. One can then show that

$$\sum_{s=1}^{\infty} P'(n \mid n;\, s) = 1, \tag{65}$$

or, in other words: *each state is bound to recur with probability 1.* This is the statistical analogue of the "Wiederkehrsatz." One can show furthermore that the mean recurrence time, namely, the quantity

$$\theta_n = \sum_{s=1}^{\infty} s\tau P'(n \mid n;\, s)$$

is equal to

$$\tau \frac{(R+n)!(R-n)!}{(2R)!} 2^{2R}. \tag{66}$$

This is the statistical analogue of a "Poincaré cycle," and it tells us, roughly speaking, how long, on the average, one will have to wait for the state "$n$" to recur.

If $R+n$ and $R-n$ differ considerably, $\theta_n$ is enormous. For example, if $R=10000$, $n=10000$, $\tau=1$ second, we get

$$\theta = 2^{20000} \text{ seconds (of the order of } 10^{6000} \text{ years!).}$$

If on the other hand, $R+n$ and $R-n$ are nearly equal, $\theta_n$ is quite short. If in the above example we set $n=0$ we get (using Stirling's formula)

$$\theta \sim 100\sqrt{\pi} \text{ seconds} \sim 175 \text{ seconds.}$$

It was Smoluchowski who advanced the rule [12] that if one starts in a state with a long recurrence time the process will appear as irreversible. In our example if one starts with 20000 balls in one box and none in the other, one should observe, for a long time, an essentially irreversible flow of balls. On the other hand, if the mean recurrence time is short, there is no sense to speak about irreversibility.

We now give the proofs of (65) and (66). We shall base our considerations on a formula which Professor Uhlenbeck used for similar purposes in some of his unpublished notes. The formula in question is:

$$P(n \mid m; s) = P'(n \mid m; s) + \sum_{k=1}^{s-1} P'(n \mid m; k)P(m \mid m; s - k). \tag{67}$$

To convince oneself of the validity of this formula we divide all possible ways of reaching "$m$" from "$n$" in $s$ steps into classes according to when "$m$" has been reached for the first time. We then observe that starting from "$n$" one can reach "$m$" in $s$ steps in the following $s$ mutually exclusive ways:

(1) "$m$" is reached for the first time after $s$ steps.

(2) "$m$" is reached for the first time in 1 step and then, starting from "$m$" it is again reached in $s-1$ steps.

(3) "$m$" is reached for the first time in 2 steps and then, starting from "$m$", it is reached again in $s-2$ steps, and so forth. We note furthermore that the probability that "$m$" will be reached for the first time in $k$ steps and then, starting from "$m$," it will be reached again in $s-k$ steps, is

$$P'(n \mid m; k)P(m \mid m; s - k). \tag{68}$$

This completes the proof of (67).

It should be emphasized that the justification of using the product of probabilities in (68) rests upon the fact that in our process the past is independent of the future. In other words, once we know that the system starts, say, from "$m$," its subsequent behavior is independent of the way in which "$m$" was reached in the first place.

We introduce now the generating functions

$$h(n \mid m; z) = \sum_{s=1}^{\infty} P(n \mid m; s)z^s \tag{69}$$

$$g(n \mid m; z) = \sum_{s=1}^{\infty} P'(n \mid m; s)z^s, \tag{70}$$

and note that (67) is equivalent to

$$h(n \mid m; z) = g(n \mid m; z) + h(m \mid m; z)g(n \mid m; z),$$

or

$$g(n \mid m; z) = \frac{h(n \mid m; z)}{1 + h(m \mid m; z)}. \tag{71}$$

In particular,

$$g(n \mid n; z) = \frac{h(n \mid n; z)}{1 + h(n \mid n; z)} = 1 - \frac{1}{1 + h(n \mid n; z)}, \tag{72}$$

and we also note that

$$\frac{dg(n \mid n; z)}{dz} = \frac{\dfrac{dh(n \mid n; z)}{dz}}{(1 + h(n \mid n; z))^2}. \tag{73}$$

From the definition of $g(n \mid n; z)$, we obtain

$$\lim_{z \to 1} g(n \mid n; z) = \sum_{s=1}^{\infty} P'(n \mid n; s) \tag{74}$$

$$\tau \lim_{z \to 1} \frac{dg(n \mid n; z)}{dz} = \sum_{s=1}^{\infty} s\tau P'(n \mid n; s). \tag{75}$$

It is from these formulas that we shall derive (65) and (66). We have, using (62)

$$h(n \mid n; z) = \frac{(-1)^{R+n}}{2^{2R}} \sum_{j=-R}^{R} \sum_{s=1}^{\infty} \left(\frac{jz}{R}\right)^s C_{R+j}^{(-n)} C_{R+n}^{(j)},$$

and since

$$1 = \frac{(-1)^{R+n}}{2^{2R}} \sum_{j=-R}^{R} C_{R+j}^{(-n)} C_{R+n}^{(j)},$$

we obtain

$$1 + h(n \mid n; z) = \frac{(-1)^{R+n}}{2^{2R}} \sum_{j=-R}^{R} \frac{1}{1 - \dfrac{j}{R} z} C_{R+j}^{(-n)} C_{R+n}^{(j)}. \tag{76}$$

All terms in the sum on the right hand side of (76) are regular at $z = 1$ except the term corresponding to $j = R$, which has a simple pole at that point. Thus we can write

$$1 + h(n \mid n; z) = p(z) + \frac{(-1)^{R+n}}{2^{2R}} C_{2R}^{(-n)} C_{R+n}^{(R)} \frac{1}{1 - z},$$

where $p(z)$ is regular at $z = 1$. We see that

$$\lim_{z \to 1} (1 + h(n \mid n; z)) = \infty$$

and hence, using (72) and (74)

$$\sum_{s=1}^{\infty} P'(n \mid n; s) = 1.$$

It is easy to see that

$$\frac{(-1)^{R+n}}{2^{2R}} C_{2R}^{(-n)} C_{R+n}^{(R)} = \frac{1}{2^{2R}} \frac{(2R)!}{(R+n)!(R-n)!},$$

and, denoting this expression by $\omega$, we have (for $|z| > 1$)

$$\frac{dg(n \mid n; z)}{dz} = \frac{(1-z)^2 p'(z) + \omega}{[(1-z)p(z) + \omega]^2},$$

and hence

$$\lim_{z \to 1} \frac{dg(n \mid n; z)}{dz} = \frac{1}{\omega}.$$

This together with (75) yields (66).

The above considerations can be extended to more general processes. However, Markoffian processes (*i.e.*, processes for which (68) is valid) are still the only ones for which one can also calculate the "fluctuation" of the recurrence time, namely, the quantity

$$(77) \qquad \sum_{s=1}^{\infty} s^2 \tau^2 P'(n \mid n; s) - \theta_n^2.$$

Without going into the details, let us mention that (77) can be calculated in terms of

$$\lim_{z \to 1} \frac{d^2 g(n \mid n; z)}{dz^2}.$$

The fluctuation (77) gives us a measure of stability of the mean recurrence time inasmuch as it permits us to estimate how likely (or unlikely) it is to get a specified deviation of the actual recurrence time from the mean. It may seem that since the generating function $g(n \mid n; z)$ is known explictly it should be easy to get an explicit expression for $P'(n \mid n; s)$. This, however, is not the case. We have not succeeded in finding such an expression, except for $P'(0 \mid 0; s)$, and even then we had to use a different method. We shall give a brief description of this method. Let

$$P(n \mid m; 1) = p_{nm}.$$

Then,

$$P'(n \mid n; s) = \sum_{m_1, \cdots, m_{s-1}}' p_{nm_1} p_{m_1 m_2} \cdots p_{m_{s-1} n},$$

where the accent on the summation sign indicates that $m_j \neq n$, $j=1, 2, \cdots, s-1$. Now let

$$\epsilon_i = \begin{cases} 0 & \text{if } i = n \\ 1 & \text{if } i \neq n. \end{cases}$$

Noticing that

$$\epsilon_i^2 = \epsilon_i,$$

we can write

$$P'(n \mid n; s) = \sum_{m_1, \cdots, m_{s-1}} p_{nm_1}\epsilon_{m_1}p_{m_1m_2}\epsilon_{m_2}p_{m_2m_3} \cdots \epsilon_{m_{s-2}}p_{m_{s-2}m_{s-1}}\epsilon_{m_{s-1}}p_{m_{s-1}n},$$

where the summation is now extended over all $m_j$. If $B$ is the matrix

$$((p_{nm})),$$

and $B_1$ the matrix

$$((\epsilon_n p_{nm} \epsilon_m)),$$

we see that

$$P'(n \mid n; s) = (n, n) \text{ element of } BB_1^{s-2}B. \tag{78}$$

We may note that $B_1$ is obtained from $B$ by crossing out the $n$th row and the $n$th column of the latter, and replacing them by a row and column consisting entirely of zeros. If $B_1$ can be explicitly diagonalized, that is, written in the form

$$B_1 = P_1 \begin{pmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ & & \cdot \\ 0 & & & \cdot \end{pmatrix} Q_1,$$

where

$$P_1 Q_1 = I$$

one can calculate $P'(n \mid n; s)$, explicitly using (78).

We have applied this method to the Ehrenfest model, but only in the case when the middle (zeroth) row and column of $B$ are replaced by a row and column consisting entirely of zeros have we been able to diagonalize explicitly the resulting matrix $B_1$. The diagonalization proceeds very much as in Section 4, but it has proved necessary to distinguish between the cases when $R$ is even or odd. In case $R$ is even, we were able to derive the formula

$$P'(0 \mid 0; s) = -\frac{1}{2^{2R-1}} \frac{R+1}{2R} \sum \left(\frac{j}{R}\right)^{s-2} C_{R-j}^{(-1)} C_{R-1}^{(j)}, \qquad s \geqq 2, \tag{79}$$

where the summation is extended over all odd integers $j$ between $-R$ and $R$. The details of the derivation are somewhat tedious and will not be reproduced here. Formula (79) furnishes a partial solution to a question left open by Wang and Uhlenbeck [1].

#### References

1. An extensive list of references to the physical literature can be found in the following articles: G. E. Uhlenbeck and L. S. Ornstein, On the theory of Brownian motion, Phys. Rev. 36 (1930) pp. 823–841, M. C. Wang and G. E. Uhlenbeck, On the theory of Brownian motion II, Rev. Mod. Phys. 17 (1945) pp. 323–342. For a very complete summary of earlier results see the important paper of Smoluchowski, Drei Vortrage über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen, Phys. Zeit. 17 (1916) pp. 557–571 and 585–599.

2. References to the work of Wiener, Kolmogoroff, Doob and Feller are given in the second article in [1]. See also P. Lévy, Sur certains processus stochastiques homogènes, Comp. Math. 7 (1939) pp. 283–339, R. Fortet, Les fonctions aléatoires du type de Markoff associées à certain équations paraboliques, Jour. de Math. 22 (1943) pp. 177–243.

3. For a brief discussion of the nature of the friction coefficient $f$ see the first article in [1].

4. For proofs of this theorem the reader is referred to H. Cramér, Mathematical Methods of Statistics, Princeton University Press (1946), in particular, pp. 198–203.

5. For a complete presentation of the matrix method as applied to Markoff chains see M. Fréchet, Traité du Calcul des Probabilités, Tome I, Fasc. III Second Livre, Paris, Gauthier-Villars (1936).

6. For the proof of the continuity theorem in its most general form see Cramér's book [4] pp. 96–100.

7. Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem, Phys. Zeit. 8 (1907) pp. 311–314.

8. Das Ehrenfestsche Model der H-Kurve, Phys. Zeit. 27 (1926), pp. 306–313.

9. M. Kac, Bull. Am. Math. Soc. 52 (1946) p. 621 (abstract).

10. See footnote 27 in the second paper of [1].

11. See the paper [8] and for a derivation based on an entirely different principle H. Steinhaus, La théorie and les applications des fonctions indépendantes au sens stochastique, Actualités Scientifiques et Industrielles 738 (1938) Paris, Hermann et Cie, pp. 57–73, in particular, pp. 61–64.

12. See the third article in [1] p. 568.

# VIBRATION MODES OF TAPERED BEAMS

EDMUND PINNEY, University of California

The vibration modes of harmonically oscillating thin beams are obtained by solving the differential equation

$$(d^2/dx^2)(EId^2y/dx^2) - \rho\omega^2 y = 0 \tag{1}$$

subject to certain boundary conditions. In this differential equation, $x$ is the distance along the beam, $y$ is the amplitude of the transverse vibration, $E$ is Young's modulus for the beam material, $I$ is the cross-sectional moment of inertia of the beam, $\rho$ is its linear density, and $\omega$ is the circular frequency of vibration.

When $EI$ and $\rho$ are constant along the beam the solution of (1) is simple, and may be expressed in terms of trigonometric and hyperbolic functions. The frequency $\omega$ will be a root of a simple transcendental equation. Such theories are to be found in most textbooks on vibration theory.

When $EI$ and $\rho$ vary very slowly with $x$, solutions to (1) may be obtained by writing $EI=E_0I_0[1+\xi(x)]$, $\rho=\rho_0[1+\eta(x)]$, $y=y_0(x)+\zeta(x)$, where $E_0I_0$ and $\rho_0$ are constants, where $y_0(x)$ is a linear combination of

$$\text{(2)} \qquad \operatorname{sh}(kx), \quad \operatorname{ch}(kx), \quad \sin(kx), \quad \cos(kx),$$

where the quantities $\xi$, $\eta$, $\zeta/y_0$ are much smaller than unity, and where

$$\text{(3)} \qquad k^4 = \rho_0\omega^2/E_0I_0.$$

Substituting into (1) and neglecting terms of higher order than the first in $\xi$, $\eta$, $\zeta$, we obtain

$$\text{(4)} \qquad d^4\zeta/dx^4 - k^4\zeta = f(x),$$

where

$$\text{(5)} \qquad f(x) = -(d^2/dx^2)(\xi d^2y_0/dx^2) + k^4y_0\eta.$$

Equation (4) has a particular solution

$$\text{(6)} \qquad \zeta(x) = \tfrac{1}{2}k^{-3}\int^x f(t)[\operatorname{sh} k(x-t) - \sin k(x-t)]dt.$$

The general solution is obtained by adding to this an arbitrary linear combination of the functions in (2).

If $EI$ and $\rho$ may be represented by a finite number of terms of a Fourier series, then $y$ may be obtained from (1) as a Fourier series whose coefficients are the solutions of certain algebraic equations [1; §VII]. This method is useful for beams of greater taper than those to which the analysis of the preceding paragraph may be applied. Another method of great practical importance is due to Myklestad [2; Ch. VI].

When $EI$ and $\rho$ are certain particular functions of $x$, it is possible to obtain exact solutions of (1) in terms of known functions. One case of considerable interest is that in which

$$\text{(7)} \qquad EI = E_0I_0x^{\mu+2}, \qquad \rho = \rho_0x^{4\beta+\mu-2},$$

where $E_0I_0$ and $\rho_0$ are constants. The case where $\beta=0$ leads to solutions of the form $x^p$ where $p(p-1)(p+\mu)(p+\mu-1)=k^4$. If one of the four conditions

$$\text{(8)} \qquad \beta = 1/2, \qquad \beta = -1/2, \qquad \beta = \mu/2, \qquad \beta = -\mu/2$$

holds, the vibration modes of the beam may be expressed in terms of Bessel functions.

To show this, we insert (7) into (1), getting

(9) $$x^2(d^2/dx^2)(x^{\mu+2}d^2y/dx^2) - k^4x^{4\beta+\mu}y = 0.$$

If we write

$$\theta = x\frac{d}{dx},$$

it follows that

$$x^2\frac{d^2}{dx^2} = \theta(\theta - 1)$$

and that

$$\theta(x^n f) = x^n(\theta + n)f.$$

Using these, we may write (9) in the form

(10) $$\theta(\theta - 1)(\theta - \mu)(\theta - \mu - 1)y = k^4x^{4\beta}y.$$

By [3; §4.5], the equation

(11) $$(\theta + \alpha)(\theta + \alpha - 2\beta)(\theta + \alpha - 2\beta\nu)(\theta + \alpha - 2\beta - 2\beta\nu)y = \beta^4c^4x^{4\beta}y$$

has the solution

(12) $$y = x^{\beta\nu-\alpha}[AJ_\nu(cx^\beta) + BY_\nu(cx^\beta) + CI_\nu(cx^\beta) + DK_\nu(cx^\beta)].$$

Comparing (10) and (11), we see that solutions may be obtained in the following cases:

(13)

| *Type* | $\nu$ | $\beta$ | $\beta\nu-\alpha$ | $c$ |
|---|---|---|---|---|
| I | $\mu$ | $1/2$ | $\mu/2$ | $2k$ |
| II | $-\mu$ | $-1/2$ | $\mu/2+1$ | $2k$ |
| III | $1/\mu$ | $\mu/2$ | $1/2$ | $2k/\mu$ |
| IV | $1/\mu$ | $-\mu/2$ | $\mu+1/2$ | $2k/\mu$ |

Beams of type *I* are of particular interest because the exponent for *EI* is 2 greater than that for $\rho$, a situation often encountered in practice. Examples are the wedge, corresponding to $\mu=1$, and the cone, corresponding to $\mu=2$. For modes of type *I*

(14) $$y = F_\mu(A, B, C, D; x),$$

where

(15) $$F_\mu(A, B, C, D; x) = x^{\mu/2}[AJ_\mu(2kx^{1/2}) + BY_\mu(2kx^{1/2}) + CI_\mu(2kx^{1/2}) + DK_\mu(2kx^{1/2})].$$

By [3; §§3.2, 3.56, 3.71], it is known that

(16) $$F_\mu'(A, B, C, D; x) = kF_{\mu-1}(A, B, C, -D, x),$$

and

(17) $F_{\mu+1}(A, B, C, D; x) + xF_{\mu-1}(A, B, -C, -D; x) = (\mu/k)F_\mu(A, B, -C, D; x).$

It follows from (16) that the vibration frequencies for a free-free beam (end conditions: $y''=0$, $y'''=0$) of type $I$ are the same as those of a similar clamped-clamped beam (end conditions: $y=0$, $y'=0$) of type $I$ in which $\mu$ is replaced by $\mu-2$.

For beams of types I, II, III, and IV, (12) provides the basis for the solution of a large number of beam vibration problems. The determination of the natural period of vibration of such a beam usually leads to a fourth order determinant whose elements are Bessel functions. The roots can be calculated numerically without very great difficulty if the Bessel functions are tabluated. Forced vibration problems of such beams are similar in theory and complexity.

In the preliminary design of aircraft this theory may be useful in obtaining approximate vibration modes of airplane wings to be used in preliminary flutter calculations. Many wings have tip chords that are small in comparison with their root chords. Such winds may, with little error, be considered to taper to a point at a distance $a$, say, from the root of the wing. The wing should be approximately of type I.

To avoid singularities, take $B=D=0$ in (15). Also $y=0$, $y'=0$ when $x=a$ for a wing that may be considered to be cantilevered at the root. From (15) and (16), we have

(18) $$F_\mu(A, 0, C, 0; a) = 0, \qquad F_{\mu-1}(A, 0, C, 0; a) = 0.$$

By (3), (15), if $z_1, z, \cdots$, are roots of

(19) $$J_\mu(z)/J_{\mu-1}(z) = I_\mu(z)/I_{\mu-1}(z),$$

then the vibration frequencies, $\omega_1, \omega_2, \cdots$, are given by

(20) $$\omega_n = (z_n^2/4a)(E_0I_0/\rho_0)^{1/2}, \qquad n = 1, 2, \cdots.$$

The ratio $C/A$ may then be computed from (18).

For wings that cannot be considered to taper to a point, this determination may still be useful in providing a first approximation for the frequency calibration.

By (13), the vibration modes may be expressed in terms of elementary functions when $\mu=n+\frac{1}{2}$ for beams of types I and II, and when $\mu=2/(2n+1)$ for beams of types III and IV, $n$ being an integer.

**References**

1. B. F. Ruffner, The use of Fourier series in the solution of beam-column problems, Bulletin Series no. 21, Engineering Experiment Station, Oregon State College.

2. N. O. Myklestad, Vibration Analysis, McGraw-Hill Co., 1944.

3. G. N. Watson, Theory of Bessel Functions, Cambridge University Press, 1944.

# TETRAHEDRONS HAVING A COMMON FACE*

VICTOR THÉBAULT, Tennie, Sarthe, France

The main theorem of this paper is: *Consider two tetrahedrons* $T \equiv ABCD$, $T' \equiv A'BCD$, *having a common face* $BCD$. *Let the perpendiculars at* $A$ *to faces* $CDA$, $DAB$, $ABC$ *of* $T$ *meet the planes of the faces* $A'CD$, $A'DB$, $A'BC$ *of* $T'$ *in* $A_1$, $A_2$, $A_3$, *and the perpendiculars at* $A'$ *to faces* $A'CD$, $A'DB$, $A'BC$ *of* $T'$ *meet the planes of faces* $CDA$, $DAB$, $ABC$ *of* $T$ *in* $A'_1$, $A'_2$, $A'_3$. *Then the perpendiculars from* $A$ *to plane* $A_1A_2A_3$ *and from* $A'$ *to plane* $A'_1A'_2A'_3$ *meet in a point in the common face* $BCD$.

If we designate by $\beta$, $\gamma$, $\delta$ the angles made by planes $A'CD$, $A'DB$, $A'BC$ with the face $BCD$, and further designate by $a$ and $a'$, $b$ and $b'$, $c$ and $c'$ the dihedral angles having for edges $BC$ and $DA$, $\cdots$, of $T$, the equations of the perpendicular from the vertex $A$ to the plane $A_1A_2A_3$ are:

$$\frac{y}{\sin c' \cot (c' - \beta)} = \frac{z}{\sin b' \cot (b' - \gamma)} = \frac{t}{\sin a \cot (a - \delta)}.$$

This perpendicular meets the plane of face $BCD$ in a point $A''$ whose normal coördinates with respect to the triangle $BCD$ are:

$$(y', z', t') \sim \cot (c' - \beta), \cot (b' - \gamma), \cot (a - \delta).$$

Because of symmetry, the perpendicular from $A'$ to the plane $A'_1A'_2A'_3$ meets the plane of face $BCD$ in the same point $A''$. Hence the theorem.

If the tetrahedron $T$ is orthocentric and $A' \equiv H$, the feet of the altitudes of triangle $BCD$ being $B_1$, $C_1$, $D_1$, then

$$\frac{y'}{\tan B_1AB} = \frac{z'}{\tan C_1AC} = \frac{t'}{\tan D_1AD}.$$

But in the triangle $D_1AD$,

$$AH = \frac{DD_1}{\tan D_1AD} \quad \text{and} \quad \frac{y'}{BB_1} = \frac{z'}{CC_1} = \frac{t'}{DD_1}.$$

Hence $A''(y', z', t')$ coincides with the centroid of $BCD$. This provides the solution for problem 4150 [1945, 102].

Our theorem also answers the following question which we have raised (*Annales de la Société Scientifique de Bruxelles*, 1925, p. 302) and which has heretofore remained unanswered. Two given tetrahedorns $T \equiv ABCD$ and $T' \equiv A'B'C'D'$ are homological. The perpendiculars at $A$ to the faces $CDA$, $DAB$, $ABC$ of $T$ meet the planes of faces $A'C'D'$, $A'D'B'$, $A'B'C'$ of $T'$ in $A_1$, $A_2$, $A_3$, and the perpendiculars at $A'$ to the faces $A'C'D'$, $A'D'B'$, $A'B'C'$ of $T'$ meet the planes of faces $CDA$, $DAB$, $ABC$ of $T$ in $A'_1$, $A'_2$, $A'_3$. In the same way we obtain, relative to the vertices $B$, $C$, $D$ of $T$ and $B'$, $C'$, $D'$ of $T'$, the triads of

* Translation from the French by J. B. Lennes, University of Oklahoma.

points ($B_1, B_2, B_3$) and $B'_1, B'_2, B'_3$), $\cdots$. The perpendiculars from $A, B, C, D$ to the planes $A_1A_2A_3$, $B_1B_2B_3$, $\cdots$, meet in a point $O$, and the perpendiculars from $A', B', C', D'$ to the planes $A'_1A'_2A'_3$, $B'_1B'_2B'_3$, $\cdots$, meet in a point $O'$. Are $O$ and $O'$ corresponding points in the homology $(ABCD, A'B'C'D')$?

The answer is affirmative, for it is sufficient to show that lines $AO$ and $A'O'$, $BO$ and $B'O'$, $CO$ and $C'O'$, $DO$ and $D'O'$ meet the plane of homology $(P)$ in the same points, as shown by the preceding theorem. It can be added that the point $O$ is on the perpendicular to plane $(P)$ through the isogonal conjugate $O_1$ of the center of homology with respect to the tetrahedron $T$, and that $O'$ is on the perpendicular to $(P)$ through the isogonal conjugate $O'$ of the center of homology with respect to $T'$.

In the general case first treated above, the perpendicular $AX$ from a point $P\equiv A'$ whose normal coördinates in $T$ are $(x_1, y_1, z_1, t_1)$ to the plane $A_1A_2A_3$ has the following equations:

$$\frac{y}{\dfrac{\cos c'}{x_1}+\dfrac{1}{y_1}}=\frac{z}{\dfrac{\cos b'}{x_1}+\dfrac{1}{z_1}}=\frac{t}{\dfrac{\cos a}{x_1}+\dfrac{1}{t_1}},$$

and meets the plane of face $BCD$ in a point $P'$ whose coördinates are:

$$\frac{y}{\dfrac{\cos c'}{x_1}+\dfrac{1}{y_1}}=\cdots=\frac{3V}{\dfrac{A}{x_1}+\dfrac{B}{y_1}+\dfrac{C}{z_1}+\dfrac{D}{t_1}},$$

$A, B, C, D$ being the areas of the faces and $V$ the volume of $T$.

(a) If $P$ is on Cayley's cubic surface

$$\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{t}=0,$$

the line $AX\equiv AP'$ is parallel to plane $BCD$ which is normal to plane $A_1A_2A_3$. In particular, this is the case when $P$ coincides with the midpoint of one of the 28 line-segments joining the centers of the spheres tangent to the four planes of the faces of $T$.

(b) The equations of the perpendiculars $BY$, $CZ$, $DT$ from $B$, $C$, $D$ to the planes $B_1B_2B_3$, $C_1C_2C_3$, $D_1D_2D_3$ (paragraph 2) are similar in form to that of $AX$. The following relations are necessary and sufficient in order that the four lines $AX$, $BY$, $CZ$, $DT$ shall be generators of a hyperboloid:

$$\frac{\cos c'}{x_1}+\frac{1}{y_1}=\frac{\cos c'}{y_1}+\frac{1}{x_1},$$

$$\frac{\cos b'}{x_1}+\frac{1}{z_1}=\frac{\cos b'}{z_1}+\frac{1}{x_1},$$

$$\frac{\cos a}{x_1}+\frac{1}{t_1}=\frac{\cos a}{t_1}+\frac{1}{x_1}.$$

Hence $x_1=y_1=z_1=t_1$.

The point $P$ coincides with the center $I$ of the sphere inscribed in $T$ and the hyperboloid is the one formed by the lines joining the vertices to the points of tangency of the opposite faces with the inscribed sphere. These lines are concurrent if

$$\cos \tfrac{1}{2}a \cos \tfrac{1}{2}a' = \cos \tfrac{1}{2}b \cos \tfrac{1}{2}b' = \cos \tfrac{1}{2}c \cos \tfrac{1}{2}c',$$

that is, if the tetrahedron is isogonic.*

As a consequence we have the theorem: *In any tetrahedron $T\equiv ABCD$ let the perpendiculars at $A$ to the faces $CDA$, $DAB$, $ABC$, meet the bisecting planes $ICD$, $IDB$, $IBC$, $\cdots$, in $A_1$, $A_2$, $A_3$, $\cdots$. Then the perpendiculars from $A$, $B$, $C$, $D$, to the planes $A_1A_2A_3$, $B_1B_2B_3$, $\cdots$, pass through the points of contact of the inscribed sphere with the opposite faces.*

All the properties apply to the triangle. Given a triangle $ABC$ and a point $P$ in the same plane. The perpendiculars at $A$ to $AB$ and $AC$ meet $BP$ and $CP$ in $A_1$ and $A_2$; we have the analogous points $B_1$ and $B_2$, $C_1$ and $C_2$. The perpendiculars from $A$, $B$, $C$ to $A_1A_2$, $B_1B_2$, $C_1C_2$ meet $BC$, $CA$, $AB$ in $X$, $Y$, $Z$. Let $(AB, AP)=\alpha$, $(AP, AC)=\alpha'$, $(BC, BP)=\beta$, $(BP, BA)=\beta'$, $(CA, CP)=\gamma$, $(CP, CB)=\gamma'$. We have

$BX/XC=\tan\gamma/\tan\beta'$, $CY/YA=\tan\alpha/\tan\gamma'$, $AZ/ZB=\tan\beta/\tan\alpha'$,
and

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{\tan\alpha\tan\beta\tan\gamma}{\tan\alpha'\tan\beta'\tan\gamma'}. \tag{1}$$

Now, if $A'$, $B'$, $C'$ are the orthogonal projections of $P$ on $BC$, $CA$, $AB$, we have also

$$\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B}=\frac{\tan\alpha\tan\beta\tan\gamma}{\tan\alpha'\tan\beta'\tan\gamma'}. \tag{2}$$

When point $P$ describes the cubic of Darboux circumscribed about the triangle $ABC$, the lines $AA'$, $BB'$, $CC'$ are concurrent; the two members of equation (2) have the value $-1$, and, by (1), the lines $AX$, $BY$, $CZ$ meet in a point $Q$. Conversely, if lines $AX$, $BY$, $CZ$ are concurrent, point $P$ describes the cubic of Darboux.†

The locus of point $Q$ is the cubic of Lucas, and the perpendiculars to $BC$, $CA$, $AB$ from $X$, $Y$, $Z$ meet in the isogonal conjugate of $P$ with respect to triangle $ABC$.

* See Nathan Altshiller-Court, Modern Pure Solid Geometry, New York, 1935, p. 290, art. 879.

† We do not know if this method of generating the cubics of Darboux and Lucas has been noticed heretofore. (V.T.)

In particular, if in a triangle $ABC$, $P$ coincides with the orthocenter, points $X$, $Y$, $Z$ are the midpoints of $BC$, $CA$, $AB$, and $Q$ coincides with the centroid.

When $P$ describes the circumscribed circle, $AX$, $BY$, $CZ$ are parallel to $BC$, $CA$, $AB$.

Finally, if $P$ coincides with the center of one of the tritangent circles, $X$, $Y$, $Z$ are the points of contact of this circle with $BC$, $CA$, $AB$, and $Q$ is the Gergonne point of the triangle, or one of its associated points.

---

## INSTRUCTION AND RESEARCH IN APPLIED MATHEMATICS*

The history of mathematics reveals a continuous interplay of ideas between the mathematician and the experimental or practical scientist. In the last half-century this interplay has decreased, to the detriment, it may be believed, of both parties. The war caused a lively cooperation, but unless this continues under peacetime conditions, the prospect for the future is serious and warrants earnest consideration.

The widening gap between the mathematician and other scientists may be traced to several causes. The most obvious of these is the great activity in all branches of science, which has necessitated a high degree of specialization in each branch, and consequently less opportunity for roaming interests. However, side by side with the expansion of science, increased understanding of basic theory acts as a simplifying compensation. Such simplification is achieved largely through mathematical methods, of increasing depth and generality, and it would be indeed unfortunate if mathematicians should not be in a position to contribute towards this simplification.

As our educational system is at present organized, there are relatively few opportunities for a student of mathematics, undergraduate or graduate, to become acquainted with the mathematical structures of the theories underlying other sciences. His needs in this connection are not identical with those of students pursuing other branches of science as their major interest; many practical details, of importance to them, are not essential for his purpose, and he can go further and faster by concentration on the mathematical aspects of the subject. It is suggested that the study of some branches of other sciences from the mathematical standpoint (and that is a meaning commonly attached to the words "applied mathematics") should be regarded, wherever feasible, as an essential part of the training of mathematics majors, undergraduate or graduate. Apart from the general advantage to science which we might hope to see as a result of this procedure, it seems only fair to offer a young mathematician the opportunity of pursuing a career in applied mathematics if his natural inclinations are so directed. Unless he has the opportunity of viewing the field of applied

* Prepared by the special committee of the American Mathematical Society on applied mathematics, and approved by the Council of the Society and the Board of Governors of the Association.

mathematics from the mathematical standpoint, it is unlikely that he will be attracted to it if his interests are basically those of a mathematician.

Consequently it is suggested that departments of mathematics throughout the country should consider the feasibility of enlarging their offerings in the direction of applied mathematics, both on the undergraduate and graduate levels, insofar as the departments are qualified to offer such instruction. Many branches of applied mathematics have not only well established axiomatic mathematical structures, but are also fruitful fields of research, involving mathematical ideas and techniques of the highest order. Attention may in particular be directed to mathematical statistics, theoretical mechanics (including elasticity and fluid dynamics), statistical mechanics and thermodynamics, heat conduction, electromagnetic theory, relativity, quantum mechanics, genetics, and the theory of high polymers.

In some institutions the problem is already solved in part by the activities of other departments, containing members well qualified in mathematics. Nevertheless, since each of the subjects listed above has not only its mathematical side, but also an even greater experimental or practical side, it appears likely that full justice can be done to the mathematical theories only by a specialization on that aspect. Needless to say, the task of maintaining and increasing an interplay between mathematics and the other sciences is one that can be dealt with only by full cooperaton between all the parties concerned. It is to be expected that scientists in other fields would welcome the desire of young mathematicians to increase their knowledge of the mathematical structures of those fields, for the sake of a wider mathematical perspective and in the case of those qualified to pursue research, with a view to solving outstanding problems and simplifying basic theory by mathematical methods of increased generality and power.

A similar gap in the background of our students lies in the field of computation. Even if we carefully distinguish mathematical computation and computation engineering, there is much of importance concerning numerical computation and its organization for hand computer, IBM equipment, relay calculator or electronic computer, which will be of value to the student whether he is to become a topologist, a flutter analyst, or a mathematical logician. Of equal importance is the field of algebraic and analytic computation and its possibilities of mechanization. With the exception of a few institutions where the engineers have begun to work on computation engineering, this whole field is at the disposal of the departments of mathematics.

Consequently, it is suggested that departments of mathematics throughout the country should consider the feasibility of enlarging their offerings in computation, algebraic, analytic, or numerical, at both the undergraduate and graduate levels.

# THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

G. W. MACKEY, Harvard University

The following results of the seventh annual William Lowell Putnam Mathematical competition held May 24, 1947, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to the reader only by number.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were C. W. Hewlett, Jr., J. J. Newman, W. F. Stinespring; to each of these, a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Yale University, New Haven, Connecticut. The members of the team were Murray Gell-Mann, Murray Gerstenhaber, and Henry Otto Pollak; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Columbia University, New York, New York. The members of the team were Alex Heller, Harold Lehrer, and Maxwell Rosenlicht; to each of these, a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Pennsylvania, Philadelphia, Pennsylvania. The members of the team were Joachim Ehrman, Donald F. Hunt, and William Turanski; to each of these, a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were: C. W. Hewlett, Jr., Harvard University; Maxwell Rosenlicht, Columbia University; W. F. Stinespring, Harvard University; William Turanski, University of Pennsylvania; Eoin L. Whitney, University of Alberta. Each of these will receive a prize of forty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, were: Leonard Geller, Brooklyn College; Harry Gonshor, McGill University; Julian Kielson, Brooklyn College; John F. Nash, Jr., Carnegie Institute of Technology; Henry Otto Pollak, Yale University. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: Brooklyn College, Brooklyn, New York, the members of the team being Melvin Hausner, Robert Margolies, and George Shapiro; University of California, Berkeley, California, the members of the team being Marvin P. Epstein, George E. Gourrich, and Roger A. Stafford; Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being George W. Hinman, John F. Nash, Jr., and Robert I. Van Nice; Northwestern University, Evanston, Illinois, the members of the team being Donald B. MacMillan, James E. Murrin,

and Ernest Tilden Parker; Queen's University, Kingston, Ontario, the members of the team being Robert W. Butcher, Thomas G. Donnelly, and Donald W. Dunn; University of Toronto, Toronto, Canada, the members of the team being W. T. Sharp, Miss M. J. Straus, and W. J. D. Lewis.

Fifteen individuals were given honorable mention. The names are listed in alphabetical order. G. F. D. Duff, Toronto University; Joachim Ehrman, University of Pennsylvania; Murray Gell-Mann, Yale University; Murray Gerstenhaber, Yale University; George E. Gourrich, University of California; Melvin Hausner, Brooklyn College; Donald B. MacMillan, Northwestern University; J. J. Newman, Harvard University; John F. Riordan, Massachusetts Institute of Technology; William Riordan, Massachusetts Institute of Technology; George Shapiro, Brooklyn College; W. T. Sharp, University of Toronto; Roger A. Stafford, University of California; James E. Storer, Cornell University.

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: University of Alberta, Boston College, Brooklyn College, University of California (Berkeley), University of California at Los Angeles, Carleton College, Carnegie Institute of Technology, College of St. Thomas, University of Colorado, Columbia University, Cornell University, Harvard University, Haverford College, Loyola College (Montreal), Loyola College (New Orleans), Massachusetts Institute of Technology, McGill University, Northwestern University, Oklahoma Agricultural and Mechanical College, University of Oklahoma, University of Pennsylvania, Queen's College, Queen's University, Rutgers University, Swarthmore College, Texas Technological College, University of Toronto, Ursinus College, United States Naval Academy, Wayne University, University of Washington, and Yale University.

The following additional colleges and universities entered individual contestants only: Case School of Applied Science, Purdue University, University of Saskatchewan, and Washington University.

A total of 145 undergraduates representing 36 institutions took part in the competition.

Participants in the competition were given the following lists of problems.

## PART I. THREE HOURS

*(Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.)*

1. If $\{a_n\}$ is a sequence of numbers such that for $n \geqq 1$

$$(2 - a_n)a_{n+1} = 1,$$

prove that $\lim a_n$, as $n \to \infty$, exists and is equal to one.

2. A real valued continuous function satisfies for all real $x$ and $y$ the functional equation

$$f(\sqrt{x^2 + y^2}) = f(x)f(y).$$

Prove that

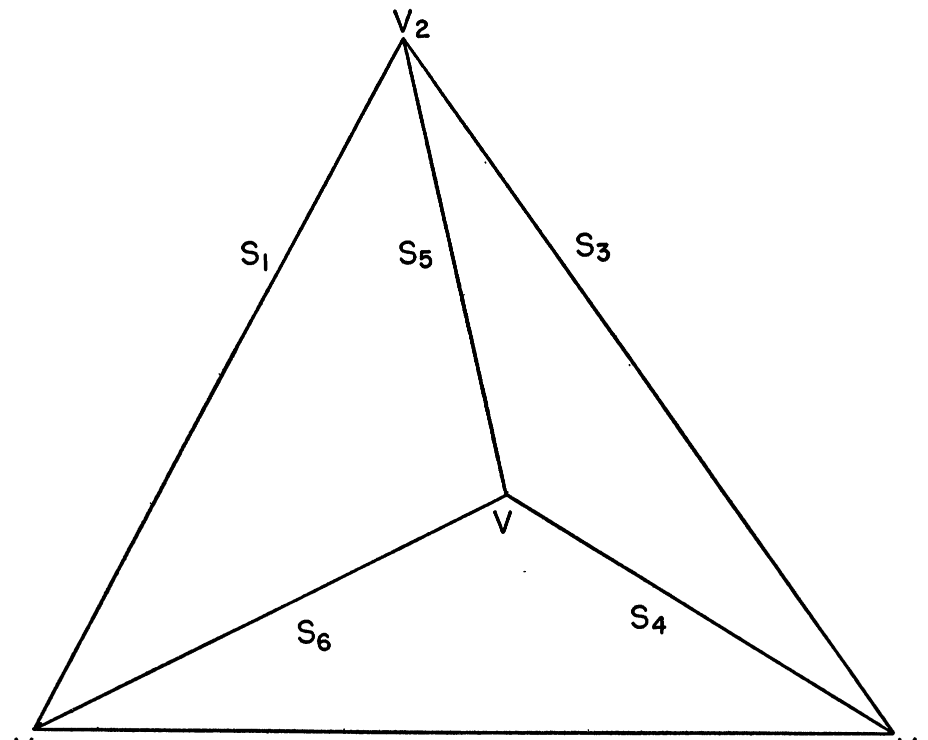
$$f(x) = [f(1)]^{x^2}.$$

3. Given this figure and any two points $Q_1$, $Q_2$ in the plane not lying on any of the segments $s_1, s_2, \cdots, s_6$, show that there does not exist a polygonal line $P$ joining $Q_1$, and $Q_2$ such that:

(1) $P$ crosses each $s_i$, $i=1, 2, \cdots, 6$, exactly once;

(2) $P$ does not intersect itself;

(3) $P$ does not pass through any vertex $V$, $V_1$, $V_2$, $V_3$.




4. A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant $(=v_0)$, determine the set $H$ of points in the plane and above the horizontal which can be hit.

5. $a_1$, $b_1$, $c_1$ are positive numbers whose sum is 1, and for $n=1, 2, \cdots$ we define $a_{n+1}=a_n^2+2b_nc_n$, $b_{n+1}=b_n^2+2c_na_n$, $c_{n+1}=c_n^2+2a_nb_n$. Show that $a_n$, $b_n$, $c_n$ approach limits as $n \to \infty$ and find these limits.

6. A three by three matrix has determinant zero, and has the further property that the cofactor of any element is equal to the square of that element.

(The cofactor of $a_{ij}$ is $(-1)^{i+j}$ multiplied by the determinant obtained by striking out the $i$th row and $j$th column.) Show that every element in the matrix is zero.

## PART II. THREE HOURS

*(Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.)*

7. Let $f(x)$ be a function such that $f(1)=1$ and for $x \geqq 1$

$$f'(x) = \frac{1}{x^2 + f^2(x)}.$$

Prove that

$$\lim_{x\to\infty} f(x)$$

exists and is less than $1+\pi/4$.

8. Let $f(x)$ be a differentiable function defined in the closed interval $(0, 1)$ and such that

$$|f'(x)| \leqq M, \qquad 0 < x < 1.$$

Prove that

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right| \leqq \frac{M}{n}.$$

9. Let $x$, $y$ be Cartesian coördinates in the plane. $I$ denotes the line segment $1 \leqq x \leqq 3$, $y=1$. For every point $P$ on $I$, let $P^*$ denote that point that lies on the segment joining the origin to $P$ and such that the distance $PP^*$ is equal to $1/100$. As $P$ describes $I$, the corresponding point $P^*$ describes a certain curve $C^*$. Let $l(I)$, $l(C^*)$ be the lengths of $I$ and $C^*$ respectively. Which one of $l(I)$, $l(C^*)$ is greater? Prove your answer.

10. Given $P(z)=z^2+az+b$, a quadratic polynomial of the complex variable $z$ with complex coefficients $a$, $b$. Suppose that $|P(z)|=1$ for every $z$ such that $|z|=1$. Prove that $a=b=0$.

11. $a$, $b$, $c$, $d$ are distinct integers such that

$$(x-a)(x-b)(x-c)(x-d) - 4 = 0$$

has an integral root $r$. Show that $4r=a+b+c+d$.

12. $C$ is a fixed point on $OZ$ and $U$, $V$ are variable points on $OX$, $OY$ respectively, where $OX$, $OY$, $OZ$ are mutually orthogonal lines. Find the locus of a point $P$ such that $PU$, $PV$, $PC$ are mutually orthogonal.

# MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

## ON THE NUMBER OF PATHS IN A FINITE PARTIALLY ORDERED SET

E. W. CHITTENDEN, University of Iowa

Let $P$ be a finite *partially ordered* system* with *initial* elements $a_1, a_2, \cdots, a_m$ and *terminal* elements $b_1, b_2, \cdots, b_n$. Let $N$ be a matrix $(x_{ij})$, $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, where $x_{ij}$ is the number of different *paths* from $a_i$ to $b_j$, where a *path* is a *maximal* simply ordered subset of $P$ which contains $a_i$ and $b_j$. We show that the matrix $N$ can be computed in terms of a series of incidence matrices† which we proceed to define. The initial elements of $P$, $a_1, \cdots, a_m$, are elements of *rank* zero. An element $p$ of $P$ is of rank 1 if it is not of rank zero and all its predecessors are of rank zero. In general an element $p$ of $P$ is of rank $r+1$ if it has a predecessor of rank $r$ and no predecessor of higher rank. If $p'$ is of rank $r'$ and $p''$ of rank $r''>r'+1$, $p''$ is a successor of $p'$, and there is no element $p$ of $P$ of rank $r'<r<r''$ between $p'$ and $p''$, we agree that $p''$ may be counted among the ranks $r'<r<r''$ and that in passing from rank $r$ to rank $r+1$, $p''$ is regarded as its own successor. Thus as there is actually only one path from $p'$ to $p''$ in $P$, the convention just adopted permits us to admit the existence of a path from $p'$ in rank $r'$ to $p''$ in rank $r+1$ without actually increasing the total number of paths in the system.

The incidence matrix $A_r=(e_{ij})$ is defined by the rule: If $p_{ir}$ is an element of $P$ of rank $r$ and is followed by $p_{j,r+1}$, then $e_{ij}=1$. Otherwise $e_{ij}=0$. Let $k$ be the maximal rank of any element of $P$. Then the $b_j$ are of rank $k$ and $N$ is the matrix product

$$N = \prod_{r=0}^{r=k} A_r.$$

The proof is easily made by induction. Evidently $A_0$ is the number of paths from elements of rank zero to their successors, and $A_r$ is the number of paths from an element of rank $r$ to an element of rank $r+1$. Hence if $N_r$ is the matrix giving the number of paths from rank 0 to rank $r$, we see that the $i$th row of $N_r$ will give the number of paths from $a_i$ to each element $p$ of rank $r$. The elements $(e_{ij})$ of $A_r$ which are different from zero in a column give the incidences from an element of rank $r$ to a particular element $p_j$ of rank $r+1$. Thus the element $x_{ij}$ of the product matrix $N_r \cdot A_r$ becomes the number of paths from $a_i$ to $p_j$.

In the case of a checkerboard, the number of paths from one king row to the

* Garret Birkhoff, Lattice Theory, American Mathematical Society, 1940.

† O. Veblen, Analysis Situs, Colloquium Lectures, American Mathematical Society, 1916.

other is given by the matrix

$$\begin{pmatrix} 14 & 14 & 6 & 1 \\ 28 & 34 & 21 & 6 \\ 20 & 35 & 14 & 14 \\ 7 & 22 & 28 & 14 \end{pmatrix}.$$

Thus the number of paths, without jumps, from the second element of the starting king row to the third element of the ending king row is 21.

## NOTE ON CONJUGATE HARMONIC FUNCTIONS

Edward Kasner, Columbia University
John De Cicco, Illinois Institute of Technology

In many textbooks on functions of a complex variable, the problem of finding a harmonic function $\psi(x, y)$ conjugate to a given harmonic function $\phi(x, y)$ is solved in the following manner. Since the functions $\phi$ and $\psi$ are conjugate-harmonic, they obey the Cauchy-Riemann equations

$$\phi_x = \psi_y, \qquad \phi_y = -\psi_x, \tag{1}$$

and also each of the functions satisfies the Laplace equation, that is

$$\phi_{xx} + \phi_{yy} = 0, \qquad \psi_{xx} + \psi_{yy} = 0. \tag{2}$$

Hence the function $\psi$ is found by integrating the equation

$$d\psi = -\phi_y dx + \phi_x dy, \tag{3}$$

which is exact since the given function $\phi$ is harmonic. Moreover all such functions $\psi$ differ from one another merely by a real constant.

We shall show how to find all such functions $\psi$ without resorting to integration. The advantage of our method is that it enables one to prove certain theorems concerning the conjugates of rational, algebraic, or entire harmonic functions.

In the first place, $\phi(x, y)$ is harmonic and hence is an analytic function of $x$ and $y$. Therefore by the obvious relations, $u = x+iy$ and $v = x-iy$, we can substitute $x = (u+v)/2$ and $y = (u-v)/(2i)$ into this function. The result is

$$\phi(x, y) = \phi\left[\frac{u+v}{2}, \frac{u-v}{2i}\right] = \mu(v) + \lambda(u), \tag{4}$$

where $\mu(v)$ is an analytic function of $v$ with coefficients the conjugates of the corresponding ones of the analytic function $\lambda(u)$.

Since we are searching for the second component $\psi(x, y)$ of the conformal transformation: $X = \phi(x, y)$, $Y = \psi(x, y)$, we know that $\phi + i\psi = 2f(u)$ and $\phi - i\psi = 2g(v)$, where $g(v)$ is an analytic function in $v$ with coefficients the conjugates of the corresponding ones of the analytic function $f(u)$. Thus we have

$$\phi = g + f, \qquad \psi = i(g - f). \tag{5}$$

Comparing (4) and (5), we find

(6) $$f = \lambda - ic/2, \qquad g = \mu + ic/2,$$

where $c$ is a real constant. Therefore the solution for $\psi$ is

(7) $$\psi = i(\mu - \lambda) + c.$$

Thus from (7), the function $\psi$ may be deduced immediately.

As an example, consider the problem of finding the conjugate of the harmonic function $\phi = x/(x^2+y^2)$. Performing the obvious substitutions, we find

$$\phi = x/[(x + iy)(x - iy)] = (u + v)/(2uv) = 1/(2v) + 1/(2u),$$

so that $\lambda = 1/(2u)$ and $\mu = 1/(2v)$. Substituting into (7), we have

$$\psi = (i/2)(1/v - 1/u) + c = i(u - v)/(2uv) + c = - y/(x^2 + y^2) + c.$$

The related analytic function is $\phi + i\psi = 1/u + ic$.

As another example, let us find the conjugate of the harmonic function $\phi = \text{arc tan } y/x$. First recall that $\text{arc tan } m = (1/2i) \log [(1+im)/(1-im)]$. Hence we have

$$\phi = \text{arc tan } y/x = (1/2i) \log [(x + iy)/(x - iy)] = (1/2i)(\log u - \log v).$$

Thus $\lambda = (1/2i) \log u$ and $\mu = (-1/2i) \log v$. By (7), we find

$$\psi = (- 1/2)(\log u + \log v) + c = (- 1/2) \log uv + c = - \log (x^2 + y^2)^{1/2} + c.$$

The related analytic function is $\phi + i\psi = -i \log u + ic$.

It is evident that the conjugate of a harmonic polynomial is also a harmonic polynomial. However it is not so obvious that the conjugate of a rational harmonic function is also rational.

*The conjugate of a rational harmonic function is rational. Thus if one component of an analytic function is a rational function of $(x, y)$, then the analytic function is rational in $u = x + iy$.*

For if $\phi(x, y)$ is a rational function of $(x, y)$, obeying the Laplace equation, it follows from (4) that $\lambda(u)$ and $\mu(v)$ are both rational functions. Hence by (7), it follows that $\psi$ is rational also. Finally the function $\phi + i\psi$ is a rational function of $u = x + iy$.

*The conjugate of an algebraic harmonic function is algebraic. Therefore an analytic function of $u = x + iy$ is algebraic if only one component is given to be an algebraic function of $(x, y)$.*

This result is also a consequence of (4) and (7).

*Similarly the conjugate of an entire harmonic function is entire. Hence an analytic function of $u = x + iy$ is entire if only one component is given to be an entire function of $(x, y)$.*

For if $\phi(x, y)$ is an analytic function of $(x, y)$ convergent for all finite values, then $\lambda(u)$ and $\mu(v)$ converge for all finite values of $u$ and $v$. Hence by (7), $\psi(x, y)$ is an analytic function of $(x, y)$ convergent for all finite values. Therefore $\phi + i\psi$ is an entire function of $u = x + iy$.

## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics.*

### END-POINT MAXIMA AND MINIMA

C. O. OAKLEY, Haverford College

It sometimes happens that a very elementary maximum-minimum problem will lead to seemingly impossible or even foolish results. The following examples are of this category and belong to the type commonly known as *end-point maxima*. They are of such simple nature that they may be readily assigned to a beginning class in the calculus.* Although not original, these two particular problems do not seem to be widely known and should prove to be both interesting and instructive to teachers and students alike.

PROBLEM I. *Find the position of the point $P(x, y)$, on an ellipse, such that the distance to a focus $F$ is a maximum (minimum).*

Let the ellipse have the equation $b^2x^2+a^2y^2=a^2b^2$. Then the distance $PF$ is a continuous function of $x$ in the interval $(-a, a)$ and hence it has both a maximum and a minimum value. (Any function which is continuous in a closed finite interval possesses both a maximum value and a minimum value in that interval). Now the fortunate student, bright or dull, might begin to solve this problem as follows:

SOLUTION 1. Let the ellipse be represented parametrically by $x=a\cos\theta$, $y=b\sin\theta$ and let $F$ be the right hand focus $(c, 0)$. Then

$$L \equiv \overline{PF}^2 = (a\cos\theta - c)^2 + b^2\sin^2\theta,$$

and

$$\frac{dL}{d\theta} = 2(a\cos\theta - c)(-a\sin\theta) + 2b^2\sin\theta\cos\theta = 0,$$

from which it follows that $\sin\theta=0$, or $\cos\theta=a/c$. The solutions of $\sin\theta=0$ yield the answers while $\cos\theta$ cannot equal $a/c$ since $a/c>1$.

But even an able student might attack the problem differently—and come to grief.

SOLUTION 2. Now $PF=a-ex$. It's derivative is $-e\neq 0$. And yet $a-ex$ is obviously least when $x=a$ and greatest when $x=-a$, if the condition $-a\leqq x\leqq a$ is kept in mind. Even though the square of the distance is used, $L\equiv\overline{PF}^2=(a-ex)^2$, the unwary student is still in trouble for $dL/dx=-2e(a-ex)=0$ yields $x=a/e$, a point outside of the ellipse (on the directrix, indeed). The func-

* See J. L. Walsh, A Rigorous Treatment of the First Maximum Problem in the Calculus, this MONTHLY, vol. 54, (1947), pp. 35–36.

tion $L$ *does* have a (relative) minimum at $x=a/e$ (what about the maximum?) but there is no (real) $y$-coördinate for the corresponding point $P$ on the ellipse. Whether $PF$ or $\overline{PF}^2$ is used, the end-points of the interval determine the (absolute) maximum and the (absolute) minimum values wanted.

Even the method that most beginning students are likely to employ will result in the same difficulty:

SOLUTION 3.

$$L \equiv \overline{PF}^2 = (x-c)^2 + y^2,$$
$$= (x-c)^2 + \frac{b^2}{a^2}(a^2 - x^2);$$
$$\frac{dL}{dx} = 2(x-c) - 2\frac{b^2}{a^2}x = 0,$$

or again,

$$x = a^2/c = a/e.$$

PROBLEM II. *A man is in a boat at $P$ one mile from the nearest point $A$ on shore. He wishes to go to $B$ which is farther down the shore $M$ miles from $A$. If he can row $r$ miles an hour and walk $w$ miles an hour, toward what point $C$ should he row in order to reach $B$ in least time?*

Usually this problem is given with simple numerical values that present no difficulty. The shore line is assumed to be straight; let $x=AC$. Then the total time $t$ that it takes to go from $A$ to $B$ is readily determined as

$$t = \frac{1}{r}(1+x^2)^{1/2} + \frac{1}{w}(M-x),$$

whence

$$\frac{dt}{dx} = \frac{x}{r}(1+x^2)^{-1/2} - \frac{1}{w} = 0,$$

and

$$x = \frac{r}{\sqrt{w^2 - r^2}}.$$

The first astonishing thing about this "answer" is that it is independent of the distance $M$. If $r<w$ and if $r<M\sqrt{w^2-r^2}$, routine testing of the derivative will show that a minimum time is attained. But if $r<w$ and $r>M\sqrt{w^2-r^2}$, then the derivative vanishes for a value of $x$ that lies outside of the interval $AB$ and the problem has an end-point minimum, that is, the man should row directly toward $B$. Finally if $r\geq w$, the man should again row directly to $B$.

Plotting the graphs of the functions involved in these problems is a great aid in clarifying the points at issue.

DISCUSSION. Problems involving end-point maxima and minima usually arise only when the independent variable is restricted to a certain interval in order to insure reality of the function or for certain physical reasons such as the avoidance of negative values of distance or time. In Problem I we found that the answers were either relative maxima and minima or end-point maxima and minima according to the particular choice of the independent variable. This is indeed a general situation: extreme values of the one type can be converted into those of the other type by an appropriate choice of a parameter or, what amounts to the same thing, by an appropriate transformation on the independent variable. Indeed let the function to be maximized be $y=f(x)$ and let the range of $x$ be $(a, b)$. Then put $x=a\cos^2\theta+b\sin^2\theta$. This transformation restricts $x$ to the interval $(a, b)$ but permits the new independent variable $\theta$ to vary from $-\infty$ to $+\infty$. It is readily seen that now $dy/d\theta=(dy/dx)2(b-a)\sin\theta\cos\theta$ which vanishes at $\theta=0$ and $\pi/2$, namely, at the points $x=a$ and $x=b$, as well as at any relative maxima and minima inside the interval $(a, b)$. Applying this transformation to the above exercises will convert them into the usual relative maxima and minima problems. For example in Problem II if we set $x=M\sin^2\theta$, we find $dt/d\theta$ vanishes when $\sin\theta=0$, $\cos\theta=0$ and when $\sin^4\theta=r^2/M^2(w^2-r^2)$. An analysis of these values gives the results stated above.

If the interval is an infinite one, say $(a, \infty)$, the transformation $x=a+\tan^2\theta$ will change an end-point maximum (minimum) at $x=a$ into a relative one.

*Editorial Note.* Another example of an end-point maximum has been submitted by V. L. Klee, Jr. of the University of Virginia. It is as follows:

PROBLEM. We are given a straight fence 100 feet long, and wish by adding 200 feet more to form a rectangular enclosure whose boundary contains the original fence. How shall this be done so as to enclose the greatest possible area?

Let $x$ denote the length of new fence which is aligned with the original 100 feet of fence. Then the dimensions of the enclosure are $100+x$ and $50-x$, and its area $A(x)$ is $-x^2-50x+5000$. The student may reason as follows: $D_xA=-2(x+25)$; hence we would choose $x=-25$ in order to enclose the maximum area. He will then be perplexed to note that this solution obviously does not satisfy the conditions of the problem. The correct solution, of course, is $x=0$ which is an end-point maximum.

## LETTERS TO THE EDITOR

### The Trial Integral Method

The essential idea of the method explained in my note, *The Trial Integral Method*, this MONTHLY, vol. 34 (1947) pp. 159–160, is contained in a paper by T. H. Hildebrandt, Marginal Notes, this MONTHLY, vol. 36 (1929), pp. 216–221.

M. F. SMILEY
Northwestern University

### The Equation of an Ellipse

In this MONTHLY for April (vol. 54 (1947) pp. 219–220) there appears a note by Frank Hawthorne in which the Cartesian equation of the ellipse with respect to its axes is derived, without the use of radicals, from the constancy of the sum of the focal radii. May I point out that the derivation of the equation of the ellipse in this manner appeared in 1707 in the posthumous *Traité analytique des sections coniques* (pp. 22–25) of the Marquis de L'Hospital. A concise account of this and other portions of the work of L'Hospital is given in J. L. Coolidge, *A history of the conic sections and quadric surfaces*, Oxford, 1945, p. 77.

C. B. BOYER
Brooklyn College

### The Remainder Theorem

The basic ideas of the note by R. W. Wagner, *An Application of the Remainder Theorem*, this MONTHLY, vol. 54 (1947) p. 106, were contained in a lecture by L. C. Karpinski, "Mathematical Short Cuts for Engineers and Architects," published in a lithographed form by the Multi-Color Company of Detroit, Michigan in February 1930.

Karpinski extended this idea to make an interesting and pedagogically desirable connection with a common arithmetical concept ,the "check by nines." The proof of this arithmetical trick, taught usually by rule, if at all, follows, of course, from the fact that if, as in Wagner's note, $p(x)$ represents an integer conceived of as a polynomial in 10, then $p(1)$ is both the remainder if the number is divided by $x-1$, and the sum of the digits. In this case, $x-1$ is 9. The "check by casting out nines" then follows from the fact that if $p$, $q$, $r$ are integers, then $(p+q+r)/9=p/9+q/9+r/9$ and hence the remainder of the sum should equal the sum of the remainders (with nines cast out). The extension of the check to the other fundamental operations of arithmetic is obvious.

Karpinski pointed out in his lecture that one may also derive checks by 11, 99, 101, and in a recent conversation he added that if the number is written as a polynomial in 1000 (e.g. $1{,}484{,}365=p(x)=x^2+484x+365$), then $p(-1)$ is the remainder after division by 1001 (in the example $p(-1)=-118$). Since $1001=7\cdot 11\cdot 13$, the divisibility of the remainder by any of these factors is necessary and sufficient to show the divisibility of the original integer.

P. S. JONES
University of Michigan

### Linear Differential Equations

The purpose of these remarks is to express the opinion that the note *On the Solution of Linear Differential Equations* in this MONTHLY, vol. 54 (1947) p. 160 is in violation of (a) sound mathematics (b) sound pedagogy (c) sound common sense

I will briefly restate the author's thesis, using his first example, the equation

$$y'' - 2y' + y = e^{ax}.$$

He sets down the general solution, and observes that as it stands it is not valid when $a=1$, containing as it does a denominator $(a-1)^2$. He then rigs a special form which when $a$ approaches unity does not become infinite but is indeterminate, and naïvely assumes that the limit of this expression is the solution when $a=1$. This is to assume the validity of a theorem which certainly has not been proved at this stage of the game, namely, that if a solution of $f(x, y, y', a)=0$ is bounded, it is a continuous function of $a$. The result, in the case before us, happens to be correct, but the process is unjustifiable.

What does the conscientious mathematician do when a general result becomes meaningless in a special case? He does not try to patch it up by ingenuity but goes back to the original problem and works it through. In this instance whether or not we like to use the symbolic operator $D$, we have a well-defined process for solving linear equations with constant coefficients, namely the repetition of the relationship:

$$\text{If} \qquad y' - ay = f(x), \qquad y = e^{ax}\left[\int e^{-ax}f(x)dx + c\right].$$

Direct application of this rule yields easily the desired solution of the equation $y''-2ay''+a^2y=e^{ax}$ in a neat and simple form.

Pedagogically, the operations which I suggest are straightforward and direct. The author remarks that the usual methods seem to involve pulling something out of thin air, but it seems to me that on the contrary his method involves pulling a rabbit out of a previously prepared hat. What student could see the sense in setting up his peculiar form of the general solution, or would understand it well enough to be able to handle another similar situation? The process which I suggest, on the other hand, goes back to sound and well-established principles. This argument leads naturally and logically to my third point, sound mathematics and pedagogy are bound up with common sense. We attack any given problem on its own merits, not as an offshoot of some other problem.

R. A. JOHNSON
Brooklyn College

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 781. *Proposed by P. D. Thomas, Navy Department, Washington, D.C.*

A heavy ball is gently dropped into a vase full of water. A section through the vertical axis of the vase is a semiellipse, the height being the semimajor axis, the diameter being then the minor axis. The size of the ball is such as to cause the maximum displacement. (1) Find the radius of the ball. (2) Show that the plane of the circle of tangency bisects the height of the submerged segment of the ball. (See E 687 [1946, 334].)

E 782. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

Show that the product of two numbers each of the form $x^2+kxy+y^2$, where $k$, $x$, $y$ are integers, $k$ fixed, is also of that form. If $x$ and $y$ are relatively prime, are all factors of $x^2+kxy+y^2$ also of the same form?

E 783. *Proposed by C. D. Olds, San Jose State College*

Given a parallelogram and its diagonals. Let each side of the parallelogram be divided into $n$ equal parts and let lines be drawn through the points of division, parallel to the sides and to the diagonals of the parallelogram. Find the total number of triangles in the resulting figure.

E 784. *Proposed by R. E. Gaines, University of Richmond*

Show that the locus of the intersection of two successive perpendicular tangents to a logarithmic spiral is another logarithmic spiral.

E 785. *Proposed by R. J. Walker, Cornell University*

Each of $n-1$ tanks, $T_1, \cdots, T_{n-1}$, holds $V$ gallons of water, and an $n$th tank, $T_n$, holds $V$ gallons of a salt solution containing $M$ pounds of salt. Liquid is circulated at the rate of $g$ gallons per minute from $T_n$ to $T_{n-1}$, $T_{n-1}$ to $T_{n-2}, \cdots,$ $T_2$ to $T_1$, $T_1$ to $T_n$. How much salt is in $T_n$ after $t$ minutes?

### SOLUTIONS

E 743, Solution II [1947, 285]. *Correction.* In the third line of the solution the word "positive" should be replaced by "negative."

#### A Cryptarithm

E 751 [1947, 38]. *Proposed by Alan Wayne, Flushing, N. Y.*

Find the digits represented by the letters in the following addition, if no two

different letters represent the same digit:

$$\begin{array}{r} FORTY \\ TEN \\ TEN \\ \hline SIXTY \end{array}$$

*Solution by A. Chulick, Hofstra College.* Since the sum of column (1), (right to left), is $Y$ and no digit can be carried, $N$ is zero. Likewise, since the sum of column (2) is $T$, or $T$ plus 10, $E$ is either zero or five. But $N$ is zero; therefore $E$ is five. A digit must be carried from column (4). That digit is 1, for the sum of column (4) can be at most 11. It follows that $I$ is either zero or one. But $N$ is zero; therefore $I$ is one, and $O$ is nine. The sum of column (3) must be at least 21, for $X$ cannot be either zero or one, and at most 1 can be carried from column (2); hence $T$ must be either seven or eight. If $T$ is seven $R$ must be eight and $X$ three. But $F$ and $S$ are consecutive digits. The only possibilities are two and three, and three and four; therefore $X$ cannot be three. $T$ must be eight, $R$ seven, and $X$ four. The only consecutive digits available are two and three; hence $F$ is two and $S$ is three. $Y$ is six. The solution is unique and we have $29786+850+850=31486$.

Also solved by Richard Andree, Maurice Anthony, LeRoy Babcock, Murray Barbour, T. E. Berry, Daniel Block, R. A. Bradley, W. G. Brady, Paul Brock, W. E. Buker, M. I. Chernofsky, P. L. Chessin, W. H. Coulter, R. E. Crane, J. A. Cromelin, J. H. Cross, J. E. Darraugh, E. de la Garza, Monte Dernham, William Douglas, R. L. Duncan, Virginia Felder, J. H. Ferguson, Barkley Fritz, I. M. Gardoff, M. A. Geisler, E. St. John Gough, E. L. Harp, Jr., R. H. Hoskins, J. M. Kingston, Mrs. V. L. Klee, Jr., William Kruskal, Elmer Latshaw, H. N. Leifer, G. E. McAllister, Burnett Meyer, Leo Moser, Roger Osborn, S. T. Parker, C. R. Perisho, C. F. Pinzka, P. A. Pisa, A. O. Qualley, P. W. A. Raine, A. P. Rhodes, G. G. Roberts, E. D. Schell, David Sohn, G. W. Walker, Davis Wellinger, Hazel Schoonmaker Wilson, M. S. Constable and the proposer.

*Editorial Note.* Problems of this nature are appropriately called *cryptarithms*, and constitute a special class of the more general problems known as *arithmetical restorations*. See Ball-Coxeter, *Mathematical Recreations and Essays*, pp. 20–26 and Kraitchik, *Mathematical Recreations*, pp. 79–80. Although the above problem is easy, cryptarithms can be very difficult. Devotees, however, prefer *charming* cryptarithms, and to be charming a cryptarithm should (1) make sense in the given letters as well as the solved digits, (2) involve all the digits, (3) have a unique solution, and (4) be such that it can be broken by logic, without recourse to trial and error. Judged by these standards the above cryptarithm is very charming indeed.

The proposer has composed a number of these cryptarithms, some of which have appeared in *The Cryptogram*, publication of the American Cryptogram Association. Thus he has offered $SEVEN+SEVEN+SIX=TWENTY$ (unique

solution), $SEVEN+THREE+TWO=TWELVE$ (unfortunately possessing two solutions, since $N$ and $O$ may be interchanged), $TWENTY+FIFTY+NINE+ONE=EIGHTY$ (unique solution but requiring an undesirable amount of trial and error), $SCAN+THESE=DIGITS$ (unique solution), and $RODE+MAIL+ADDED+EMBLEM=OFFERED$ (having, as a literary accessory, the key word *FORMIDABLE*, whose successive letters correspond to the ordered set of digits $0, 1, \cdots, 9$).

Dernham and McAllister both suggested that if anyone wants more, let him $SEND+MORE=MONEY$.

Of the proposed problem W. H. Coulter wrote:

"Two sums that oddly cleave as one
Though far apart like Earth and Sun."

### Two Six-Point Circles Associated with a Right Triangle

E 752 [1947, 38]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that in a right triangle the twelve points of contact of the inscribed and escribed circles form two groups of six points situated on two circles which cut each other orthogonally at the points of intersection of the circumcircle with the line joining the midpoints of the legs of the triangle.

*Solution by J. H. Butchart, Arizona State College.* Let $ABC$ be the right triangle, $C$ the vertex of the right angle, and let $X$, $Y$, $Z$ be the points of contact of the incircle with the sides $a$, $b$, $c$ opposite the vertices $A$, $B$, $C$ respectively. Let $X_1$, $X_2$, $X_3$ be the points of contact of the escribed circles opposite $A$, $B$, $C$ respectively with side $a$, and similarly define $Y_i$ and $Z_i$. Let $P$, $Q$ be the points where the line bisecting $a$, $b$ at $A'$, $B'$ meets the circumcircle. Then, using the relations like $AZ=s-a$, $BX_3=s-a$, where $s$ is the semi-perimeter of triangle $ABC$, we can show that the sets of points $P$, $Y$, $Q$, $Y_3$; $P$, $X$ $Q$, $X_3$; $X$, $X_3$, $Y_3$, $Y$ are concyclic. For instance, $(PA')(A'Q)=(CA')(A'B)=a^2/4=(XA')(A'X_3)$. Since $XX_3$, $YY_3$ meet at $C$, which is not on $PQ$, it is impossible for circles $(PYQY_3)$ and $(PXQX_3)$ to be distinct. Similarly $Z_1$, $Z_2$ lie on circle $(XX_3Y_3Y)$ and the center is the intersection of the perpendicular bisectors of $XY$ and $Z_1Z_2$, *i.e.*, an end, $K'$, of the diameter of the circumcircle perpendicular to $AB$. The points $Z$, $Z_3$, $X_1$, $Q$, $Y_1$, $X_2$, $P$, $Y_2$ are proved concyclic in a similar manner, and $K$, the other end of the diameter through $K'$, is proved the center of this circle by showing that $KQ$ and $KZ$ are equal. Since the radii $KQ$, $K'Q$ of the two circles are perpendicular, the circles are orthogonal.

Also solved by the proposer.

*Editorial Note.* It is believed that the two six-point circles of this problem are new in the geometry of the right triangle.

### The Elliptical Race Track

E 753 [1947, 38]. *Proposed by L. M. Kelly, University of Missouri*

How can one convince a class in elementary analytics that if the inside of a

from another curve $C$, is called *a parallel* to $C$. A curve and any of its parallels have the same normals and the same evolute. The parallel curves of a non-circular ellipse are curves of the eighth degree. See, *e.g.*, Ex. 3, art. 372, of Salmon's *Conic Sections*. R. C. Yates has devised a linkage for describing curves parallel to an ellipse. See this MONTHLY [1938, 607].

### Skew Ordered Sequences

E 754 [1947, 39 and 1947, 163.] *Proposed by S. T. Thompson, Tacoma, Wash.*

A finite sequence of positive integers will be said to be *skew ordered* if either each integer in an even position of the sequence is greater than or each such integer is less than its immediate neighbors. If the eight integers $1, \cdots, 8$ are placed in random order in a sequence, what is the probability that the sequence will be skew ordered?

I. *Solution by J. B. Kelly, Hampton, Va.* Let $Q_n$ be the number of skew ordered arrangements of $n$ distinct integers. There is no loss in generality in supposing that the integers are $1, 2, \cdots, n$. Suppose that the integer $n$ occurs in the $k$th position. If we have a skew ordered arrangement, the sequences on either side of $n$ must be skew ordered. The sequence to the left of $n$ must have its last element less than its next to last element. Once the integers in this sequence are chosen, its elements may be arranged in $\frac{1}{2}Q_{k-1}$ different ways so as to satisfy this condition. The sequence to the right of $n$ must have its first element less than its second element. Once the integers in this sequence are chosen, its elements may be arranged in $\frac{1}{2}Q_{n-k}$ different ways so as to satisfy this condition. There are $\binom{n-1}{k-1}$ ways of choosing the integers in the sequence to the left of $n$ and once these integers are chosen, the integers in the sequence to the right of $n$ are determined. Thus the number of skew ordered permutations of $n$ distinct integers for which the greatest integer (here $n$) occurs in the $k$th position is $\frac{1}{4}\binom{n-1}{k-1}Q_{k-1}Q_{n-k}$. It follows that

$$Q_n = \frac{1}{4}\sum_{k=1}^{n}\binom{n-1}{k-1}Q_{k-1}Q_{n-k}. \tag{1}$$

In applying this formula, it is necessary to make the convention that $Q_0=Q_1=2$. Let $P_n$ be the probability that a given permutation of $n$ distinct integers will be skew ordered. Evidently $P_n=Q_n/n!$, and relation (1) becomes

$$P_n = \frac{1}{4n}\sum_{k=1}^{n} P_{k-1}P_{n-k}, \tag{2}$$

where again we make the convention that $P_0=P_1=2$. Calculating $P_8$ by successively calculating $P_2, P_3, \cdots, P_7$ by means of (2) we obtain $P_8=277/4032$.

II. *Solution by Frederick Mosteller, Harvard University.* The probability

that a finite sequence of $n$ unequal integers will be skew ordered when all permutations are equally likely can be translated directly into statistical terminology. It is the probability that the sequence will have no *run up* or *run down* of length greater than or equal to 2. This problem was treated by P. S. Olmstead, *Distribution of sample arrangements for runs up and down.* Annals of Mathematical Statistics, vol. 17 (1946), pp. 24–33.

For $n=3$ to 14 we derive from Olmstead's Table 2

| $n$ | Probability of skew ordering |
|---|---|
| 3 | 0.66666667 |
| 4 | 0.41666667 |
| 5 | 0.26666667 |
| 6 | 0.16944444 |
| 7 | 0.10793651 |
| 8 | 0.06870040 |
| 9 | 0.04373898 |
| 10 | 0.02784447 |
| 11 | 0.01772647 |
| 12 | 0.01128499 |
| 13 | 0.00718426 |
| 14 | 0.00457364 |

Since for $n$ even the number of arrangements with runs of length no more than one is just twice Euler's number $E_n$, and for $n$ odd the number of arrangements is twice the tangent number, we might use the approximation

$$4\left(\frac{2}{\pi}\right)^{n+1}.$$

(See, *e.g.*, Milne-Thompson, *Calculus of Finite Differences*, p. 147.)

For $n=4$ this approximation agrees to 0.002, for $n=6$ to 0.0001, for $n=8$ to 0.000003, and for $n=9$ to 0.000001. The exact answer for $n=8$ is 277/4032.

# ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.*

## PROBLEMS FOR SOLUTION

4259. *Proposed by Richard Bellman, Princeton University*

If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1 + x^{n_k}} = x \prod_{k=1}^{\infty} (1 + x^{n_k}), \qquad |x| < 1,$$

show that, except perhaps for order,

$$n_k = 2^k.$$

4260. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle $ABC$ inscribe two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose sides are parallel to the medians. Show that (1) the triangles $ABC$, $A_1B_1C_1$, $A_2B_2C_2$, have the same centroid and the same Brocard angle; (2) the triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in an ellipse concentric and homothetic to the inscribed Steiner ellipse, the ratio of homothety being $1/\sqrt{3}$.

4261. *Proposed by F. J. Dyson, Trinity College, Cambridge, England*

The number of partitions of an integer $n$ into a sum of positive integral parts is denoted by $p(n)$. The result of subtracting the number of parts in a partition from the largest part is a positive or negative integer called the rank of the partition. Ramanujan proved that $p(5n+4)$ is always divisible by 5, and $p(7n+5)$ by 7. Show that the number of partitions of $5n+4$ whose ranks are congruent modulo 5 to a given residue is the same whichever of the five residues is chosen, and the number of partitions of $7n+5$ whose ranks are congruent modulo 7 to a given residue is the same whichever of the seven residues is chosen.

4262. *Proposed by L. A. Santaló, Rosario, Argentina*

Let $C$ be a rectifiable plane curve of length $L$, contained within a given circle of radius $R$. Prove that there is a circle of radius $\rho \geqq R$ which cuts $C$ in $n$ points, where

$$n \geqq L/\pi R. \tag{1}$$

In particular there is a line which cuts $C$ in $n$ points, where $n$ satisfies (1). If

$\rho < R$, the inequality (1) must be replaced by

$$n \geqq \frac{4L\rho}{\pi(R+\rho)^2}. \tag{2}$$

See L. A. Santaló, A theorem and an inequality referring to rectifiable curves, *American Journal of Mathematics*, 1941, p. 635.

4263. *Proposed by Howard Eves, Oregon State College, and Paul Halmos, Syracuse University*

Criticize the following alleged proof of the continuum hypothesis.

Let $X$ be the set of all infinite sequences of 0's and 1's, and let $E$ be an arbitrary uncountable subset of $X$. Corresponding to any finite sequence, $\{a_1, \cdots, a_k\}$, of 0's and 1's, write $E(a_1, \cdots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to $E$ and begin with $\{a_1, \cdots, a_k\}$. Since $E=E(0)+E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1=0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If $a_i$ has already been defined for $i=1, \cdots, k$, so that $E(a_1, \cdots, a_k)$ is uncountable, then write $a_{k+1}=0$, or 1 according as $E(a_1, \cdots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \cdots\}$ has the property that for any $k$ it is true that $E(a_1, \cdots, a_k)$ is uncountable. Write $E^*$ for the union of all $E(a_1, \cdots, a_k)$, for $k=1, 2, 3, \cdots$; then $E^*$ is a subset (in fact an uncountable subset) of $E$.

For certain positive integers $k$ it is true that both $E(a_1, \cdots, a_k, 0)$ and $E(a_1, \cdots, a_k, 1)$ are uncountable; in fact this must happen for an infinite number of $k$'s. (Otherwise, for a sufficiently large $k$, $E(a_1, \cdots, a_k)$ would not be uncountable, contrary to its construction.) Let $k_1, k_2, k_3, \cdots$ be the integers for which this is true, and write, for any $\{x_1, x_2, x_3, \cdots\}$ in $E^*$,

$$y_n = x_{k_n+1};$$

then $\{y_1, y_2, \cdots\}$ is an infinite sequence of 0's and 1's. From the way in which the $k_n$ are defined it follows that every possible sequence of 0's and 1's occurs as a $y$ sequence, and that consequently the sequences $\{x_1, x_2, \cdots\}$ in $E^*$ correspond (in possibly a many to one manner) to a set (viz. the set of all $y$ sequences) having the power of the continuum. It follows that the cardinal number of $E^*$ (and hence of $E$) cannot be less, and since $E$ is a subset of $X$ it cannot be greater. In other words it has been proved that every uncountable subset of a set having the power of the continuum has also the power of the continuum.

4248 [1947, 232], corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given a tetrahedron $ABCD$, place a sphere $(S)$ of given radius in such a manner that the volume of the polar tetrahedron of $ABCD$ with respect to $(S)$ will be a relative minimum.

## SOLUTIONS

### A Configuration of n Spheres

3895 [1938, 631]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Four spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ pass respectively through the vertices $A_1, A_2, A_3, A_4$ of a tetrahedron and intersect in pairs on the corresponding edges. Straight lines $A_5A_1$, $A_5A_2$, $A_5A_3$, $A_5A_4$ which join an arbitrary point $A_5$ to the vertices of the tetrahedron cut the respective spheres again in $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and these four points with $A_5$ lie on a sphere $(\omega_5)$. Continuing in this way, the lines $A_nA_1, A_nA_2, \cdots, A_nA_{n-1}$ which join an arbitrary point $A_n$ to the points of a preceding set $A_1, A_2, \cdots, A_{n-1}$ cut the respective spheres $(\omega_1), (\omega_2), \cdots, (\omega_{n-1})$ in the points $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ which with $A_n$ lie on a sphere $(\omega_n)$. (1) Show that the spheres $(\omega_1), (\omega_2), \cdots, (\omega_n)$ meet in a point $P$. (2) Show that the points diametrically opposite to $A_1, A_2, \cdots, A_n$ on the corresponding spheres lie in a plane through $P$.

*Note by Proposer.* The part (2) is an extension of a theorem by Bouvaist-Delens, *Annales de la Société Scientifique de Bruxelles*, 1937, p. 159.

*Solution by the Proposer.** (1). A solution of this first part has already been given for $n=5$ (this MONTHLY, 3819 [1939, 239]). The same reasoning extended by adding one more point at a time shows that the spheres $(\omega_1), (\omega_2), \cdots, (\omega_n)$ meet in a point $P$ common to the first four spheres.

(2) To solve the second part it is convenient to establish a preliminary

THEOREM. *If three spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$ pass respectively through the vertices $A_1, A_2, A_3$ of a triangle $A_1A_2A_3$, intersect in pairs on $A_1A_2, A_2A_3, A_3A_1$, and have in common a point $P$ not in the plane of triangle $A_1A_2A_3$, then the point $P$ and the points $A_1', A_2', A_3'$ diametrically opposite $A_1, A_2, A_3$ on the given spheres are coplanar.*

The plane $A_1A_2A_3$ cuts the spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$ in three circles $(E_1)$, $(E_2)$, $(E_3)$, which have a common point $P'$.† The points $A_1', A_2', A_3'$ diametrically opposite $A_1, A_2, A_3$ on $(\omega_1)$, $(\omega_2)$, $(\omega_3)$ project orthogonally on the plane $A_1A_2A_3$ into points $a_1', a_2', a_3'$ diametrically opposite $A_1, A_2, A_3$ on $(E_1)$, $(E_2)$, $(E_3)$. The triangle $a_1'a_2'a_3'$ has its sides perpendicular to the corresponding sides of triangle $A_1A_2A_3$. The sides $A_2'A_3'$, $A_3'A_1'$, $A_1'A_2'$ of triangle $A_1'A_2'A_3'$ are therefore perpendicular to the homologous sides $A_2A_3$, $A_3A_1$, $A_1A_2$ of triangle $A_1A_2A_3$.

Likewise the orthogonal projections $a_1''$, $a_2''$, $a_3''$ of $A_1, A_2, A_3$ on the plane $A_1'A_2'A_3'$ coincide with the points diametrically opposite $A_1', A_2', A_3'$ on the circles $(E_1')$, $(E_2')$, $(E_3')$ cut by this plane from the spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$. The sides of $a_1''a_2''a_3''$, perpendicular to the corresponding sides of $A_1'A_2'A_3'$, intersect $(E_1')$, $(E_2')$, $(E_3')$ on $A_2'A_3'$, $A_3'A_1'$, $A_1'A_2'$. These three circles meet in the point $P$, which is the second point of intersection of the spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$

* Translation by W. E. Byrne, Virginia Military Institute.

† R. A. Johnson, Modern Geometry, 1929, Art. 184.

since $P$ is not in the plane $A_1A_2A_3$. The theorem is thus proved.

The solution of (2) is merely a corollary. By the Miquel-Roberts theorem* the four spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ of the present problem have in common a point $P$. The points $A_1'$, $A_2'$, $A_3'$, $A_4'$ diametrically opposite $A_1$, $A_2$, $A_3$, $A_4$ on $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ are in the same plane $(\pi)$ passing through $P$, since the points $(A_1', A_2', A_3', P)$, $(A_2', A_3', A_4', P)$, $(A_3', A_4', A_1', P)$, $(A_4', A_1', A_2', P)$ are coplanar by our preliminary theorem.

The lines $A_5A_1$, $A_5A_2$, $A_5A_3$, $A_5A_4$ intersect $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ respectively in $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$. By the Miquel-Roberts theorem applied to the tetrahedron $A_5A_2A_3A_4$, the spheres $A_5\alpha_2\alpha_3\alpha_4$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ meet at $P$. Hence the point $A_5'$ diametrically opposite $A_5$ on the sphere $(\omega_5) \equiv A_5\alpha_2\alpha_3\alpha_4$ is in the plane $(\pi)$. Step by step it is clear that the points $A_6', \cdots, A_n'$ diametrically opposite to $A_6, \cdots, A_n$ on $(\omega_6), \cdots, (\omega_n)$ are also in the plane $(\pi)$.

*Note.* Part (2) generalizes an interesting theorem due to R. Bouvaist, who obtained it by analytical means. It was proved also in various ways by P. Delens.† The reasoning may be extended step by step to the polygon $A_1'A_2' \cdots A_n'$ to prove part (2). The theorem is:

*Given a tetrahedron $A_1A_2A_3A_4$ and four spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ passing respectively through $A_1$, $A_2$, $A_3$, $A_4$ and intersecting in pairs on the corresponding adjacent edges, these spheres have a point $P$ in common (S. Roberts), and the points $A_1'$, $A_2'$, $A_3'$, $A_4'$, diametrically opposite to $A_1$, $A_2$, $A_3$, $A_4$ on the spheres $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_4)$ are in a plane passing through $P$.*

### A Theorem on the Tetrahedron

4150 [1945, 102]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$ with the orthocenter $H$, the perpendiculars at $A$ to the faces $ACD$, $ABD$, $ABC$ meet respectively the planes $HCD$, $HBD$, $HBC$ in $(A_1, A_2, A_3)$, and similarly for the points $(B_1, B_2, B_3)$, and so on. Show that the planes $A_1A_2A_3$, $B_1B_2B_3, \cdots$, are perpendicular to the medians of $ABCD$.

*A solution* will be found in the Proposer's article, *Tetrahedrons having a common face*, p. 395, this issue of the MONTHLY.

### Barycentric Coördinates

4192 [1946, 103]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $A_1A_2A_3A_4$, if $P$ is the centroid of its antipedal tetrahedron $ABCD$, its barycentric coördinates are inversely proportional to the squares of its distances to $A_1$, $A_2$, $A_3$, $A_4$; and conversely.

---

* J. L. Coolidge, *A Treatise on the Circle and the Sphere*, 1916, p. 240. The priority of the theorem usually attributed to S. Roberts, seems to belong to Miquel, who gave an elementary demonstration of it in 1838 (Journal de Liouville, t. III, p. 522, theorem VI). Miquel obtained it by a stereographic projection.

† P. Delens, Ann. de la Soc. Scient. de Bruxelles, 1937, p. 159.

*Solution by R. Goormaghtigh, Bruges, Belgium.* The following generalization will be proved: *Let $A_1A_2A_3A_4$ be a tetrahedron and $B_1B_2B_3B_4$ the antipedal tetrahedron of an arbitrary point $P$; if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are the barycentric coördinates of $P$ as to $A_1A_2A_3A_4$ and as to $B_1B_2B_3B_4$, then the ratios $\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3, \beta_4/\alpha_4$ are proportional to the squares of the distances $PA_1, PA_2, PA_3, PA_4$.*

The volume of the tetrahedron $PA_2A_3A_4$ is

$$\tfrac{1}{6}PA_2\cdot PA_3\cdot PA_4 \sin (P - A_2A_3A_4).$$

Further, the areas $\sigma_1, \cdots$ of the faces $B_2B_3B_4, \cdots$ are proportional to the sines of the supplements to the solid angles $(B_1-B_2B_3B_4), \cdots$. But the solid angle $(P-A_2A_3A_4)$ is the supplement to $(B_1-B_2B_3B_4)$, and so on. Hence $\alpha_1, \cdots$ are proportional to $\sigma_1/PA_1, \cdots$ and to $\beta_1/\overline{PA}_1^2, \cdots$; and this proves the theorem. The Proposer's theorem is an immediate corollary.

It follows also that the products $\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \alpha_4\beta_4$ are proportional to the squares of the faces of the tetrahedron $B_1B_2B_3B_4$.

### Collinear Orthopoles

4195 [1946, 160]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

There are ten ways to divide six points on a circle into two groups of three so as to form pairs of triangles having no common vertex. The midpoints of the segments joining the orthopoles of a given straight line with respect to each pair of triangles are ten collinear points.

*Solution by Ou Li, Yenching University, Peiping, China.* Let the given circle be the unit circle and let $t_i$ $(i=1, 2, \cdots, 6)$ be the complex coördinates of the six points $A_i$ on it. Denote the elementary symmetric functions of, say, $t_1, t_2, t_3$ by $s_j^{123}$ $(j=1, 2, 3)$; and those of $t_1, t_2, \cdots, t_6$ by $\sigma_k$ $(k=1, 2, \cdots, 6)$. The given straight line has an equation in the form

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1$$

where $a$ is constant. The orthopoles of this line with respect to the triangles $A_1A_2A_3$, $A_4A_5A_6$ are, respectively,*

$$x_{123} = \frac{1}{2}\left(a + s_1^{123} + \frac{\bar{a}}{a} s_3^{123}\right), \qquad x_{456} = \frac{1}{2}\left(a + s_1^{456} + \frac{\bar{a}}{a} s_3^{456}\right).$$

The midpoint of these two orthopoles will be

$$M = \frac{1}{4}\left[2a + \sigma_1 + \frac{\bar{a}}{a}(s_3^{123} + s_3^{456})\right].$$

Now let $M'$ be the analogous midpoint for the triangles $A_1A_2A_4$ and $A_3A_5A_6$.

* See for instance R. Goormaghtigh, A Study of a Quadrilateral Inscribed in a Circle, this MONTHLY 1942, p. 178.

The clinant of the line $MM'$ becomes

$$\frac{M - M'}{\overline{M} - \overline{M}'} = \left(\frac{\bar{a}}{a}\right)^2 \sigma_6.$$

Since this is symmetric in the $t_i$, it is evident that all the ten midpoints are collinear.

Solved also by H. E. Fettis and the Proposer.

### Isosceles n-Points

4254 [1947, 345]. *Proposed by Paul Erdös, Syracuse University*

We have seven points in the plane. Prove that we can always select three which do not form an isosceles triangle. For six points this does not necessarily hold. (If $A$, $B$, $C$ are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

*Editorial Note.* This is essentially the same as the Proposer's problem E 735 [1947, 227–229], which see for solution and comments.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*An Introduction to Mathematical Genetics.* By Lancelot Hogben. New York, W. W. Norton and Co., 1946. 12+260 pages. $5.00

Genetical theories and various breeding practices lead to pleasing mathematical problems. They have intrigued abstract algebraists to construct special "genetic algebras," and some of the recurrence relations and functional equations describing the genetical structure of succeeding generations have not yet been solved. Readers of this MONTHLY will be more interested in the fact that genetical theories provide perhaps the easiest and best accessible examples illustrating how even very elementary mathematical reasoning leads to interesting insights and applications. It is a fairly common practice to use the Mendelian laws as a first application for the elementary rules of combining probabilities. The equally elementary and intuitive applications of simple difference and other recurrence equations, of matrix computation, and so forth, are less known. Mathematicians, and in particular mathematics teachers, will, therefore, greet a book on mathematical genetics with great interest.

Unfortunately, the book under review will disappoint them inasmuch as it is clearly not intended for them. It is written for students of biology and practical

breeders who have difficulties with simple fractions but wish to understand the results of a quantitative analysis of breeding systems, of mutations, selection, and so forth. Accordingly, the choice of material, though natural and justified, is not ideal from the peculiar point of view of the mathematician. The mathematical tools and ideas are hidden so far as possible and the access to the book is made difficult by the standard practice in genetics of using as many technical terms as possible and not defining a single one. It is possible that students of biology would be acquainted with these terms long before they desire to consult Hogben's book. Nevertheless, one page of explanations would have opened the book to many other readers.

As has been indicated, the book relies on most elementary methods. An integral, a differential, and a statistical test of significance occur once or twice, but in a way calculated not to disturb a reader familiar only with simple algebra. All probability considerations are translated into the language of mixing and selecting cards. The notion of a sequence $u_n$ is carefully analyzed, but it may mislead some readers that $u_n$ usually refers simultaneously to any generation and to the initial or zero generation. In practice, $u_0$ is arbitrary and $u_n(n>0)$ is determined from a recurrence relation. Hence, identifying $u_n$ with $u_0$ is not legitimate.

The content is as follows. Chapter I, Gene Frequencies, Genotypic Frequencies, and Systems of Mating. This chapter is mostly concerned with the card-pack model and the simplest rules. Chapter II, Basic Types of Algebraic Series in Genetical Theory, dedicated to recurrence relations, and limiting values. Chapter III, First Steps in the Calculus of Finite Differences, including approximate extrapolation. Chapter IV, Binomial Series (by which the author means the binomial probability distribution). Chapter V, Non-assortative Mating in the Absence of Selection or Mutation. The term non-assortative stands for the usual "random" to which the author objects. This chapter treats the inheritance of one or several characters, sex-linked or not. Chapter VI, Selection. Chapter VII, Assortative Mating and Consanguinity, covering various systems of breeding. Chapter VIII, Mutation Pressure and Isolate Effects. The Appendices describe the simplest statistical techniques: I, Significance Tests for Mendelian Ratios, and II, The Estimation of Linkage and Determination of Variance Formulas for Gene-Frequency Analysis by the Method of Maximum Likelihood.

W. FELLER

*Die Mathematische Denkweise.* By Andreas Speiser. Basel, Birkhauser, 1945. 132 pages. S. Fr. 14.50.

In this book we have the philosophy of a mathematician. It is written with the enthusiasm of a distinguished mathematician who penetrates the arts and the world in his peculiar way. It will transmit, I imagine, this enthusiasm to every mathematician who is not only a craftsman but possessed by the sacred fire as the poet and philosopher ought to be.

The first section shows the rôle of symmetry in ornamentics. Examples with beautiful pictures are given from Egypt and Crete, from Arabic and Renaissance art.

The following section gives a description of the complex of motives in a piece of music with characterization of the motive by the number of its measures. The author gives examples from music by Mozart, Beethoven, Verdi and other composers.

Very impressive is the third section, in which the author investigates Dante's Divine Comedy in regard to natural philosophy which is essentially new Platonic. Here one finds a cosmic building which is the result of very scanty experience and a construction guided by the paramount desire for symmetry and harmony. If we compare this with the natural philosophy of today, we find the same tendency; however, the substructure of experience has grown to an immense extent and has so many interrelations that it represents in itself already a gigantic design.

The fourth part is about the philosophy of mathematics by Proclus Diadochus. This philosophy is essentially based on Plato's idea that mathematics, liberated from all the little devices of the craft, is a great ornament of divine origin. At the end of this section we find the following hymnic words of Proclus:

"This, therefore, is Mathematics: she reminds you of the invisible forms of the soul; she gives life to her own discoveries, she awakes the mind and purifies the intellect; she brings to light our intrinsic ideas, she abolishes oblivion and ignorance which are ours by birth. . . . "

The average mathematical reader will, probably, appreciate only with difficulty the chapter on numbers and space in the Neoplatonic doctrine. As the author justly remarks, numbers were something much more alive, much more real with the Greeks than with us. One may, perhaps, add that in the development of human thought the concept of the general number is a rather late achievement and that the Greeks were close enough to the birth of this concept to feel the wonder of its nature. In these theories as in the cabala the numbers are not used to describe the order of the universe. They are felt as *creating* this order. In trying to penetrate this realm of thought a mathematician may have the feeling that he comes out of the clear day into an almost uncanny and mystical dark.

The section on Goethe's Theory of Colors follows. The author shows that Goethe tries to bring a system of symmetry into the realm of colors. In this way, through an apparent psychical order, he achieves a connection of the physical phenomena with the realm of beauty and art.

The last section deals with astrology. Here we are introduced into the attempt to analogize the mathematical order of the stars with the somewhat chaotic fate of human beings and human society. The remarkable thing is that from the order of the heavenly bodies at the time of the birth, that is, from the horoscope, the great astrologers constructed human characters of a peculiar force. Sometimes, this construction formed the personality and life of the man

for whom the horoscope was cast.

The book has an epilogue, consisting of an oration which Speiser gave at the tercentenary of Kepler's death. Here, it is especially important to be reminded of the fact that Kepler's firm belief in the mathematical structure of the universe showed him the way and strengthened his amazing power of concentration and endurance to achieve his immortal work on the orbits of the planets.

It is interesting to realize that Speiser, in his discussions, dwells mainly on long bygone times, when the physical world and the world of the human soul were felt, by the wise men as well as by the common people, as one.

M. DEHN

*Concise Analytic Geometry*. By C. H. Sisam, New York, Henry Holt and Co., 1946. 9+155 pages. $2.00.

This book comprises the section on analytic geometry contained in the author's more extensive *College Mathematics*, which was reviewed in the February, 1947 issue of this MONTHLY (volume 54, number 2, page 119).

The definitions and proofs of theorems have been formulated with care and accuracy and no essential topic has been omitted. In spite of the concise treatment the explanations, illustrative examples, and exercises are quite adequate. Polar coördinates are introduced early and the polar equations of straight lines, circles, and conics are derived. The application of polar coördinates to more complicated loci is deferred to a later chapter. The last three chapters provide a brief introduction to solid analytic geometry. Answers to the odd numbered exercises are given in the back of the book.

In the opinion of the reviewer the book contains ample material for a three semester hour course in Analytic Geometry and it is believed that instructors who have experienced difficulty in finding a suitable text for such a course would do well to examine this book carefully.

H. P. EVANS

**New Books Received**

*Antennae: An Introduction to Their Theory*. By J. Aharoni. New York, Oxford University Press, 1946. $8.50.

*College Algebra*. Third Edition. By W. L. Hart. Boston, D. C. Heath and Co., 1947. 8+416 pages. $3.00.

*College Algebra*. Alternate Edition. By Paul Rider. New York, The Macmillan Company, 1947. 15+407 pages. $2.50.

*Engineering Problems Manual*. Fourth Edition. By F. C. Dana and L. R. Hillyard. New York and London, McGraw-Hill Book Co., 1947. 14+419 pages. $3.25

*Mathematics of Accounting*. Third Edition, By A. B. Curtis and J. H. Cooper. New York, Prentice-Hall, Inc., 1947. 10+550 pages. $6.00 trade edition, $4.50 text edition.

*Preparatory Business Mathematics*. By L. L. Smail. New York, Ronald

Press Co., 1947. 10+244 pages. $2.75.

*Multiple-factor Analysis*. By L. L. Thurstone. University of Chicago Press, 1947. 19+535 pages. $7.50.

*Plane Trigonometry*. By E. B. Mode. New York, Prentice-Hall, Inc., 1947. 10+216 pages.

*Relativity. The Special and General Theory*. By Albert Einstein. New York, Henry Holt and Co., 1930. Reprint, New York, Peter Smith, 1947. 13+168 pages. $2.50.

*Lecciones de Analisis Infinitesimal*. By F. J. Duarte. Caracas, Tipografia Americana, 1943. 605 pages.

*An Introduction to Business Statistics*. Second Edition. By J. R. Stockton. Boston, D. C. Heath and Co., 1947. 7+478 pages. $4.00.

*Six-Place Tables*. Seventh Edition. By E. S. Allen. New York, McGraw-Hill Book Co., 1947. 23+232 pages. $2.50.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### CLUB REPORTS, 1946–47

#### Mathematics Club, Berea College

Professor W. R. Hutcherson of Berea College, was in charge of a mathematics and astronomy exhibit in connection with a science exhibit held at Berea, Kentucky on March 29, 1947. Many favorable remarks were made by visitors who classed the display in mathematics ahead of those of some of the sciences. Extensive use of placards, maps, blackboards and models was made in the following exhibits:

*History of Mathematics*: Development and trends
*Computing Machines*: The mechanical brain
*Statistics*: Applications and graphs (bar graphs, curves)
*Mechanical Drawing*: Blueprinting and layout work
*Mathematics and Music*
*Slide Rule*: Uses and types
*Mathematics Tables and Applications of Calculus*
*Surveying*: Instruments, and maps
*Astronomy*: Telescope, relative star sizes, and pictures
*Magic in Mathematics*: Magic squares, perpetual calendar, and puzzles
*Modern Geometry*: Law of duality and Desargues' Theorem
*Calculus*: Mechanics, and friction problems

Members of the club were divided into groups according to their interests and each group had charge of planning one of the exhibits and then demonstrated it during the period for exhibition.

**Mathematics Club, Rutgers University**

The *Mathematics Club* of Rutgers University, which had suspended its activity during the war, resumed operation on December 5, 1946, and regular semi-monthly meetings were held during the balance of the school year.

The following papers were presented:

*Farey series*, by Professor H. S. Grant

*Pythagorean triangles*, by Professor C. R. Phelps

*Maxima and minima*, by Dr. L. M. Court

*The actuarial profession*, by Professor R. Walter

*Constructibility*, by George Y. Cherlin

*Mathematics in industry*, by Professor F. G. Fender

The officers for 1946–47 were: President, George Y. Cherlin; Vice-President, Robert Coursen; Secretary-Treasurer, C. F. Pinzka. The Faculty Advisor was Professor C. R. Phelps.

**Kappa Mu Epsilon, William Jewell College**

*Mu Sigma Alpha*, the mathematics honorary fraternity at William Jewell College was organized by a group of interested students in January, 1943. It has assumed a place of distinction and respect among the honor societies on the campus. During the year 1946–47, the programs consisted of the following speeches made by members of the club and guest speakers:

*Contribution of Einstein toward the atomic bomb and his mass-velocity relationship*, by Dr. W. A. Hilton

*The invention and development of the concept of zero*, by Professor L. O. Jones

*Fallacies in mathematics*, by Nicholas Housley

*Short-cuts of mathematics*, by Professor C. O. Van Dyke

*Geometric designs in lighting*, by LeRoy Heaton

*Telling time by the stars*, by Lloyd Elrod

*Improvement of high school mathematics courses*, by Paul Curau

*The case for mathematics*, by Woody Rixey.

During the year 1946–47, several of the students became interested in becoming affiliated with a national mathematical fraternity. *Kappa Mu Epsilon* was petitioned for membership, and on May 7, 1947, the *Missouri Gamma* Chapter of *Kappa Mu Epsilon* was duly installed at William Jewell College by the *Missouri Beta* Chapter from Warrensburg under the direction of Dr. Claude Brown. The officers for 1946–47 were: President, Mary Ruth Carney; Vice-President, Truett Neese; Treasurer, LeRoy Heaton; Corresponding Secretary, Professor L. O. Jones; Secretary, Edwin Watson.

The officers elected for the year 1947–48 are: President, LeRoy Heaton; Vice-President, Maynard Cowan; Secretary, Walter Binns; Corresponding Secretary, Paul Swedburg; Treasurer, Lloyd Elrod.

**Mathematics Club, Haverford College**

During the past academic year, the following papers were prepared and informally discussed:

*Elements of topology*, by Dr. C. B. Allendoerfer

*Squaring the circle*, by Geert Prins

*The four-color problem*, by James Thorpe

*Fermat's last theorem*, by Murray Freeman

*Fibonacci's series*, by Norman Brous

*Ham sandwich theorems*, by Dr. John W. Tukey of Princeton University

*The cattle problem of Archimedes*, by Daniel Wagner.

The members of the club who distinguished themselves by presenting solutions for the monthly prize problems were: Murray Freeman, William Warner, Daniel Wagner, and Thomas Crolius.

The officers of the club for the year 1946–47 were: President, Geert Prins; Secretary, Leon Robbins, Jr.

**Pi Mu Epsilon, St. Louis University**

The *Missouri Gamma* Chapter of *Pi Mu Epsilon* has evidenced a very strong interest in the promotion of scholarship. At its three regular meetings, members of the chapter contributed scholarly papers on the following subjects:

*History of logarithms*, by Miss Margaret Willerding

*Transcendental numbers*, by Mr. Edwin Karlowicz

*Mechanism and use of integrating machines*, by Mr. Herman Plew, Jr.

A summary of each of these papers will appear in the regular issue of the Chapter news magazine, *The Missouri Gamma News*. Each of the meetings was followed by a social hour.

The principal and final function of the year took place on April 8 at Parks Air College. It consisted in the initiation of new members, the election of the new Director, Mr. Oliver F. Anderhalter, the Annual Lecture, and the Tenth Annual Banquet. The lecturer, Professor N. A. Court of the University of Oklahoma, spoke on *Mathematical Asides*. Rev. Patrick J. Holloran, S. J., President of Saint Louis University, addressed the group at the banquet. He expressed his personal satisfaction at the activity and growth of *Pi Mu Epsilon*. He gave a vote of thanks to Professor Regan for being the inspiration for much of this development.

*Missouri Gamma News* is a Chapter bulletin which is now in its second year of publication. It reviews the principal activities of the Chapter during the year and the plans for the year ahead. A summary by the author of the paper read at the regular meetings is included in the publication. News items about members of the Chapter, the plans of the Mathematics Department and of the allied sciences can be found in the Chapter news.

Officers for the year 1946–47 were: Director, Mr. Herbert Gebhart; Vice-Director, Miss Margaret Willerding; Secretary, Miss Frances Higgins; Faculty Advisor and Corresponding Secretary, Professor Francis Regan.

## NEWS AND NOTICES

EDITED BY HARRY POLLARD, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to Harry Pollard, White Hall, Cornell University, Ithaca, New York.*

### STANFORD UNIVERSITY MATHEMATICS EXAMINATION

The second Stanford University Mathematics Examination (see this MONTHLY, vol. LIII, no. 7, pp. 406–409 (1946)) was held April 19, 1947, 2:00 p.m. to 5:00 p.m., in thirty-nine high schools in California; 196 students took part. The following problems were proposed.

1. To number the pages of a bulky volume the printer used 1890 digits. How many pages has the volume?

2. Among grandfather's papers a bill was found:

72 turkeys \$—67.9—

The first and last digit of the number that obviously represented the total price of those fowls are replaced here by blanks, for they have faded and are now illegible.

What are the two faded digits and what was the price of one turkey?

3. Determine $m$ so that the equation in $x$

$$x^4 - (3n + 2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression.

4. Let $\alpha$, $\beta$ and $\gamma$ denote the angles of a triangle. Show that

$$\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma$$

and

$$\sin 4\alpha + \sin 4\beta + \sin 4\gamma = -4 \sin 2\alpha \sin 2\beta \sin 2\gamma.$$

The writer of the best paper, Miss Maxine Steineke, Los Altos, California, a student at Palo Alto High School, Palo Alto, California, received a \$500 scholarship at Stanford University. The following contestants received "Honorable Mention":

Mr. Raymond M. Hendricks, Santa Barbara, California, a student at Santa Barbara High School.

Mr. Jerold Bernard Weller, Los Angeles, California, student at Alexander Hamilton High School, Los Angeles.

### PERSONAL ITEMS

Paul Alexandroff has been elected a foreign member of the National Academy of Sciences.

K. J. Arrow of Columbia University has been appointed research associate with the Cowles Commission for Research in Economics at the University of Chicago.

Professor W. L. Ayres of Purdue University has been appointed Dean of the

School of Science.

Dr. L. M. Brown of the University of Edinburgh, Dr. J. M. Hyslop of the University of Glasgow, Dr. J. M. Jackson of University College, Dundee, and Dr. A. J. Macintyre of the University of Aberdeen have been elected fellows of the Royal Society of Edinburgh.

Dean R. D. Carmichael of the Graduate School, University of Illinois, has retired with the title emeritus.

Miss Mary L. Cartwright of the University of Cambridge has been elected a fellow of the Royal Society.

Professor T. G. Cowling of the University College of North Wales has been elected a fellow of the Royal Society.

Associate Professor J. B. D. Derksen of the Netherlands School of Economics at Rotterdam has joined the Statistical Office of the United Nations at Lake Success.

Dr. Mary P. Dolciani of Cornell University has been awarded the Sigma Delta Epsilon post doctoral fellowship for 1947–48 and will study at the Institute for Advanced Study.

T. K. Glennan of the Ansco Division of General Aniline and Film Corporation, Binghamton, New York, has been appointed President of the Case Institute of Technology. He was wartime director of the U. S. Navy Underwater Sound Laboratory.

Professor C. C. MacDuffee of the University of Wisconsin has been awarded the degree of Doctor of Science at Colgate University.

Professor O. E. Neugebauer of Brown University and Professor Hassler Whitney of Harvard University have been elected to membership in the American Philosophical Society.

Messrs. Paul Olum of Harvard University and E. H. Spanier of the University of Michigan have been awarded Frank B. Jewett fellowships for 1947–48 by the American Telephone and Telegraph Company.

Professor P. A. Smith of Columbia University has been elected to membership in the National Academy of Sciences.

Associate Professor Max Astrachan of Antioch College has been promoted to a professorship.

Dr. W. D. Baten is now Chief, Operations Analysis Branch (A-5), Air Defense Command, Mitchell Field, New York.

Professor C. C. Bramble of the Naval Postgraduate School, Annapolis, Maryland, has accepted a position as Director of Computation and Ballistics at the Naval Proving Ground, Dahlgren, Virginia.

Professor Talmon Bell of Sterling College has retired as chairman of the Mathematics Department with the title of Professor Emeritus.

Assistant Professor Lipman Bers of Syracuse University has been promoted to an associate professorship.

Professor Garrett Birkhoff of Harvard University served as Walker Ames Professor at the University of Washington during the spring quarter.

Associate Professor I. S. Carroll of Syracuse University has been promoted to a professorship.

Dr. Myrtle Collier, Chairman of the Department of Mathematics of Immaculate Heart College, has retired with the title of Professor Emeritus.

Dr. John Dyer-Bennet of Purdue University has been promoted to an assistant professorship.

Professor Samuel Eilenberg of Indiana University has been appointed to a professorship at Columbia University.

Dr. Cleota G. Fry of Purdue University has been promoted to an assistant professorship.

Assistant Professor Abe Gelbart of Syracuse University has been promoted to an associate professorship. He will be on leave for the academic year 1947–48 to attend the Institute for Advanced Study.

Dr. W. H. Gottschalk of the University of Pennsylvania has been promoted to an assistant professorship.

Associate Professor S. H. Gould ofthe University of Toronto has been appointed to an assistant professorship at Purdue University.

Assistant Professor May N. Harwood of Syracuse University has been promoted to an associate professorship.

Associate Professor M. H. Heins of Brown University has been promoted to a professorship.

Associate Professor Magnus R. Hestenes of the University of Chicago has been appointed to a professorship at the University of California at Los Angeles.

Associate Professor H. K. Hughes of Purdue University has been promoted to a professorship.

Assistant Professor Mark Kac of Cornell University has been promoted to a professorship.

Dr. Irving Kaplansky of the University of Chicago has been promoted to an assistant professorship.

Assistant Professor M. W. Keller of Purdue University has been promoted to an associate professorship.

Assistant Professor J. L. Kelley of the University of Chicago has been appointed to an associate professorship at the University of California at Berkeley.

Dr. Carl Kossack of the Navy Department has been appointed to an associate professorship at Purdue University.

R. R. Kuebler of Dickinson College, Carlisle, Pennsylvania, has been promoted to an assistant professorship.

Professor C. G. Latimer of the University of Kentucky has been apponted to a professorship at Emory University.

Professor L. L. Lowenstein of Alfred University has been appointed professor and head of the department at Kent State University.

Professor Saunders MacLane of Harvard University has been appointed to a professorship at the University of Chicago. He will be on leave as a Guggen-

heim Fellow until the summer quarter of 1948.

G. E. Markle of the University of Detroit has been promoted to an assistant professorship.

R. D. Mindlin of Columbia University has been promoted to a professorship.

Associate Professor Harriet F. Montague of the University of Buffalo has been promoted to a professorship.

M. G. Moore of Bradley University, Peoria, Illinois, has been promoted to an associate professorship.

Assistant Professor Andrewa R. Noble of Montana State University has been appointed to an associate professorship at Pacific University, Forest Grove, Oregon.

Nilan Norris of Hunter College has been promoted to an assistant professorship.

J. I. Northam of the University of Wisconsin has been appointed to an assistant professorship at Kansas State College, Manhattan, Kansas.

Edward Paulson has been appointed lecturer in mathematics at the University of Washington.

Assistant Professor A. M. Peiser of Rutgers University has resigned to accept a position as mathematician with Hydrocarbon Research, Inc., of New York.

Dr. Clarence Perisho of McCook Junior College has been appointed assistant professor of chemistry and mathematics at Nebraska Wesleyan University, Lincoln, Nebraska.

Dr. Sam Perlis of Purdue University has been promoted to an assistant professorship.

Professor H. B. Phillips of the Massachusetts Institute of Technology has retired.

Assistant Professor R. S. Phillips of New York University has been appointed to an associate professorship at the University of Southern California.

E. K. Ritter of the Postgraduate School, United States Naval Academy, has been promoted to an associate professorship.

Assistant Professor C. K. Robbins of Purdue University has been promoted to an associate professorship.

Associate Professor Arthur Rosenthal of the University of New Mexico has been appointed to a professorship at Purdue University.

Professor A. C. Schaeffer of Stanford University has been appointed to a professorship at Purdue University.

Dr. Edith R. Schneckenburger of Michigan State Normal College has been appointed to an assistant professorship at the University of Buffalo.

Dr. I. E. Segal has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor Ruth G. Simon of Berea College has been appointed to an assistant professorship at Morningside College, Sioux City, Iowa.

Dr. Jerome C. Smith of Lafayette College, Easton, Pennsylvania, has been

promoted to an assistant professorship.

Assistant Professor B. M. Stewart of Michigan State College has been promoted to an associate professorship.

Professor A. H. Taub of the University of Washington has been granted a leave of absence to accept a Guggenheim Fellowship. He will spend the year at Princeton University.

Bradford Tye of Bethany College has been promoted to an assistant professorship.

Assistant Professor G. B. Van Schaack of Union College has been appointed to an assistant professorship at Washington University.

Assistant Professor G. L. Walker of Temple University has been appointed to an assistant professorship at Purdue University

Assistant Professor A. D. Wallace of the University of Pennsylvania has been appointed to a professorship at Tulane University.

Professor Andre Weil of the University of Sao Paulo has been appointed visiting professor at the University of Chicago.

Associate Professor E. L. Welker of the University of Illinois has been appointed associate in mathematics in the Bureau of Medical Economic Research of the American Medical Association.

D. W. Western of Brown University has been promoted to an assistant professorship.

Dr. G. W. Whitehead of Princeton University has been appointed to an assistant professorship at Brown University.

Professor R. M. Winger of the University of Washington has been appointed Executive Officer of the Department of Mathematics.

Dr. Hyman J. Zimmerberg of Rutgers University has been promoted to an assistant professorship.

Assistant Professor H. S. Zuckerman of the University of Washington has been promoted to an associate professorship.

Professor Antoni Zygmund of the University of Pennsylvania has been appointed to a professorship at the University of Chicago.

The following appointments to instructorships are announced:

Brown University: Dr. A. B. Blakers, Mrs. R. C. Buck, Dr. Kathleen Butcher.

Cornell University: Dr. Christine Williams, Dr. Bertram Yood, Dr. Bryant Tuckerman, Jr.

General Motors Institute, Flint, Michigan: J. A. Straw

Rutgers University: C. D. Firestone

United States Naval Academy: J. R. Gorman

University of California: Dr. E. W. Barankin

University of Chicago: Edwin J. Akutowicz, J. Bruce Crabtree

University of Washington: Dr. Fumio Yagi

University of Wisconsin: Dr. A. M. Mark

Wellesley College: Mrs. Jerry Cowen

C. C. Carter, Bluffs, Illinois, honorary life member of the Association, lost his life in an accident May 5, 1947.

Professor P. J. Daniells of the University of Sheffield died May 25, 1946.

Professor Emeritus C. J. Keyser of Columbia University died May 8, 1947 at the age of eighty-six years.

The death of M. A. D. Kinsman has been reported.

Professor L. G. Owens, formerly of the University of Rangoon, died February 23, 1947.

# THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## MEETING OF THE NORTHERN CALIFORNIA SECTION

The ninth annual meeting of the Northern California Section was held at the University of San Francisco on Saturday, January 25, 1947. Professor W. H. Myers, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was eighty-one including the following twenty-three members of the Association: H. M. Bacon; G. A. Baker; T. J. Bass, Jr.; Mable L. Beckwith; W. G. Brady; E. L. Fitzgerald, S.J.; S. A. Francis; Alice Graeber; W. R. Hanson; Emma V. Hesse; D. H. Lehmer; Sophia L. McDonald; E. D. Miller; F. R. Morris; W. H. Myers; C. D. Olds; George Pólya; Edris P. Rahn; E. B. Roessler; Pauline Sperry; Gabor Szegö; K. J. Waider; L. A. Walker.

At the business meeting the following officers were elected for the coming year: Chairman, Professor George Pólya, Stanford University; Vice-Chairman, Professor G. C. Evans, University of California; Secretary-Treasurer, Professor E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Mrs. Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor L. E. Reukema of the Department of Electrical Engineering, University of California, gave an hour's address during the morning session.

The following papers were presented:

1. *Vibration modes of tapered beams*, by Dr. Edmund Pinney, University of California, introduced by the Secretary.

The vibration modes of thin beams whose bending stiffness and linear densities may be expressed in simple powers of the distance along the beam may, in certain cases, be obtained in terms of Bessel functions. Four one-parameter families of beams are shown to have this property. These include the wedge and the cone. The results may be applied to finding bending modes for airplane wings while still in the preliminary design stage.

2. *Estimating electrostatic capacity from geometric data*, by Professor George Pólya, Stanford University.

Professor Pólya's paper appeared in this MONTHLY, April, 1947.

3. *Electronics in our modern world*, by Professor L. E. Reukema, University of California, introduced by the Chairman.

The speaker recalled that electronics includes the science and industry connected with vacuum tubes and their circuits. The tubes may be the tiny ones found in radio receivers, or tubes capable of passing many thousands of amperes or of generating pulses of millions of watts of radio-frequency power. They may act as rectifiers, amplifiers, oscillators, modulators, or detectors. Without vacuum tubes, long-distance telephones, world-wide radio communication, sound motion pictures, television, and radar would all be impossible. Vacuum tubes also guide our planes through the skies, our ships over the sea, and our missiles to enemy targets. They allow our planes to land through the densest fog. They automatically aim and fire our guns. In medicine and surgery electronic equipment has made notable contributions.

It is truly amazing what powers the vacuum tube gives to mankind. It makes possible the accurate measurement of distances as much smaller than the diameter of a raindrop as the diameter of the raindrop is smaller than that of a sphere 50 miles across. It enables one to weigh particles too small to be seen with the finest microscope, to detect energy in space in such tiny quantities that it would have to be received continuously for over a year to amount to as much energy as a fly expends in raising itself one inch, to measure currents so small that they would have to flow for over a billion years to pass as much electricity as flows through the filament of an ordinary electric light in a second. It enables a single television transmitter to hold a million or more receivers in synchronism to within one sixteen millionth part of a second. In the next few years it will probably supply jobs to millions of men in completely new industries, and will in thousands of ways increase our safety, comfort, and material wealth, and happiness.

4. *The impact of high speed computing on mathematics*, by Professor D. H. Lehmer, University of California.

This paper dealt with the several projects now under way in high speed computing devices, and the influences they are likely to have on mathematical problems and mathematical points of view.

5. *Refresher courses for returning students*, by S. A. Francis, San Mateo Junior College.

Mr. Francis discussed the topic of refresher courses as related to the field of mathematics. The speaker based his discussion on actual experiences with teaching such courses in classes under the auspices of E.S.M.W.T. during the war period from 1943 to 1945, and for returning students majoring in engineering and science at Stanford University and at San Mateo Junior College. Many sections of refresher courses were organized at San Jose State College, San Mateo Junior College, San Francisco Junior College, and at the University of California

to accommodate returning students who asked for such courses in the fields of algebra, trigonometry, analytic geometry, and calculus. The consensus of opinion of those engaged in the teaching of such classes is that the courses were of great value to the students as well as an encouraging and challenging experience for teachers of mathematics.

6. *Some problems in aerodynamics*, by Professor J. G. Herriot, Stanford University, introduced by the Chairman.

Professor Herriot gave a brief description of a wind tunnel, noting that the speed which can be attained is limited by the size of the test model even when sufficient power is available. In order to explain this he gave a simple mathematical proof of the fact that in subsonic flow a decrease in the cross-sectional area of the flow produces an increase in velocity whereas in supersonic flow the reverse is true. It was also shown how the non-linear differential equation of compressible flow can be linearized under the assumption of small perturbations of the velocity. It is then possible to deduce the properties of a compressible flow field from those of a known incompressible flow field. A number of other problems of current interest were mentioned briefly. These included pressure distributions over bodies of revolution in subsonic and supersonic flow, and mixed subsonic and supersonic flows.

7. *Some remarks on mathematical education*, by Professor Sophia L. McDonald, University of California.

E. B. ROESSLER, *Secretary*

## MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday morning, February 14, 1947. Professor O. H. Hamilton, Chairman of the Section, presided.

Sixty-two persons attended the meeting, including the following twenty-one members of the Association: E. F. Allen, Arthur Bernhart, J. C. Brixey, H. N. Carter, N. A. Court, A. H. Diamond, O. H. Hamilton, J. O. Hassler, W. N. Huff, H. V. Huneke, J. E. LaFon, P. E. Lewis, G. E. Meador, H. A. Palmer, W. L. Shepherd, W. T. Short, D. R. Shreve, H. W. Smith, C. E. Springer, R. W. Veatch, J. H. Zant.

At the business session the following officers were elected: Chairman, D. R. Shreve, University of Tulsa; Vice-Chairman, W. T. Short, Oklahoma Baptist University; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following ten papers:

1. *On the representation problem for projective algebras*, by Professor J. C. C. McKinsey, Oklahoma A. and M. College, introduced by Professor A. H. Diamond.

C. J. Everett and S. Ulam have recently given postulates for projective algebras, and have solved the representation problem for all complete atomic projective algebras. The author showed that every projective algebra is isomorphic to

a sub-algebra of a complete atomic projective algebra; this result, combined with that of Everett and Ulam, provides a solution of the representation problem for arbitrary projective algebras. The method of proof is similar to the method used by M. H. Stone to establish a representation theorem for Boolean algebras. If $A$ is a prime ideal, then $A_x$ is defined to be the prime ideal which contains all elements of the form $a_x$, for $a$ in $A$. If $A$ and $B$ are prime ideals, then $A\Box B$ is defined to be the class of all prime ideals $C$ which contain all elements of the form $a\Box b$, where $a$ is in $A$ and $b$ is in $B$. These definitions were extended to permit operations on classes of prime ideals.

2. *Some properties of algebras and their subalgebras*, by Professors A. H. Diamond and J. C. C. McKinsey, Oklahoma A. and M. College.

The problem considered was that of sub-algebras, $S$, generated by $n$ elements of an algebra $A$ for a given value of $n$, such that every $S$ is a Boolean algebra but $A$ is not Boolean. A proof was given that $n$ is not less than 2. This result was used to prove a theorem of Wajsberg that every set of axioms for the sentential calculus must contain at least one axiom in at least three variables.

3. *Distance in the complex domain*, by Professor W. T. Short, Oklahoma Baptist University.

A point $P$ in the complex domain was represented by the coördinates $(x+ui, y+vi)$. The distance between two such points $P_1$ and $P_2$ was defined by $d^2=(x_2-x_1)^2+(y_2-y_1)^2+(u_2-u_1)^2+(v_2-v_1)^2$, where the real part $r^2=(x_2-x_1)^2+(y_2-y_1)^2$, and the normal part $n^2=(u_2-u_1)^2+(v_2-v_1)^2$. The vector joining the two points is then $r+ni$, where $i^2=-1$. The vector is in the plane determined by the perpendiculars at $P_1$ and $P_2$. The vector in the complex domain consists of two parts. The real part $r$ is the resultant of the two real vectors. To find the normal vector $n$, we take the resultant of the two normal vectors in the real plane. This length normal to the real plane is the normal vector.

4. *The inverse geometry of Clifford numbers*, by Professor E. F. Allen, Oklahoma A. and M. College.

The number $z=x+ry$, $r^2=\pm k^2$, $k$ real, was used to solve geometric problems. In particular it was used to demonstrate a generalization of the concept of the perpendicularity of two lines.

5. *A contribution to the four-color problem*, by Professor Arthur Bernhart, University of Oklahoma.

Though a map may not be 4-colorable, yet each hemisphere of a minimal map must be colorable, subject to certain boundary conditions on the equatorial ring. For an $n$-ring there are $[3^{n-1}+3(-1)^n+2]/8$ color schemes. In the case of an 8-ring this leads to homogeneous equations in 274 unknowns. The algebraic solution is facilitated by a magic square arrangement. Results include a new reducible configuration: two hexagons guarded by two pentagons.

6. *A comment on the approximate solution of certain boundary value problems by submatric inversion*, by R. R. Reynolds, Oklahoma A. and M. College, introduced by Professor O. H. Hamilton.

Matric analysis was applied to the approximate solution and the improve-

ment of the approximate solution of certain boundary value problems.

7. *Mathematical analysis of artificial pulse lines*, by P. J. Miller, Oklahoma A. and M. College, introduced by Professor H. S. Smith.

This was a preliminary report on work which is still in progress.

8. *A conic in the tangent plane to a surface*, by W. W. Dolan, University of Oklahoma, introduced by Professor C. E. Springer.

The polar reciprocal of any line $l$ at a point $P$ of a surface, with respect to a Darboux quadric at the point, is a line in the tangent plane. If $l$ varies so as to generate a quadric cone, the reciprocal line envelops a conic. Two theorems appear: (1) the reciprocal conic will be a hyperbola, parabola, or ellipse according as the line of centers of the Darboux quadrics is outside, on, or inside the cone generated by $l$; (2) the point $P$ will be outside, on, or inside the reciprocal conic according as the cone generated by $l$ has two lines, one line, or no line in common with the tangent plane.

Analytic treatment in a properly restricted case reveals that the reciprocal conic is an ellipse whose axes are simple functions of the principal radii of normal curvature of the surface at the point.

9. *On a differential equation satisfied by a certain set of polynomials*, by Professor W. N. Huff, University of Oklahoma.

The set $\{y_n(x)\}$ of polynomials given by the generating function

$$g(x, t) = f(xt)\phi(t) = \sum_{n=0}^{\infty} y_n(x)t^n$$

with

$$f(xt) = \sum_{n=0}^{\infty} a_n \frac{(xt)^n}{n!} \quad \text{and} \quad \phi(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}, \quad (a_n \neq 0,\ b_0 \neq 0),$$

was briefly considered. A relation for which these polynomials are of $A$ type $k$ in according with the definition of Sheffer was given. Finally a differential equation of infinite order satisfied by the set $\{y_n(x)\}$ was obtained along with a necessary and sufficient condition that the differential equation be of finite order.

10. *Diagnostic testing and remedial instruction in freshman mathematics for engineers*, by Professor D. R. Shreve, University of Tulsa.

A report on a three-year program of diagnostic testing and remedial instruction in freshman mathematics for engineers as carried out at Purdue University (1938–1942) was discussed.

JOHN C. BRIXEY, *Secretary*

## MARCH MEETING OF THE MICHIGAN SECTION

The spring meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan at Ann Arbor on Satday, March 22, 1947. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts and Letters. Morning

and afternoon sessions and a luncheon-business meeting were held, at all of which the Chairman, Professor E. E. Ingalls, presided.

About ninety persons attended the meeting including the following forty-seven members of the Association: N. H. Anning, J. W. Baldwin, J. M. Barbour, F. A. Beeler, W. M. Borgman, J. W. Bradshaw, J. B. Brandeberry, R. V. Churchill, Nathaniel Coburn, A. H. Copeland, C. C. Craig, Wayne Dancer, P. S. Dwyer, C. M. Erikson, C. H. Fischer, J. S. Frame, V. G. Grove, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, E. E. Ingalls, L. G. Johnson, L. S. Johnston, P. S. Jones, Wilfred Kaplan, G. P. Loweke, E. D. McCarthy, L. E. Mehlenbacher, A. L. Nelson, C. V. Newsom, George Piranian, G. Y. Rainich, E. D. Rainville, M. O. Reade, C. C. Richtmeyer, E. H. Rothe, L. J. Rouse, T. R. Running, Hans Samelson, W. F. Smith, T. H. Southard, B. M. Stewart, C. G. Stripe, I. L. Stright, D. R. Sudborough, R. M. Thrall, C. P. Wells, and R. L. Wilder.

The following officers were elected for the coming year: Chairman, Professor G. C. Bartoo, Western Michigan College of Education; Secretary-Treasurer, L. J. Rouse, University of Michigan.

At the morning and afternoon sessions the following program of eight papers was presented:

1. *The geometrical representation of group* $C_2$, by Dr. Mary H. Payne, University of Detroit, introduced by Professor L. S. Johnston.

It is well known that the unitary matrices form a double-valued representation of rotations in three-space, and that the entire group $C_2$ is a double-valued representation of the true Lorentz group, consisting of rotations in three-space and ordinary Lorentz transformations. The functions on which the matrices of the group $C_2$ act are two component vectors of the forms

$$\cos\frac{\theta}{2}\,e^{i(\phi+\psi)/2}, \qquad \sin\frac{\theta}{2}\,e^{i(-\phi+\psi)/2}$$

each multiplied by a constant. Dr. Payne showed that these two components may be interpreted as the relativistic distances of some point on the light cone from each of two reference points.

2. *"Kasner" monogenic functions*, by Professor Emeritus V. C. Poor, University of Michigan, introduced by the Secretary.

This is a restricted class of polygenic functions dependent upon the existence of the Cioranescue derivative, a two-dimensional directed space derivative. This note is an analogue to the R. J. Haskell paper on areolar monogenic functions in the *Bulletin of the American Mathematical Society*, vol. 52, 1946, pp. 332–337.

3. *Music and ternary continued fractions*, by Professor J. M. Barbour, Michigan State College.

The problem of dividing the octave (2:1) into a larger number of equal parts in terms of which the major third (5:4) and the perfect fifth (3:2) might

be expressed with nearly integral coefficients may be attacked by approximating the ratios of log 2:log 3/2:log 5/4 simultaneously by ternary continued fractions. The common division into twelve semitones gives the ratios 12:7:4. Some other known approximations are 19:11:6, 31:18:10, and 53:31:17. Slow convergence of the ternary continued fractions is desired in order to obtain ratios involving relatively small numbers. Hence in dividing to obtain the partial denominators for the left and right hand fractions it is well to divide by the larger remainder. For any right term $S_n = qS_{n-1} + S_{n-2}$, whereas for the left term $S_n = qS_{n-1} + S_{n-k-3}$ where $k$ is the number of intervening right terms since the last left term.

4. *Transformations of sequences into regions*, by Dr. George Piranian, University of Michigan.

The core of a sequence $S_n = x_n + iy_n$ is a certain non-empty set in the closed complex plane; in the case of a bounded sequence it reduces to the least convex set containing all limit points of the sequence (*cf.* Knopp, *Zur Theorie der Lemilierungsverfahren*, Math. Zeitschr., vol. 31, 1929, pp. 95–127). In the formal transformation $t_n = \sum_{k=0}^{\infty} a_{nk} s_k$ of an arbirtary sequence $s_n$ by an arbitrary matrix $a_{nk}$, some of the numbers $t_n$ may fail to exist; but the core $c_n$ of the sequence $w_{np} = \sum_{k=0}^{p} a_{np} s_p$ $(p = 0, 1, \cdots)$ always exists. The sequence of cores $c_n$ is used to define a convex region $\Gamma$ to be associated with the formal transformation of the sequence $s_n$. If the transformed sequence exists in the classical sense, the region $\Gamma$ is the core of the transformed sequence.

5. *Multinomial theorem*, by Professor G. P. Loweke, Wayne University.

By proper arrangement of terms within the summation, the expansion of a multinomial of $m$ terms may be expressed as

$$(A + B + C + \cdots + M)^n = \sum \sum_{\beta=\gamma}^{\beta=[(n-\gamma-\delta-\cdots-\mu)/2]} \sum_{\gamma=\delta}^{\gamma=[n/3]} \sum_{\delta=\epsilon}^{\delta=[n/4]} \cdots$$

$$\cdot \sum_{\kappa=\mu}^{\kappa=[n/m-1]} \sum_{\mu=0}^{\mu=[n/m]} \frac{n!}{(n-\beta-\gamma-\delta-\cdots-\mu)!\beta!\gamma!\cdots\mu!} A^{n-\beta-\gamma-\cdots-\mu} B^{\beta} C^{\gamma} \cdots M^{\mu}$$

where the brackets in the upper limit indicate the integral part of the fraction, and the number of the summation signs will be $m$ when $m < n$ and $n$ when $n < m$. Expressed in this form the binomial theorem becomes,

$$(a + b)^n = \sum \sum_{\beta=0}^{\beta=n/2} \frac{n!}{(n-\beta)!\beta!} a^{n-\beta} b^{\beta}.$$

This summation for the multinomial expansion can be used to derive the terms of the expansion directly, or if an arbitrary selection of terms is employed, the evaluation of the limits only will indicate the number of characteristic terms which arise in the solution.

6. *Undergraduate program in mathematics*, by Professor C. V. Newsom, Oberlin College.

7. *A machine shop problem*, by Professor B. M. Stewart, Michigan State College.

A geometric solution is provided for the angular adjustments, $A$, $B$, of a drill vise which is to hold a rectangular parallelepiped so that a vertical drill will bore a hole whose projections on a pair of intersecting faces of the parallelepiped will make specified angles, $a$, $b$, with respect to the line of the intersection of these faces. The solution is found as follows: $A = 90° - a$; $B = \text{arc tan } (\cos a \tan b)$. A nomogram is provided for the latter relation.

8. *On graphical methods for elementary differential equations*, by Professor Wilfred Kaplan, University of Michigan

The purpose of this paper is to indicate how interest can be added to the course in elementary differential equations by a closer correlation between analytical and graphical methods. As illustrations, some significant properties of the graphs of linear, homogeneous, and exact equations, are pointed out.

L. J. ROUSE, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Muncie, Oct. 17, 1947
IOWA, Fairfield, April 1, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Beloit, May 8, 1948

# FOURIER'S SERIES

## *The Genesis and Evolution of a Theory*


By

RUDOLPH E. LANGER

Professor of Mathematics

in the

University of Wisconsin


*The First*

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER


Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 54 Number 7

August-September, 1947

THE AMERICAN

MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 7

PART II


# FOURIER'S SERIES

## *The Genesis and Evolution of a Theory*


R. E. LANGER


Number 1
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS


AUGUST-SEPTEMBER 1947

ONE DOLLAR A COPY

# The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)




---


THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)




---


Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

## TABLE OF CONTENTS

## THE HERBERT ELLSWORTH SLAUGHT MEMORIAL PAPERS

The editorial committee is happy to present this, the first of the Slaught Memorial Papers. In view of the very great interest which Professor R. E. Langer has had in these Papers from their very inception, it is particularly fitting that he should be the author of the first of the series.

At this time it may be appropriate to give a brief account of the origin of these Papers, and an indication of their purpose. This can perhaps best be done by quoting from the reports of two different committees of the Association. In the MONTHLY for February, 1940, there appears the report of a "committee to review the activities of the Mathematical Association of America." Along with other recommendations, this report, which was prepared by Professor Langer as chairman, suggested that the Association establish a series of expository pamphlets which might well take the form of a memorial to Professor Slaught. Subsequently, another committee was appointed, with Professor C. V. Newsom as chairman, to make specific recommendations for the establishment of such a series. The report of this committee appeared in the MONTHLY for February, 1941.

From the report of the "Langer Committee" we may quote as follows:

"The encouragement and sponsorship of expository and critical writing is one of the objectives of the Association which enjoys the unanimous support of the members. There is a ready welcome and a general demand for more readable scholarly papers on all kinds of mathematical subjects from the classical to the modern, from the elementary to the advanced, on theory, on applications, on history, or on philosophy. In the past there have, of course, been the Carus monographs, and from time to time excellent papers in the MONTHLY. There seems, however, to be at the present little or no means for the ready publication of writings which in length are intermediate betwen the relatively few pages of a journal paper, and the relatively many pages of a complete monograph. Such papers, say in length between twenty and a hundred pages, could be profitably written on subjects in many categories, including among others, elementary introductory expositions of theories and their applications, more advanced expositions and interpretations of modern viewpoints and theories, philosophical essays and criticisms, broad historical accounts of important schools, or biographical accounts of individuals."

The report of the "Newsom Committee" said in part:

"The principal conclusion reached by this investigation is that there is a widespread interest in additional expository writing of the type discussed in the report of the 'Langer Committee,' and that those who are now sponsoring series of expository monographs would welcome the creation of additional opportunities for the publication of studies pertaining to mathematical subjects. In truth, the members of the committee have been impressed with the enthusiasm which has been displayed by those who have given opinions relative to the possibility

of a new publication program sponsored by the Association. Syntheses of modern investigations in many fields of mathematics seem to be wanted by college men who do not have an opportunity to follow developments in the mathematical literature. Instructors in our junior colleges and secondary schools who may have a limited preparation in mathematics are seeking easily accessible accounts of some of the older theories. Some correspondents have expressed the belief that there is an amazing dearth of readable mathematical material for college students who have studied little beyond the calculus. And finally, some have emphasized that the interest of the American public in mathematical attainments and methods needs to be cultivated; this interest is attested to by the recent wide sale of a few popular books upon mathematics."

In accordance with the recommendations of the "Newsom Committee," the Board of Governors authorized a series of expository pamphlets to be known as the "Herbert Ellsworth Slaught Memorial Papers." The long delay in the actual appearance of the first of these Papers was largely caused by the demands of the war which left little or no time for the writing of mathematical expositions.

The Slaught Memorial Papers are to be published in the form of supplements to the MONTHLY and, at least for the present, are being sent free to all subscribers. The success of this project will depend on the interest of mathematicians generally and, more particularly, upon the co-operation of competent scholars who will be willing to devote sufficient effort to the difficult but worth-while task of writing elementary expositions of their respective fields of interest.

The editorial committee through the undersigned will welcome suggestions from any interested persons and, in particular, will be glad to hear from prospective authors of expository articles which might be suitable for publication in this series.

N. H. McCOY

# FOURIER'S SERIES

## THE GENESIS AND EVOLUTION OF A THEORY

R. E. LANGER

### PREFACE

In choosing to present in this exposition some chapters from the theory of the representation of arbitrary functions in infinite series, I have done so in the belief that this subject has an unusually broad appeal. For in singular measure it serves both theoretical and practical ends. The pure analyst finds in it a wealth of structure and subtle inter-relationship, while the applied mathematician and the related scientist find in it, no less, a tool of almost endless flexibility and use.

The simpler formal elements of the theory of trigonometrical infinite series are, it may be assumed, in some measure familiar to all who aspire to a level of mathematical attainment above the elementary one. Presentations of them in text-book form are common, and many of them eminently readable. It is not my purpose to duplicate any such expositions of fact and procedure, but rather to present here other matters less usually considered. These I have, in the main, centered about two focal theses, namely first, a sort of case history of the inception of the theory and its development to the stage attained by Fourier, and second, a generalization of the theory in which the trigonometrical form of it is subordinated to the status of a mere special case.

In its modern form the theory of Fourier's series and its applications to problems of physics admit of presentation in a direct and logical manner that is, on the whole, strikingly economical in design. The reasoning is straight-forward and to the point, and has at almost every turn an aspect of complete inevitability. The trigonometric formulas invariably appear to fit the needs at issue with such precision and neatness as could not have been more so had they been specifically tailored to the purpose. So completely is this true, that it seems no far cry to the suggestion that the whole structure might be the creation of some single master architect, who, in his genius, could draw to hand the exact and unerring means for an orderly and consummate unfolding of all the whole essential machinery of thought and analysis. Of erstwhile possible deficiencies, no trace is left revealed.

Well developed mathematical theories are prone to seem like that, and in the deceptiveness of this there is weakness as well as strength. The craftsman, whose concern with the theory is motivated by the mere search for a tool, naturally has small interest in the cruder forms of it that are now obsolete. For the student whose concern is with ideas no less than with facts, on the other hand, the too finished result is often concealing as well as revealing. The confusion of germinal ideas, the labor and stumbling of the early advance, the frustrations in imprecise notions—all these are matters which for a speedy mastery of the facts are well

left aside. But precisely in these as in none other are to be discerned the creative imagination, the initial inductions and the logical strategy by which the final result was shaped. When, as in the present instance, the ingenuity and the technical exploits which gave the impetus and direction were those of such masters as the Bernoullis, Euler, d'Alembert, Lagrange and Fourier, it need hardly be feared that a review of the course of the developments will prove to be an unrewarding venture. In the growth of mathematics, much more than in its refined and polished results, is its living character evidenced.

Generalization is the medium through which the mathematician constantly seeks the enlargement of his conceptions and understanding. The vista revealed by an extant theory may be broad, but it is broadened further by generalization. And from the more expansive viewpoint the scene may be revealed not only more amply but also more distinctly. A greater simplicity in the intrinsic plan may be discernible, for many features originally judged to be quite essential may be shown, on the contrary, to be in fact merely incidental or fortuitous. This is quite the case with the Fourier theory. Its dependence upon the trigonometric formulas of combination is so conspicuous as to seem to be the very essence of it. And if in the related theories of representations in series of Bessel functions or Legendre polynomials *etc.* other combination formulas are basic, these in turn generally seem, if anything, even more specialized. A true generalization, from which the Fourier theory may be drawn forth as a special case, is the theory of ordinary differential boundary problems in which the fundamental interval of the variable is one upon which the differential equation is without singular points.

The discussion which I have given here is intended to serve these two purposes. In the first part the theme is historical. It centers about the incipience and the classical development of the theory, and is in fact a digest of some works by different masters through which conspicuous advances were made. This comes to its terminus with the discussion of Fourier's deductions, and therewith the historical thread is definitively dropped. In the second part, which is devoted to the generalization, the purpose is purely expository. The material is there set forth in as elementary a manner as I found possible, not with the generality in which it exists in the literature, but with such generality as seemed to be adequate to the display of its essential character. The fact that this larger theory embraces that of Fourier, and the manner in which it does so, is shown at appropriate points by drawing the trigonometric formulas forth as specializations obtainable without any peculiar implementations from the more general relationships derived.

I believe that in the main the paper will be readable for students who in mathematics have gone but little beyond a good course in the calculus. The simplest facts about infinite series and differential equations, the formulas for the trigonometric functions in terms of exponentials, and such, have been assumed. Beyond that all pertinent deductions have been included until the closing chapters are reached. Incidental material has, in part, been relegated to

appendices. In the final chapters the elementary theory of functions of a complex variable, and in particular the theory of residues is a requisite.

In taking material from the literature, especially in the earlier parts of the paper, I have felt under no obligation to hold to the letter of the originals. The excerpts are, therefore, distinctly not to be regarded as facsimiles or verbatim reports. Although it was my intention to preserve the spirit, many formal changes were made, in part to bring the contributions from diverse sources under a consistent scheme of notation, but also in part to eliminate discursive material and to avail myself of such advantages as the presentation to modern readers might afford. The sources in the literature from which excerpts were made, or at which more extensive deductions may be found, have been indicated in the text, and are listed at the end of the paper. I have, however, made no attempt whatsoever to be complete in this matter.

# PART I

## CHAPTER 1

**Introduction.** It is perhaps true that few mathematical doctrines are built around facts which on first acquaintance seem so surprising as those of the infinite trigonometric series. That some such series represent functions is obvious. Simple examples are easily constructible in any abundance. How broad or how deep the adaptability of these series for functional representation may be, is, on the other hand, far from easy to see. The terms of the series are but elementary functions of a very simple type—the sines and cosines of the multiples of an angle. The theory of power series does not promise much by analogy, since those series can represent only such functions as have all the regularity which unlimited differentiability assures. It might well be expected, therefore, that the property of trigonometric representability attaches only to the functions of a quite restricted class. Historically that was the opinion which originally held sway, and which was very generally maintained. It was, in fact, so long maintained and so tenaciously, even by the greatest of mathematical masters, and in the face of most insistent evidence to the contrary, that the final breach with it took on quite definitely the character of an emancipation.

The concept of the function lies at the very heart of mathematical analysis. As it is now currently accepted it is a notion of very great breadth, covering very general interdependencies of variables upon each other. During the eighteenth century this concept was not only much more restricted, but precisely what its content and delimitations were had not yet been brought to any complete or clear formulation. Imprecise notions rarely fail to breed confusion, and in this respect the functional notion of that time was in no way an exception. It was differently conceived by different investigators. And these latter then disagreed among each other because unconsciously they talked at cross purposes. The written words flowing from different pens had different meanings. While, for instance, the function and the analytic formula were one to d'Alembert, the function was thought of as a graph by Euler, and probably meant something else again to still another.

The basis of the functional notion was originally drawn, of course, from observations upon concrete examples—in the main from such functions as we now designate as of elementary type, such as present themselves in the simpler applications of mathematics to the problems of physics. Such functions are almost invariably expressible by formulas. They generally have comparatively simple, orderly, and continuous graphs, and the identity of any two of them is restricted to isolated values of the variable. Inasmuch as this category includes no examples of distinct functions whose graphs have an entire arc in common, it was no more than natural then to consider that generally the course of a function over any interval was determinative and completely identifying, so that the graph over its entire range of definition was to be thought of as unambiguously fixed. Functional relationships such as are now commonly dealt with, in which

the variables are related by different laws in different parts of an interval, were not thought of then as subsumed in any single function at all, but were regarded rather as a composite of a plurality of functional fragments. The possibility of representing such a conglomerate by a single formula was not even conceived of.

The eighteenth century stands out in mathematical history as an era of great genius. Through the work of an astonishing array of masters the science was extended and broadened by the opening of many new fields. Technical skill attained to extraordinarily high levels and new ideas were crowded one upon the other. And yet through this period the facts of the trigonometric series withheld themselves. Euler, d'Alembert, Lagrange and others walked upon the very edge of them without falling upon them. A more conspicuous example of the confining effects of preconceptions is hardly to be found. The break with all this remained to become the accomplishment of the next century, the personal achievement of Fourier. Once the step to a broader conception of the function had been made, the results of computations upon trigonometric series could be given a much more inclusive interpretation. As is now generally familiar, such series may represent functions which are not only discontinuous, but which may be quite arbitrary in the sense that over different portions of the interval they may accord with laws that need have no logical relation with each other. Computations upon even a few of the initial terms of the series often reveal these facts quite clearly.

As Fourier announced his famous theorem it was to the following effect:

*Any single-valued function $f(x)$ defined over an interval $-l<x<l$, is representable over this interval by a series of sines and cosines in the manner*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l}\right]. \tag{1.1}$$

*In this representaion the coefficients are those which are computable from the function $f(x)$ by the formulas*

$$\begin{aligned} a_k &= \frac{1}{l}\int_{-l}^{l} f(s) \cos\frac{k\pi s}{l}\, ds, \\ b_k &= \frac{1}{l}\int_{-l}^{l} f(s) \sin\frac{k\pi s}{l}\, ds. \end{aligned} \tag{1.2}$$

*If the interval over which the representation is to maintain is only $0<x<l$, then either sines or cosines alone suffice, the series being in the one case*

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l}, \tag{1.3}$$

*with the coefficients*

$$b_k = \frac{2}{l}\int_{0}^{l} f(s) \sin\frac{k\pi s}{l}\, ds, \tag{1.4}$$

*and in the other case*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l}, \tag{1.5}$$

*with the coefficients*

$$a_k = \frac{2}{l} \int_0^l f(s) \cos \frac{k\pi s}{l}\, ds. \tag{1.6}$$

This theorem, in this utter generality, is now known, of course, to be not strictly true, even with the modern notion of the term function. The restrictions which must be applied are, however, in no way gross. They are, on the contrary, so subtle that in the pursuit of them much further clarification of analytic notions was achieved. The rôle which the trigonometric series have played in the development of precise conceptions is, on these many accounts, one of unusual interest to the student of the evolution of mathematical ideas. [1]

## CHAPTER 2

**Of mathematical applications to physics.** By the time of the year 1725, a decade after the death of Leibniz and near the close of Newton's life, familiarity with the formal processes of the calculus had become widely disseminated, and facility in the use of these new techniques had been developed to a very substantial degree. In particular their effectiveness as instruments for the treatment of problems in mechanics had been generally recognized. This science was, therefore, under assiduous study. The more immediate of its problems—in the main those centering themselves around the motions of single mass particles—had already been pretty effectively brought to their solutions. The forefront of interest had, therefore, already been pushed beyond them to matters of greater complexity, such as presented themselves in connection with the motions of bodies with several or many degrees of freedom, or even with the reactions of flexible continuous mass distributions. Problems in the vibrations of elastic bodies in particular had begun to receive attention, and in the ensuing period these were to preoccupy an increasing number of investigators. The scientific literature of the following half century is, therefore, heavily interspersed with memoirs bearing upon this field. One may readily conjecture that the problems to be found there must have seemed almost endless in their abundance and variety. They must, moreover, have exerted a very strong fascination, if for no other reason, then because of the evident suggestion that in them lay an important key to an analytic mastery over the manifestations of nature.

Any review of the activities of the time show this to have been so. Important and difficult works were produced in great variety—among them, to name a few, investigations upon the oscillations of plates in vacuum or immersed in fluids, upon rods suspended from fixed or flexible mountings, upon jointed pendulums,

upon heavy dangling chains *etc. etc.* But among all these the researches upon the motions of tautly stretched elastic strings or wires were to be especially significant. Though these were perhaps of no greater mechanical importance than the discussion of many another problem, they did contemporaneously assume a quite disproportionate prominence, and they still do so in retrospect. They became, namely, a conspicuous point of impact for many divergent conceptions and opinions, and in the rôle thereby thrust upon them they became crucial to the development not only of mechanics but of mathematics over a much wider range. Inasmuch as the ideas at issue are in large measure our essential concern in this discussion, we shall have to give a considerable modicum of our attention to this problem of the string in subsequent pages.

From the very start of investigations upon them it seems to have been assumed that continuous material bodies could, for the purposes of analysis, be approximated by systems of discrete mass particles. The conception of an extensive body as composed of particles is a very natural one. Quite apart from whether this viewpoint was intended in the end to be philosophically maintained or not, it was apparently seen to suggest more strongly than any other a practicable mode of procedure. It requires no vivid imagination to picture the discrete system as merging into the continuous one as the size of the individual particles is diminished indefinitely while their number and density is correspondingly increased. If in the analysis of a finite approximating system the formulas can be so framed and dealt with that results are deducible from them without any actual specification of the number of the particles that are involved, then this number, retaining its generality, may be assumed to figure in the results in the way of a free parameter. This parameter may then logically be made the crux of limiting considerations, and through this means the physical transition from the discrete to the continuous configuration may be thought of as implemented mathematically by a passage from the finite to the infinite.

It may well be recognized that success in the execution of any such subtle program as this would be contingent both upon superior insight and a high level of technical skill on the part of the investigator. Such is the case, no less, with almost any application of mathematics to a phenomenon of nature. The actual responses of physical bodies to ponderable influences are invariably of a discouraging degree of complexity, and this is generally due much less to the influences that are primarily under scrutiny than to the many others that are inevitable and yet really incidental and effectively irrelevant. Were complete recognition to be given to all these latter, the formulation of a natural problem would, almost without exception, be quite submerged in intricacies of detail. On the physical side the fortuitous distracting features might well obscure the salient ones, and mathematically they might well throw the problem far beyond the range of possible solution.

At the very outset, therefore, it is usual and necessary to regard the physical configuration not as it actually is, but as it might be were it to be disencumbered of all but its primarily intrinsic features. The result of this is at once a simplifi-

cation and an idealization. It is this which is made the subject of analysis. The idealization is, of course, a departure from the physical reality, and this fact is certainly of no secondary importance. The original problem having been replaced by another, no immediate guarantee exists, of course, that results derived for the latter have a sufficient relevancy or applicability to the former. It is evident that with too great a divergence from the original all practical purpose would thus be defeated.

This is the consideration which checks the extent to which idealizing abstractions may be made. In determining upon these a fine sense of values and a depth of understanding are the indispensable guides. Whether in any case the permissible bounds were exceeded or not, must in the end ordinarily be decided after the fact by experiment. The applicability of the mathematically deduced theory stands or falls according as at strategic points its results agree or disagree with the data of observations. The decision, either way, implies no reflection whatever upon the theory's logical soundness. Its implications bear only upon the legitimacy of the simplifications which were made in the determination of its basis, namely upon the insight with which suitability and adequacy in the initial idealizing approximations were sensed.

## CHAPTER 3

**The loaded string.** A natural phenomenon that is universally familiar and frequently observable in our surroundings, and to which, despite this, the attention is often sharply drawn, is that of the behavior of a stretched elastic string or wire in its response to a displacement from its state of equilibrium. In many cases this response is acoustically conspicuous, ranging from the hum of the heavy structural wire in the wind to the eloquent notes of the strings of a musical instrument. And then again, not rarely, the response is visually noticeable, being sometimes marked by so curious a feature as the presence of nodal points that maintain the state of rest while the string between them is in violent agitation.

The elastic string thus almost obtrudes itself upon the notice of the experimenter, and, having drawn his attention, recommends itself in many ways. Its geometrical configuration is of the simplest sort, permitting the identification of any of its points by a single dimension. Its motions are, under many circumstances, markedly regular, and respond promptly and prominently to adjust themselves to any quantitative modifications in the length, the tension, the weight or the initial state. It can hardly be looked upon as surprising, in virtue of all this, that the string should have been drawn under analysis at as early a time in the development of mechanics as that science became capable of dealing with continuous flexible bodies.

The finite discrete system of particles that most naturally approximates the continuous material string is suggested by the string of beads. More precisely, it is to be conceived of as comprised of any number $n$ of equal concentrated mass particles, that are mounted respectively at equally spaced points along a string

which is itself weightless, though strong, perfectly flexible, and elastic. Such "loaded strings" were used as the conceptual bases of a variety of significant investigations. Euler and Daniel Bernoulli based upon them, in 1732 or 1733, studies of the motions of heavy dangling chains, and Euler, among other things, regarding the particles as oscillating longitudinally, built upon them in 1746 a theory of sound. [2] Mechanically simpler and chronologically even earlier than these researches, were certain investigations of John Bernoulli upon the transverse vibrations of a string with its end points fixed. We find in a consideration of this problem a convenient point of departure for our present discussion.

Consider, therefore, a loaded string such as has been described, with loading particles having a total mass $M$. Let this string be thought of as held taut under a tension $T$, the magnitude of this being so large that the ratio $M/T$ is negligible. This last stipulation amounts, of course, to an idealization. Substantially its purpose is to discount the effects of gravity, and thus to concentrate the considerations upon those forces which spring from the tension alone. The divergence from physical actuality which this simplifying hypothesis sanctions is in many important instances of a very small amount. In the case of musical strings, for example, the ratio $M/T$ which is to be ignored is quite commonly of a magnitude no greater than one one-thousandth.

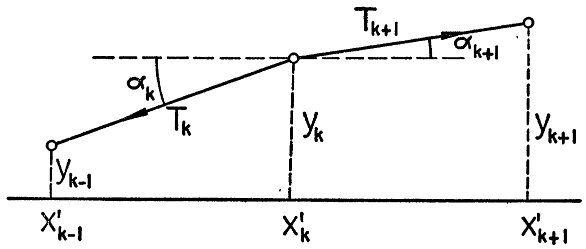
The initial state of the string in question is to be one of displacement from its equilibrium position, the forces which hold it in this state being coplanar and directed perpendicularly to the line through the string's fixed end points. The particles of the string thus lie along some plane curve. We shall choose the plane of this curve as the $(x, y)$ plane, and shall take the $x$-axis through the string's end points with the origin at one of them. Under the hypotheses made, and in this system of reference, the $x$-axis then marks the string's equilibrium position. With $l$ designating the length of the string, with $x_0=0$ its initial end, and with $x_1, x_2, \cdots, x_n$ the equilibrium abscissas of the particles, the formulas for these are to be

$$x_k = k\frac{l}{n}, \qquad k = 0, 1, 2, \cdots, n.$$

It will be noted that in this assignment one of the particles is allocated to be mounted at the point $x_n=l$ which is the terminal end of the string. The motivation for adopting this arrangement, which may well seem a bit curious, is not by any means profound. It is merely one having some formal advantages, since, as can be shown, it leads to somewhat simpler formulas. The rôle of the $n$th particle under these circumstances is, of course, an entirely passive one since the particle is constrained from all motion.

In the accompanying figure three adjacent particles and the tensions operating between them are schematically indicated. The position shown is one of displacement from the equilibrium, and the particle which in the latter state is located at the point $(x_k, 0)$ is here shown to have the coordinates $(x_k', y_k)$. Simple

mechanical considerations based upon this figure readily lead to usable results, as we now propose to show.




The segment of the string which joins the $(k-1)$th and the $k$th particles is, as is clear from the figure, of the length $(x_k' - x_{k-1}')\sec\alpha_k$. In the state of equilibrium this segment would be of the length $l/n$, and since, by the laws of elasticity, the respective lengths are to each other as the tensions, it must be concluded that

$$(x_k' - x_{k-1}')\sec\alpha_k : \frac{l}{n} = T_k : T,$$

or

$$(x_k' - x_{k-1}')Tn = lT_k\cos\alpha_k. \tag{3.1}$$

Now the forces by which the particles are initially held displaced are by hypothesis in the direction of the $y$-axis. The tensions in the string segments which balance them must, therefore, have $x$-components which annul each other in pairs, namely they must be such that

$$T_k\cos\alpha_k = T_{k+1}\cos\alpha_{k+1}, \qquad k = 1, 2, \cdots, (n-1). \tag{3.2}$$

Since the right-hand members of the equations (3.1) are thus all equal, this equality must extend to the left-hand members as well. The differences $(x_k' - x_{k-1}')$ therefore have a common value, and since their sum is the length of the string this value is evidently $l/n$. It follows that for each $k$ the relation $x_k' = x_k$ maintains, namely that each particle has when displaced the same abscissa as in equilibrium. The measures of the displacements are thus simply the ordinates $y_k$, and the relations (3.1) reduce to the forms

$$T_k\cos\alpha_k = T, \qquad k = 1, 2, \cdots, (n-1). \tag{3.3}$$

From the initial position which has been described the string is now to be thought of as released, while in the state of rest, at an instant which is to be taken as the origin of time, $t=0$. The motion into which the $k$th particle springs is, of course, then governed by Newton's law

$$\frac{M}{n}\frac{d^2y_k}{dt^2} = F_k,$$

the force $F_k$ acting upon it being shown by the figure to have the value

$$F_k = T_{k+1}\sin\alpha_{k+1} - T_k \sin\alpha_k.$$

By virtue of the relations (3.3) this formula is alternatively expressible as

$$F_k = T(\tan\alpha_{k+1} - \tan\alpha_k),$$

and in terms of the coördinates this is

$$F_k = T\left[\frac{(y_{k+1} - y_k) - (y_k - y_{k-1})}{(l/n)}\right].$$

Hence if the constant $a^2$ is defined by the relation

$$a^2 = \frac{lT}{M}, \tag{3.4}$$

the equations of motion are

$$\frac{d^2y_k}{dt^2} = \left(\frac{na}{l}\right)^2[y_{k+1} - 2y_k + y_{k-1}], \qquad k = 1, 2, \cdots, (n-1). \tag{3.5}$$

In many respects the simplest modes of vibration which the string is capable of are those in which the ordinates $y_k$ maintain constant ratios to each other. In these so-called *normal* vibrations all particles of the string traverse their positions of equilibrium in synchronism, and their displacements are expressible as functions of the time by formulas of the type

$$y_k(t) = u_k\phi(t), \qquad k = 0, 1, 2, \cdots, n, \tag{3.6}$$

in which the coefficients $u_k$ are constants, with $u_0 = 0$, $u_n = 0$, and $\phi(t)$ is common to them all. The fact that the motion originates from the state of rest is then expressed by the relation $\phi'(0) = 0$.

Now the substitution of the forms (3.6) into the equations (3.5) gives to these latter the aspect

$$u_k\phi''(t) = \left(\frac{na}{l}\right)^2[u_{k+1} - 2u_k + u_{k-1}]\phi(t).$$

From this it is clear that the second derivative $\phi''(t)$ stands in a constant ratio to the function $\phi(t)$ itself, namely that

$$\frac{d^2\phi(t)}{dt^2} = -c^2\phi(t), \qquad \phi'(0) = 0, \tag{3.7}$$

the constant $c$ being one for which the relations

$$u_{k+1} - 2u_k + u_{k-1} = -\left(\frac{cl}{na}\right)^2 u_k, \qquad k = 1, 2, \cdots, (n-1),$$

together with $u_0=0$, $u_n=0$ maintain. In terms of the coefficient $q$ given by the formula

$$q = \left(\frac{cl}{na}\right)^2 - 2, \tag{3.8}$$

the values $u_k$ must thus be solutions of the algebraic system of equations

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} + qu_k + u_{k-1} &= 0, \qquad k = 1, 2, \cdots, (n-1), \\ u_n &= 0. \end{aligned} \tag{3.9}$$

This system is neatly solvable (*c.f.* appendix I) having a non-trivial solution when and only when the coefficient $q$ has one of the set of characteristic values $q_1, q_2, \cdots, q_{n-1}$, given by the formulas (I.5). The values of $c$ which respectively correspond to these under the relation (3.8) are those of the set

$$c_\nu = \frac{2na}{l} \sin \frac{\nu\pi}{2n}, \qquad \nu = 1, 2, \cdots, (n-1), \tag{3.10}$$

and the solution $u_{\nu,k}$ of the system (3.9) which exists for the value $c_\nu$ is obtainable from the formulas (I.6). It is

$$u_{\nu,k} = A_\nu \sin \frac{k\nu\pi}{n}, \qquad k = 0, 1, 2, \cdots, n,$$

with $A_\nu$ designating any constant. Since when $c=c_\nu$ the equations (3.7) are solved by the function

$$\phi(t) = \cos(c_\nu t),$$

or by a constant multiple of this, it may be drawn from the relations (3.6) that the ordinates in any normal vibration of the loaded string must accord with the formulas

$$y_k(t) = A_\nu \sin \frac{k\nu\pi}{n} \cos\left(\frac{2ant}{l} \sin \frac{\nu\pi}{2n}\right), \qquad k = 0, 1, 2, \cdots, n. \tag{3.11}$$

A loaded string carrying $n$ particles is thus seen to be capable of sustaining $(n-1)$ distinct motions of the normal type, these being given by the formulas (3.11) in correspondence with the indices $\nu=1, 2, \cdots, (n-1)$.

Under assumptions that were somewhat more restrictive than those which we have here imposed, these normal vibrations were considered by John Bernoulli as early as the year 1728 in the cases of loaded strings in which particles up to eight in number were involved. [3]

## CHAPTER 4

**The equations of motion for the continuous string.** When the differential equations (3.5) for the motions of the loaded string of $n$ particles have once been deduced, two alternative modes of procedure for utilizing them toward the ultimate purpose of an analysis of the vibrations of a continuous string suggest themselves. On the one hand the equations may be integrated, as has already been done in the preceding chapter, and the resulting finite equations (3.11) may then be subjected to the limiting process in which the number $n$ is indefinitely increased. On the other hand this limiting process may be applied directly to the system of equations (3.5) itself, and the integration of the result may then subsequently be undertaken. Both of these procedures were carried out in the first half of the eighteenth century. As we shall see, their results are of quite dissimilar aspects. Indeed they seemed to the men of the time to be no less than contradictory, to the extent that the proponents of either method saw no alternative but to reject the other. That no real dilemma was actually involved therein at all, came to its realization only half a century or more later. The clarifications of ideas by which the way out of the quandary was ultimately found are of especial interest to us here. We propose, therefore, to pursue the analysis of the two mentioned procedures to such points, at least, as afford some surveys of their conclusions.

Returning then, to begin with, to the equations (3.11), let any positive integer $\nu$ be chosen. Once chosen, $\nu$ is to be regarded as fixed. A loaded string with particles in number greater than $\nu$, $(n>\nu)$, may then be thought of, and for the normal motions of such a string the equations (3.11) are derivable. Let the attention then be fixed upon any one of the particles of this string, and let its abscissa and ordinate be designated by $x$ and $y(t, x)$. If this particle, in the enumeration that was adopted, is the $k$th one, the equalities

$$x = k\frac{l}{n}, \qquad y(t, x) = y_k(t), \tag{4.1}$$

evidently maintain. The respective $k$th equation of the set (3.11) may then be written in the manner

$$y(t, x) = A_\nu \sin\frac{\nu\pi x}{l}\cos\left[\theta_n\frac{\nu\pi a t}{l}\right], \tag{4.2}$$

with the significance of $\theta_n$ given by the formula

$$\theta_n = \frac{\sin\left(\dfrac{\nu\pi}{2n}\right)}{\left(\dfrac{\nu\pi}{2n}\right)}. \tag{4.3}$$

Suppose now that the parameters $n$ and $k$ are increased, and indefinitely so, in any way such that the ratio $k/n$ remains fixed. The point $x$ then clearly remains invariant, and inasmuch as the formula (4.3) familiarly shows that the value $\theta_n$ approaches the limit 1 it follows that the relation (4.2) passes, as $n \to \infty$, into the limiting form

$$y(t, x) = A_\nu \sin \frac{\nu\pi x}{l} \cos \frac{\nu\pi a t}{l}\,. \tag{4.4}$$

This result is now evidently to be accepted as a formula applicable to the continuous string and representing a normal vibration of it. Inasmuch as the integer $\nu$ was initially open to an arbitrary choice, the inference that infinitely many such normal motions are possible and that they are given by the formulas (4.4) in conjunction with the indices $\nu = 1, 2, 3, \cdots$ is inevitable.

The simplest of the normal vibrations, namely that described by the formula (4.4) with $\nu = 1$ was deduced by Brook Taylor at as early a date as 1713. [4] In this motion the string vibrates without nodes and emits its fundamental tone. The existence of other normal motions, namely those associated by the formula (4.4) with other values of $\nu$ and in which the string emits its various over-tones, were known later to Daniel Bernoulli. We shall have occasion to return to this matter again.

The alternative procedure, to which we now turn, is associated most prominently with the names of d'Alembert and Euler. With the notational changes (4.1) and with the definition of $\Delta x$ by the formula $\Delta x = l/n$, the $k$th one of the equations (3.5) may evidently be written in the form

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2 \left[\frac{y(t, x + \Delta x) - 2y(t, x) + y(t, x - \Delta x)}{(\Delta x)^2}\right]. \tag{4.5}$$

Now whenever the function $y(t, x)$ is one which is twice differentiable as to $x$, its second partial derivative with respect to $x$ is obtainable as the limit of the difference quotient within brackets on the right of the equality (4.5) as $\Delta x \to 0$, namely as $n \to \infty$. Basing himself upon this observation d'Alembert deduced in 1747 the partial differential equation

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2 \frac{\partial^2 y(t, x)}{\partial x^2}, \tag{4.6}$$

for the motion of the continuous string. This result is of course still a standard. A solution of it, if it is to represent the ordinates of a string that is fastened at its end points upon the $x$-axis and that springs into motion at the time $t = 0$ from the state of rest and from the position of a curve $y = f(x)$, must, of course, also fulfill the conditions

$$
\begin{aligned}
y(t, 0) &= 0, \\
y(t, l) &= 0, \\
\left.\frac{\partial y(t, x)}{\partial t}\right]_{t=0} &= 0, \\
y(0, x) &= f(x).
\end{aligned} \tag{4.7}
$$

The problem as thus formulated was solved by d'Alembert through the following ingenious use of familiar formulas from the calculus. [5]

In terms of the abbreviations

$$\frac{\partial y}{\partial t} = p, \qquad \frac{\partial y}{\partial x} = q, \tag{4.8}$$

the differential equation (4.6) is expressible in the form

$$\frac{\partial q}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial t},$$

whereas it is familiar that generally

$$\frac{\partial q}{\partial t} = \frac{\partial p}{\partial x}.$$

By the use of these relations the standard formula

$$dq = \frac{\partial q}{\partial t}\,dt + \frac{\partial q}{\partial x}\,dx,$$

may, however, be written thus

$$dq = \frac{\partial p}{\partial x}\,dt + \frac{1}{a^2}\frac{\partial p}{\partial t}\,dx.$$

From this together with the companion formula

$$dp = \frac{\partial p}{\partial t}\,dt + \frac{\partial p}{\partial x}\,dx,$$

it may be seen at once that

$$
\begin{aligned}
d\left(\frac{p}{a} + q\right) &= \left[\frac{1}{a}\frac{\partial p}{\partial x} + \frac{1}{a^2}\frac{\partial p}{\partial t}\right] d(at + x), \\
d\left(\frac{p}{a} - q\right) &= \left[\frac{-1}{a}\frac{\partial p}{\partial x} + \frac{1}{a^2}\frac{\partial p}{\partial t}\right] d(at - x).
\end{aligned} \tag{4.9}
$$

Consider the first one of these equations. The quantity $(p/a+q)$ is a function of $t$ and $x$. These variables are in turn determinable from the combinations

$(at+x)$ and $(at-x)$. It follows from this that it is permissible to regard the quantity as a function of the variables $(at+x)$ and $(at-x)$, and that accordingly its differential is given by the formula

$$d\left(\frac{p}{a}+q\right) = \frac{\partial\left(\dfrac{p}{a}+q\right)}{\partial(at+x)}\,d(at+x) + \frac{\partial\left(\dfrac{p}{a}+q\right)}{\partial(at-x)}\,d(at-x).$$

But by the evaluation (4.9) this differential includes no term in $d(at-x)$. The coefficient of this term must, therefore, be zero, namely

$$\frac{\partial\left(\dfrac{p}{a}+q\right)}{\partial(at-x)} = 0.$$

This, however, is in effect the assertion that the quantity $(p/a+q)$ does not depend upon the variable $(at-x)$ but is a function of the remaining variable $(at+x)$ alone. A similar chain of reasoning shows that the quantity $(p/a-q)$ is a function of the variable $(at-x)$ alone, namely that with appropriate functions designated by $\phi$ and $\psi$,

$$\frac{p}{a}+q = \phi(at+x),$$

$$\frac{p}{a}-q = \psi(at-x).$$

If these relations are now multiplied respectively by the factors $(adt+dx)/2$ and $(adt-dx)/2$ and are then added, the result on the left of the equality is $pdt+q\,dx$, which is identified as $dy$. Thus

$$dy = \tfrac{1}{2}\phi(at+x)d(at+x) + \tfrac{1}{2}\psi(at-x)d(at-x).$$

In this formula each term is an exact differential. An integration is, therefore, possible and shows that

$$y(t,x) = \tfrac{1}{2}\Phi(at+x) + \tfrac{1}{2}\Psi(at-x), \tag{4.10}$$

the functions $\Phi$ and $\Psi$ being indefinite integrals of $\phi$ and $\psi$ respectively. With the attainment of the result (4.10) d'Alembert had deduced the fact that every solution of the partial differential equation (4.6) is of necessity expressible as the sum of a function of the variable $(at+x)$ and a function of the variable $(at-x)$. It is a simple matter to show conversely, by direct substitution, that if $\Phi$ and $\Psi$ are any suitably differentiable functions, the formula (4.10) does in fact give a solution of the differential equation.

To apply to the particular case of the vibrating string the solution (4.10) must furthermore conform to the conditions (4.7). Of these the first one, which when applied to the relation (4.10) assumes the form

$$\tfrac{1}{2}\Phi(at) + \tfrac{1}{2}\Psi(at) = 0,$$

shows that the function $\Psi$ must be identical with $-\Phi$. That being so the second condition takes the form

$$\tfrac{1}{2}\Phi(at + l) = \tfrac{1}{2}\Phi(at - l), \tag{4.11}$$

and since this is to be an identity in $t$ it shows that the function $\Phi$ must be periodic with the span $2l$ as a period. The third condition (4.7) reduces to the relation

$$\Phi'(x) = \Phi'(-x).$$

Upon an integration this becomes

$$\Phi(x) = -\Phi(-x), \tag{4.12}$$

and thus characterizes $\Phi$ to be an odd function. The last condition, which must maintain over the string's length, reduces then to the relation

$$\Phi(x) = f(x), \qquad 0 \leqq x \leqq l. \tag{4.13}$$

In total it is to be concluded, therefore, that for the vibrating string

$$y(t, x) = \tfrac{1}{2}\Phi(at + x) - \tfrac{1}{2}\Phi(at - x), \tag{4.14}$$

every motion of the string being so representable with an appropriate function $\Phi$. In the instance of any particular motion, in which the curve from which the string is released is $y = f(x)$, the function $\Phi$ that is concerned is determined over the interval $(0, l)$ by the relation (4.13) and is defined for all other arguments by its character of being odd and periodic.

## CHAPTER 5

**The d'Alembert-Euler-Bernoulli controversy.** [6] The method of d'Alembert in his analysis of the problem of the vibrating string was also the method chosen by Euler. Superficially, therefore, the initial memoirs of these masters, written, as they were at short intervals of each other, differed mainly in their details. In their over-all aspect they resembled each other markedly, at least insofar as their formal features were concerned. Only below the surface did the lines of thought show themselves to be divergent, as sharply so, at points, as were the two men in the characters of their genius. Euler's temperament was an imaginative one. He looked for guidance in large measure to practical considerations and physical intuition, and combined with a phenomenal ingenuity an almost naive faith in the infallibility of mathematical formulas and the results of manipulations upon them. D'Alembert was a more critical mind, much less susceptible to conviction by formalisms. A personality of impeccable scientific integrity, he was never inclined to minimize short-comings that he recognized, be they in his own work or in that of others.

To Euler the solution of the problem of the string seemed definitive. Since any and every motion originating from the state of rest would necessarily stem from some initial shape of the string, and since he was willing to accept as the function $f(x)$ any distribution of values consistent with such a shape physically realizable, he maintained the solution of d'Alembert and himself to be the completely general one. The values $f(x)$ involved in the formula (4.13) he regarded as appropriately subject, if necessary, to graphical definition. To such interpretations d'Alembert took exception. He regarded the functional symbol as standing for an expression which could be constructed by the ordinary processes of algebra and the calculus from the independent variables. In having taken the ordinates of the string to be denotable in the form $y(t, x)$, he believed that the results could apply only to such motions as might be characterized by the fact that in them the string shapes at any two instants $t_1$ and $t_2$ are obtainable from one and the same formal expression $y(t, x)$ by giving to $t$ the respective values. He saw no reason to suppose that all possible motions conform to this. Furthermore, inasmuch as the differential equation from which the solution emerges involves the derivative $\partial^2 y/\partial x^2$, he was unwilling to admit the applicability of the analysis to cases in which the function $f(x)$ is not twice differentiable. Finally, because of the relation (4.13) he insisted upon restricting the solution to instances in which the function $f(x)$ is periodic. That there might conceivably exist expressions $\Phi(x)$ and $f(x)$ yielding the same values over some specific interval but not persisting in this relationship for other values of the variable, was believed by neither d'Alembert nor Euler nor by any of their contemporaries.

A difference between d'Alembert and Euler lay in the fact that whereas the former was inclined to look upon the concepts of the function and the analytic expression as synonymous, the latter would not hold to this. Euler saw no reason, for instance, to rule out the possibility of releasing a string from the position of a curve made up of circular arcs of different radii, provided these arcs joined with each other continuously and with a continuously turning tangent line. It is evident that Euler had advanced measurably to the conception of an *arbitrary* curve. It is understandable, however, that d'Alembert should have declined to acknowledge the legitimacy of admitting such curves into consideration where the operations of the calculus were to be employed.

In 1755 a memoir of Daniel Bernoulli's upon the motions of the string turned the entire disagreement into new channels. Bernoulli, who had interested himself in acoustics, had recognized the relation between the several normal vibrations and the respective overtones which the string could be made to emit. It was a generally recognized fact at the time that a musical string ordinarily responds with a combination of its fundamental and overtones. Bernoulli had discovered that the motions involved in this do, in a very definite sense, retain their individuality—that in the entire motion the several normal vibrations are simply superposed upon each other. It was a relatively moderate step from this to the conception that all possible motions of a string are but

linear combinations of the normal vibrations, variations in the relative intensities of the overtone components producing the observable differences in the timbre of the tone. In terms of symbols, and with the use of the formulas (4.4), this comes to its formulation in the assertion that every motion of the string is expressible in the form

$$y(t, x) = \sum_{\nu=1}^{\infty} A_\nu \sin \frac{\nu\pi x}{l} \cos \frac{\nu\pi a t}{l}, \tag{5.1}$$

with appropriate constant coefficients $A_\nu$.

Neither Euler nor d'Alembert was inclined to accept this, and each made his rejoinder. Euler quickly recognized the fact that a motion representable in the form (5.1) would be one for which the initial ordinates have the values

$$\sum_{\nu=1}^{\infty} A_\nu \sin \frac{\nu\pi x}{l}. \tag{5.2}$$

To assume that all motions are here involved would, he pointed out, come to the assertion that an arbitrary function $f(x)$ could be represented by a series of the type (5.2). Since among other things any expression (5.2) is odd and periodic, it seemed to Euler that he had reduced Bernoulli's claim to a manifest absurdity. That Bernoulli's result gave solutions—special ones—he did not deny. He had, in fact, made that discovery on his own account some years earlier.

D'Alembert, on his part, not only endorsed all of Euler's objections but went in fact well beyond them. He was unwilling to concede even that any and every odd and suitably periodic function could be represented by an expression (5.2), maintaining, in particular, that the function would need to be twice differentiable since that is so of all terms of the series. Bernoulli's analysis, and especially his passage from the finite case of the loaded string to the continuous one, had been at best sketchy and fragmentary. His opponents found much that could properly be rejected in that.

On the whole the objections left Bernoulli unshaken. In replying to Euler's claim of absurdity he referred to the fact that any finite sum of $m$ terms from the expression (5.2) could, by an appropriate determination of the coefficients, be made to coincide in value with any given function $f(x)$ at any chosen set of points $m$ in number. He saw no reason, therefore, for rejecting the possibility that the series (5.2), involving infinitely many coefficients as it does, might not coincide with an arbitrary function at an infinity of points. This viewpoint was indeed a worthy one. A development of it will concern us in the following chapter.

The three cornered polemic spreads itself through the mathematical literature over a period of more than a decade. Since no one of the contenders succeeded in convincing another, the upshot of the matter at the time was negligible. Each of the disputants was in part right and in part wrong. Time has given the lion's share of its endorsements to Bernoulli.

## CHAPTER 6

**Lagrange's solution of Bernoulli's problem in curve fitting.** [7] The fact invoked by Bernoulli, that an arbitrarily given curve is representable at any finite set of abscissas by a suitable segment of a series (5.2), is one of considerable importance both from the theoretical and practical standpoints. It devolves, of course, upon the possibility of determining the coefficients $c_k$ in a formula

$$y = \sum_{k=1}^{n-1} c_k \sin \frac{k\pi x}{l},$$

so that the curve here represented may pass through a prescribed set of points $(x_\nu, F_\nu)$, $\nu = 1, 2, \cdots, (n-1)$, the abscissas of which lie upon the interval $(0, l)$. Alternatively stated it comes to the fact that with an arbitrary assignment of constants $F_\nu$ the system of equations

$$\sum_{k=1}^{n-1} c_k \sin \frac{k\pi x_\nu}{l} = F_\nu, \qquad \nu = 1, 2, \cdots, (n-1), \tag{6.1}$$

is solvable for the values $c_k$.

The solution of any linear algebraic system of equations, and hence in particular of this system, is, of course, possible by elementary procedures. Such a frontal attack upon it by the familiar method of determinants leads, however, through much tedious and protracted computation whenever the number of equations is large. The solution, moreover, is not likely to emerge from such manipulations in any neat or elegant form. At the very beginning of his career, while he was still in his early twenties, Lagrange concerned himself with this problem and gave solutions of it for both the cases in which the abscissas are equally and unequally spaced. It is the former of these which is of peculiar pertinence to our discussion, and although it appears in Lagrange's work incidentally to the wider investigation with which we shall be concerned in the next chapter, a self-contained exposition of it is possible and is to be given here. It is a prime merit of this solution that it shows clearly how it depends upon the number $n$ of points involved, and that it is therefore excellently adapted to an investigation in which this number is ultimately to be varied and to be allowed to become infinite.

Let the abscissas $x_\nu$ be identified again thus

$$x_\nu = \frac{\nu l}{n}, \qquad \nu = 0, 1, 2, \cdots, n, \tag{6.2}$$

and let any one of the integers $1, 2, \cdots, (n-1)$ be chosen and designated $j$. If the equations (6.1) are multiplied by the respective constant of an undetermined set $D_{j,\nu}$ and are then added, the result is the relation

$$\sum_{\nu=1}^{n-1}\sum_{k=1}^{n-1} D_{j,\nu}c_k \sin\frac{k\pi x_\nu}{l} = \sum_{\nu=1}^{n-1} D_{j,\nu}F_\nu.$$

Since $kx_\nu = \nu x_k$ this can be given the alternative form

$$\sum_{k=1}^{n-1} \Phi_j(x_k)c_k = \sum_{\nu=1}^{n-1} D_{j,\nu}F_\nu, \tag{6.3}$$

the function $\Phi_j$ herein being defined by the formula

$$\Phi_j(x) \equiv \sum_{\nu=1}^{n-1} D_{j,\nu} \sin\frac{\nu\pi x}{l}\cdot \tag{6.4}$$

Now each function sin $(\nu\pi x/l)$ in this relation is expressible as the product of sin $(\pi x/l)$ by a polynomial of the degree $(\nu-1)$ in cos $(\pi x/l)$ (*cf.* appendix II). The complete function $\Phi_j$ may therefore be similarly expressed, namely thus

$$\Phi_j(x) = \sin\frac{\pi x}{l}\, P_{n-2}\left(\cos\frac{\pi x}{l}\right), \tag{6.5}$$

with $P_{n-2}$ designating a polynomial of the degree $(n-2)$. The coefficients of this polynomial depend, of course, upon the multipliers $D_{j,\nu}$, and these have not thus far been specified. It is proposed now to specify them so that the function $\Phi_j(x)$ may be zero at each of the points $x_k$ with the specific exception of $x_j$, namely so that

$$\Phi_j(x_k) = 0, \qquad k \neq j. \tag{6.6}$$

Assuming this to be possible, it is clear from the equation (6.5) that each one of the $(n-2)$ values cos $(\pi x_\nu/l)$, $\nu \neq j$ must then be a root of $P_{n-2}$, and that each corresponding difference (cos $\pi x/l-$cos $\pi x_\nu/l$) must therefore be a factor. The factors are thus all accounted for, and with an appropriate constant $\alpha$ the formula (6.5) may accordingly be written

$$\Phi_j(x) = \alpha \sin\frac{\pi x}{l} \prod_{\nu=1,\nu\neq j}^{n-1}\left(\cos\frac{\pi x}{l} - \cos\frac{\pi x_\nu}{l}\right). \tag{6.7}$$

As is shown by the formula (II.1), however, a relation

$$\sin\frac{n\pi x}{l} = \sin\frac{\pi x}{l}\, p_{n-1}\left(\cos\frac{\pi x}{l}\right)$$

also maintains, and since the left-hand member of this is zero at each point $x_\nu$ without exception, the function $p_{n-1}$(cos $\pi x/l$) must admit as a factor each of

the differences $(\cos \pi x/l - \cos \pi x_\nu/l)$. The factors being thus again all accounted for, it follows that

$$\sin \frac{n\pi x}{l} = \beta \sin \frac{\pi x}{l} \prod_{\nu=1}^{n-1} \left( \cos \frac{\pi x}{l} - \cos \frac{\pi x_\nu}{l} \right),$$

with $\beta$ standing for some constant that is not zero. From this together with the evaluation (6.7) it appears that

$$\left( \cos \frac{\pi x}{l} - \cos \frac{\pi x_j}{l} \right) \Phi_j(x) = \frac{\alpha}{\beta} \sin \frac{n\pi x}{l}, \tag{6.8}$$

namely, because of the formula (6.4), that

$$\sum_{\nu=1}^{n-1} D_{j,\nu} \sin \frac{\nu\pi x}{l} \left( \cos \frac{\pi x}{l} - \cos \frac{\pi x_j}{l} \right) - \frac{\alpha}{\beta} \sin \frac{n\pi x}{l} = 0.$$

This equation may be reduced by the use of the familiar relation

$$\sin \frac{\nu\pi x}{l} \cos \frac{\pi x}{l} = \frac{1}{2} \sin \frac{(\nu + 1)\pi x}{l} + \frac{1}{2} \sin \frac{(\nu - 1)\pi x}{l},$$

and by a rearrangement of its terms, to appear in the form

$$\sum_{k=1}^{n-1} [D_{j,k+1} + q_j D_{j,k} + D_{j,k-1}] \sin \frac{k\pi x}{l} + \left[ D_{j,n-1} - \frac{2\alpha}{\beta} \right] \sin \frac{n\pi x}{l} = 0, \tag{6.9}$$

the coefficient $q_j$ being specifically

$$q_j = -2 \cos \frac{\pi x_j}{l}, \tag{6.10}$$

and $D_{j,0}$ and $D_{j,n}$ being zero.

The equation (6.9) is identically fulfilled if the multipliers $D_{j,k}$ satisfy the system

$$D_{j,0} = 0,$$

$$D_{j,k+1} + q_j D_{j,k} + D_{j,k-1} = 0, \qquad k = 1, 2, \cdots, (n-1),$$

$$D_{j,n} = 0,$$

and furthermore

$$D_{j,n-1} = \frac{2\alpha}{\beta}.$$

This system is precisely that which is discussed in the appendix I, the coefficient (6.10) being that one of the values (I. 5) for which the system admits the solu-

(I. 6) with $\nu = j$. The free coefficient $A_j$ in this may be determined, moreover, to yield the value prescribed above for $D_{j,n-1}$, and is thus found to have the value $2\alpha/\beta \sin (n-1)j\pi/l$, or because of the relations (6.2)

$$A_j = (-1)^{j+1} \frac{2\alpha}{\beta \sin \dfrac{\pi x_j}{l}} \cdot$$

The evaluation of the multipliers which thus results is

$$D_{j,k} = (-1)^{j+1} \frac{2\alpha \sin \dfrac{k\pi x_j}{l}}{\beta \sin \dfrac{\pi x_j}{l}}, \tag{6.11}$$

and with these the equation (6.3) reduces by virtue of the values (6.6) to the form

$$\Phi_j(x_j) c_j = \sum_{\nu=1}^{n-1} D_{j,\nu} F_\nu. \tag{6.12}$$

It only remains, therefore, to determine the value of $\Phi_j(x_j)$, and this may be done as follows. The formula for $\Phi_j(x)$, as it is given by the equation (6.8), is indeterminate at $x = x_j$. By an application of l'Hospital's rule, however, its limiting value is found to be

$$\frac{n\alpha \cos \dfrac{n\pi x_j}{l}}{-\beta \sin \dfrac{\pi x_j}{l}},$$

namely because of the definitions (6.2),

$$\Phi_j(x_j) = (-1)^{j+1} \frac{n\alpha}{\beta \sin \dfrac{\pi x_j}{l}} \cdot$$

This result, together with the evaluations (6.11), causes a final reduction of the relation (6.12) to the form

$$c_j = \frac{2}{n} \sum_{\nu=1}^{n-1} F_\nu \sin \frac{\nu \pi x_j}{l}, \tag{6.13}$$

and therewith the coefficients in the equations (6.1) have been determined.

## CHAPTER 7

**Lagrange and the vibrating string.** [8] In the controversy over the problem of the vibrating string Lagrange was inclined on the whole to enlist himself upon the side of Euler. To support himself in this position he undertook to re-examine afresh the behavior of the weightless loaded string with an unspecified number of particles, his explicit purpose being to elicit from this a proof that in the case of the continuous string no restrictions upon the shape of the curve marking the initial position are requisite. His method in this has become a standard one. As do the deductions of the preceding chapter, it hinges primarily upon an introduction of undetermined multipliers. By this means he carried through, as we shall see, a general integration of the differential equations for the string's motion, and thus displayed in terms of explicit formulas the dependence of the string's position at any instant upon its initial shape.

The differential equations for the particles of the loaded string of the length $l$ under the tension $T$, with $n$ particles of total mass $M$ located respectively at the points

$$x_k = \frac{kl}{n}, \qquad (k = 1, 2, \cdots, (n-1) \tag{7.1}$$

were deduced in chapter 3 and are given under (3.5). If the initial ordinates of the particles are denoted by $f_k$, and if the particles spring at $t=0$ from the state of rest in these positions, the boundary relations to which the differential equations are to be subjected are

$$\begin{aligned} y_k(0) &= f_k, \\ \left.\frac{dy_k(t)}{dt}\right]_{t=0} &= 0, \qquad k = 1, 2, \cdots, (n-1). \end{aligned} \tag{7.2}$$

Let the equations (3.5) be multiplied respectively by unspecified constants $M_k$. The addition of them then results in the single equation

$$\sum_{k=1}^{n-1} M_k \frac{d^2 y_k}{dt^2} = \left(\frac{na}{l}\right)^2 \sum_{k=1}^{n-1} M_k[y_{k+1} - 2y_k + y_{k-1}],$$

and this, under a re-grouping of its terms, together with the evaluations $M_0=0$, $M_n=0$, takes on alternatively the form

$$\frac{d^2}{dt^2}\sum_{k=1}^{n-1} M_k y_k = \left(\frac{na}{l}\right)^2 \sum_{k=1}^{n-1} [M_{k+1} - 2M_k + M_{k-1}]y_k. \tag{7.3}$$

Consider now the possibility of so choosing the multipliers $M_k$ as to make the corresponding terms of the two sums in the equation maintain a fixed ratio to each other. With a constant of proportionality $\gamma$, the condition upon the multi-

pliers is then this, that they may comprise a solution of the linear algebraic system

$$
(7.4) \qquad \begin{aligned} &M_0 = 0, \\ &M_{k+1} - 2M_k + M_{k-1} = \gamma M_k, \quad k = 1, 2, \cdots, (n-1), \\ &M_n = 0. \end{aligned}
$$

This, however, is precisely the system (I. 1) of the appendix I, with $(-\gamma-2)$ in the rôle of the coefficient $q$. The values of $\gamma$ for which the system is non-trivially solvable are thus found from the relations (I. 5) to be $(n-1)$ in number, namely $\gamma=\gamma_\nu$, $\nu=1, 2, \cdots, (n-1)$, with

$$
(7.5) \qquad \gamma_\nu = -4\sin^2\frac{\nu\pi}{2n}\cdot
$$

Upon designating by the symbols $M_{\nu,k}$ those multipliers $M_k$ which satisfy the system when $\gamma$ has the value (7.5), we find from the formulas (I. 6) the respective evaluations

$$
(7.6) \qquad M_{\nu,k} = A_\nu \sin\frac{k\nu\pi}{n}, \qquad k = 0, 1, 2, \cdots, n,
$$

the coefficient $A_\nu$ being arbitrary.

Let $\sigma_\nu(t)$ be used now as an abbreviation in the sense

$$
(7.7) \qquad \sigma_\nu(t) \equiv \sum_{k=1}^{n-1} M_{\nu,k}y_k(t).
$$

The differential equation (7.3) with its boundary relations (7.2) may then be written in the form

$$
\begin{aligned} \frac{d^2\sigma_\nu}{dt^2} &= \gamma_\nu\left(\frac{na}{l}\right)^2\sigma_\nu, \\ \sigma_\nu(0) &= \sum_{k=1}^{n-1} M_{\nu,k}f_k, \\ \left.\frac{d\sigma_\nu}{dt}\right]_{t=0} &= 0. \end{aligned}
$$

This is a differential system which is easily solvable by elementary means. Its solution is

$$
\sigma_\nu(t) = \left(\sum_{k=1}^{n-1} f_kM_{\nu,k}\right)\cos\left(\sqrt{-\gamma_\nu}\,\frac{nat}{l}\right),
$$

namely, in terms of the evaluations (7.5), (7.6), and (7.7),

$$\sum_{k=1}^{n-1} y_k(t) \sin \frac{k\pi x_\nu}{l} = \sum_{k=1}^{n-1} f_k \sin \frac{\nu\pi x_k}{l} \cos\left(\frac{2nat}{l} \sin \frac{\nu\pi}{2n}\right). \tag{7.8}$$

Inasmuch as the index $\nu$ is free to take the values $1, 2, \cdots, (n-1)$, this is a system of $(n-1)$ equations.

In structure the system (7.8) is evidently of the form (6.1) with the values $y_k(t)$ in the place of unknowns $c_k$. Its solution is therefore given by the formulas (6.13) to be

$$y_j(t) = \frac{2}{n} \sum_{\nu=1}^{n-1} \sum_{k=1}^{n-1} f_k \sin \frac{\nu\pi x_k}{l} \sin \frac{\nu\pi x_j}{l} \cos\left(\frac{2nat}{l} \sin \frac{\nu\pi}{2n}\right), \tag{7.9}$$

$$j = 1, 2, \cdots, (n-1).$$

With this result Lagrange's integration of the equations of motion is complete.

It is suggestive for our purposes to consider the formalisms of a passage from the discretely loaded string to the continuous one upon the basis of Lagrange's formulas. If in the equations (7.9) the notational changes indicated by the substitutions of $x$ and $y(t, x)$ for $x_j$ and $y_j(t)$, of $s_k$ and $f(s_k)$ for $x_k$ and $f_k$, and of $\Delta s$ for $l/n$, are made, the formula assumes the aspect

$$y(t, x) = \frac{2}{l} \sum_{\nu=1}^{n-1} \left( \sum_{k=1}^{n-1} f(s_k) \sin \frac{\nu\pi s_k}{l} \Delta s \right) \sin \frac{\nu\pi x}{l} \cos \left\{ \frac{\nu\pi at}{l} \cdot \frac{\sin\left(\frac{\nu\pi}{2n}\right)}{\left(\frac{\nu\pi}{2n}\right)} \right\}.$$

As $n$ is indefinitely increased the relation

$$\lim \frac{\sin\left(\frac{\nu\pi}{2n}\right)}{\left(\frac{\nu\pi}{2n}\right)} = 1,$$

maintains for each value of $\nu$ and from the very definition of the definite integral

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} f(s_k) \sin \frac{\nu\pi s_k}{l} \Delta s = \int_0^l f(s) \sin \frac{\nu\pi s}{l} ds.$$

These relations suggest as the limiting form of the solution (7.9) the formula

$$y(t, x) = \sum_{\nu=1}^{\infty} \left( \frac{2}{l} \int_0^l f(s) \sin \frac{\nu\pi s}{l} ds \right) \sin \frac{\nu\pi x}{l} \cos \frac{\nu\pi at}{l}. \tag{7.10}$$

It will be noted at once that this is precisely of the type of the solution (5.1) for

which Bernoulli had insisted upon holding out. In it the initial position of the string is along the curve whose ordinates are $f(x)$. At the instant $t=0$ the formula (7.10) thus reduces to the form

$$f(x) = \sum_{\nu=1}^{\infty}\left(\frac{2}{l}\int_0^l f(s)\sin\frac{\nu\pi s}{l}\,ds\right)\sin\frac{\nu\pi x}{l}. \tag{7.11}$$

The representability of an "arbitrary" function by a series of sines in precisely the manner (1.3), (1.4), would hereby seem to be unmistakably presaged.

Although Lagrange carried out a limiting analysis upon his formula (7.9), it varied in some respects from that outlined above. His comparative result was the relation

$$y(t, x) = \frac{2}{l}\int_0^l\left(\sum_{\nu=1}^{\infty}\sin\frac{\nu\pi s}{l}\sin\frac{\nu\pi x}{l}\cos\frac{\nu\pi a t}{l}\right)f(s)\,ds, \tag{7.12}$$

which differs from (7.10) in having the order of the integration and summation reversed. This form has the disadvantage of involving a series which is obviously divergent, a matter which was readily seized upon by opposing critics. Beyond that, both limiting considerations are open to criticism upon a number of accounts, for they fail to distinguish between the results of analytical operations upon an infinite series as a whole and upon the terms of the series individually. These distinctions, so essential to rigor, were but imperfectly understood at the time.

While in the manner shown the formula (7.9) could easily have led to the conclusion (7.11), it remains a fact that it did not do so. In deducing the form (7.12) Lagrange was bent upon a different purpose, wholly remote from that of proving any such a theory as would be implied by the relation (7.11). Indeed, when such a theory was announced by Fourier more than a half century later, the then aged Lagrange is said to have remained incredulous of it.

## CHAPTER 8

**Euler's determination of the coefficients.** In the latter half of the eighteenth century the properties of trigonometrical series were very much to the fore of mathematical interest, and numerous memoirs were written during that time upon one phase or another of the subject of the representability of functions by means of such series. In the main, however, these papers advanced the general theory but little. They may well be left aside in the present discussion as of only subordinate interest. A conspicuous exception to this, however, is a work of Euler's which he appears to have written in the year 1777, although its publication was deferred until 1793, some years after his death. Concerned with functions known upon some grounds or other to be representable in terms of a cosine series of the type (1.5), Euler deduced in this work the formula (1.6) for the coefficients [9].

If $f(x)$ is any function which in terms of the variable $\xi$, under the relation $\xi = \cos \pi x/l$, is expansible for $-1 \leqq \xi \leqq 1$ in a convergent power series $\sum_{j=0}^{\infty} c_j \xi^j$, then $f(x)$ clearly admits of representation by a cosine power series of the form

(8.1) $$f(x) = \sum_{j=0}^{\infty} c_j \cos^j \frac{\pi x}{l},$$

over the interval $(0, l)$. For each term in this, however, there maintains a respective trigonometric identity (*cf.* appendix III), namely

$$\cos^j \frac{\pi x}{l} = \frac{1}{2^{j-1}} \sum_{\mu=0}^{[j/2]} \binom{j}{\mu} \cos \frac{(j - 2\mu)\pi x}{l},$$

with $[j/2]$ designating the greatest integer not exceeding $j/2$. The substitution of these evaluations into the equation (8.1) and the subsequent collection of terms of like character, give the equation formally the aspect

(8.2) $$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \frac{\nu \pi x}{l}.$$

This is the type (1.5). It will be clear that a rather extensive class of functions $f(x)$ fulfills the assumptions that are basic to this reasoning.

The argument given, although it is adequate to permit an inference of the *form* of the representation (8.2), is readily seen to be quite far from being practical. It yields neither an easily applicable nor a generally lucid method by which the coefficients $a_\nu$ therein may be quantitatively evaluated. Since the series converges, it may, of course, be inferred that

(8.3) $$\lim_{\nu \to \infty} a_\nu = 0.$$

From this slender source Euler succeeded in deducing an evaluation of the contants $a_\nu$ which is directly referable to the function $f(x)$ in question.

Let the symbol $n_{\sigma,\tau}$ for any integer $n$ and any indices $\sigma$, $\tau$, be defined to have the values

(8.4) $$n_{\sigma,\tau} = \begin{cases} n, & \text{if } \sigma \equiv \tau \pmod{2n} \\ 0, & \text{if } \sigma \not\equiv \tau \pmod{2n}. \end{cases}$$

There is, then, a trigonometric evaluation (*cf.* appendix IV) to the effect that

(8.5) $$\sum_{\mu=1}^{n-1} \cos \frac{\mu \sigma \pi}{n} = \frac{-1}{2} [1 + \cos \sigma \pi] + n_{\sigma,0}.$$

From this a certain related formula can be easily deduced. If the equation (8.5) is written successively with $\sigma$ replaced by $(\nu + k)$ and by $(\nu - k)$, an addition of the results yields directly the equality

$$\sum_{\mu=1}^{n-1}\left[\cos\frac{\mu(\nu+k)\pi}{n}+\cos\frac{\mu(\nu-k)\pi}{n}\right]$$
$$= -\left[1+\tfrac{1}{2}\cos(\nu+k)\pi+\tfrac{1}{2}\cos(\nu-k)\pi\right]+n_{\nu+k,0}+n_{\nu-k,0},$$

and this is contractible in an obvious fashion into the formula

$$\sum_{\mu=1}^{n-1} 2\cos\frac{\mu k\pi}{n}\cos\frac{\mu\nu\pi}{n} = -\left[1+\cos k\pi\cos\nu\pi\right]+n_{\nu,k}+n_{\nu,-k}. \tag{8.6}$$

Consider now the representation (8.2). If this is multiplied by the factor $2\cos(\mu k\pi/n)$ and is then evaluated at $x=\mu l/n$, it yields the equalities

$$2f\left(\frac{\mu l}{n}\right)\cos\frac{\mu k\pi}{n} = a_0\cos\frac{\mu k\pi}{n}+\sum_{\nu=1}^{\infty}a_\nu 2\cos\frac{\mu\nu\pi}{n}\cos\frac{\mu k\pi}{n}\cdot$$

When these are summed with respect to $\mu$ the constants $a_0$ and $a_\nu$ appear with coefficients that are given by the formulas (8.5) and (8.6) respectively. It is thus found that

$$\begin{aligned}2\sum_{\mu=1}^{n-1}f\left(\frac{\mu l}{n}\right)\cos\frac{\mu k\pi}{n} &= \frac{-a_0}{2}\left[1+\cos k\pi\right]-\sum_{\nu=1}^{\infty}a_\nu\left[1+\cos k\pi\cos\nu\pi\right]\\ &\quad + a_0 n_{k,0}+\sum_{\nu=1}^{\infty}a_\nu\left[n_{\nu,k}+n_{\nu,-k}\right].\end{aligned} \tag{8.7}$$

This relation can be materially reduced. In the first place it will be found on the basis of the definitions (8,4) that for $k=0$, or for $k>0$ and $n>k$,

$$a_0 n_{k,0}+\sum_{\nu=1}^{\infty}a_\nu\left[n_{\nu,k}+n_{\nu,-k}\right] = n\left[a_k+\sum_{\lambda=1}^{\infty}(a_{2n\lambda-k}+a_{2n\lambda+k})\right]\cdot$$

In the second place the formula (8.2) itself yields the equation

$$f(0)+f(l)\cos k\pi = \frac{a_0}{2}\left[1+\cos k\pi\right]+\sum_{\nu=1}^{\infty}a_\nu\left[1+\cos k\pi\cos\nu\pi\right].$$

With these evaluations, however, the formula (8.7) is simplified into the form

$$2\sum_{\mu=1}^{n-1}f\left(\frac{\mu l}{n}\right)\cos\frac{k\mu\pi}{n} = na_k-\left[f(0)+f(l)\cos k\pi\right]+n\sum_{\lambda=1}^{\infty}(a_{2n\lambda-k}+a_{2n\lambda+k}). \tag{8.8}$$

Let the notational substitutions $s_\mu=\mu l/n, \Delta s=l/n$, now be introduced. These, together with a division by $n$, give to the equation (8.8) the form

$$\frac{2}{l}\sum_{\mu=1}^{n-1}f(s_\mu)\cos\frac{k\pi s_\mu}{l}\Delta s = a_k-\frac{1}{n}\left[f(0)+f(l)\cos k\pi\right]+\sum_{\lambda=1}^{\infty}(a_{2n\lambda-k}+a_{2n\lambda+k}).$$

In this $n$ is now to be permitted to become infinite. Since each term $a_{2n\lambda\pm k}$ in the final sum approaches zero by virtue of the relation (8.3), and the term in $1/n$ does likewise, the right-hand member of the equation has as its limit $a_k$. The limit of the left-hand member being a definite integral, the conclusion is that

$$\frac{2}{l}\int_0^l f(s)\cos\frac{k\pi s}{l}\,ds = a_k. \tag{8.9}$$

Herewith the problem was solved.

Once in possession of the formula (8.9), Euler recognized the more direct manner in which he might have found it, and by means of which a verification of it might be made. This is, namely, the procedure now generally familiar, of multiplying the representation (8.2) by the factor cos $(k\pi x/l)$, integrating it then term by term and applying the elementary evaluations

$$\int_0^l \cos\frac{\nu\pi x}{l}\cos\frac{k\pi x}{l}\,dx = \begin{cases} 0, & \text{if } \nu \neq k, \\ l/2, & \text{if } \nu = k \neq 0, \\ l, & \text{if } \nu = k = 0. \end{cases} \tag{8.10}$$

To this day the constants $a_k$ as given by the formulas (8.9) are still widely known as the "Euler coefficients" of the function $f(x)$. Such an attribution seems, however, to be somewhat over-generous if representations (8.2) of *arbitrary* functions are in question. Euler was consciously concerned only with such functions as were known upon other grounds to be representable in a cosine series. The crucial observation that the formulas (8.9) are significant for functions of a much wider class than those which, for instance, are representable in the manner (8.1) apparently escaped him. There is no evidence, either in this connection or in any other, that he ever receded from his opposition to Bernoulli's claim that arbitrary functions submit to trigonometric representation

## CHAPTER 9

**Fourier and the theory of heat.** In the interior of a material body heat is in general distributed in a manner that is both non-uniform and fluctuating—that is to say with temperatures that vary from point to point and from time to time. The distribution of temperatures throughout a body is, therefore, naturally determined by a function of the coördinates of position and time. What the precise form of this function is, in any particular case, depends in part upon the thermal properties of the material of which the body is constituted—its density, specific heat, and conductivity—but also in large part upon the instantaneous state in which the body finds itself at some specific time, and upon the conditions which thereafter maintain upon its surface.

In the early years of the nineteenth century Fourier devoted himself to an analysis of this temperature function $\tau$, and deduced from physical fundamentals [10] the fact that it must satisfy a partial differential equation of the form

$$\nabla^2\tau = \kappa^2 \frac{\partial \tau}{\partial t}. \tag{9.1}$$

In this $\kappa^2$ is a positive constant whose value is determined by the thermal properties of the material, whilc $\nabla^2\tau$ is the so-called "Laplacian of $\tau$." In terms of rectangular coördinates this differential expression is

$$\frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} + \frac{\partial^2\tau}{\partial z^2},$$

if all three of the coordinates $x$, $y$, $z$, are significant, or, more simply

$$\frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} \quad \text{or} \quad \frac{\partial^2\tau}{\partial x^2}$$

respectively, if the only space coördinates are $x$, $y$, or merely $x$ alone. A temperature function must accordingly solve a partial differential equation such as (9.1). In any specific instance it must be *that* solution of this equation which takes on those values which apply at some specific instant $t$, and which furthermore fulfills upon the body's surface the thermal relations that maintain there.

A simple and familiar physical formulation illustrates this and will serve also to show the relevancy of this subject of the flow of heat to the basic matter before us, namely that of the representation of arbitrary functions by the means of trigonometric series. Consider a homogeneous material bar in the shape of a right cylinder of the length $l$ and of any cross section. We may choose our coördinate system so that the direction of this bar is that of the $x$-axis with the end faces of the bar located at the points $x=0$ and $x=l$. Let it be supposed now that at some specific instant, which may be designated as $t=0$, the temperatures at all points within the bar having the abscissa $x$ have the common value $f(x)$. From this instant onward each end face of the bar is to be held constantly at the temperature zero, while the lateral surface is insulated against the passage of heat. Except in the trivial case in which $f(x)$ is everywhere zero, heat will flow within the bar, and the lines of flow will be parallel to the $x$-axis. The problem is to determine the temperature at any point of the bar at any instant subsequent to the initial one, namely to determine the function $\tau(x, t)$ for $0<x<l$, and $t>0$. The relations from which this is to be done are in this case evidently the differential equation

$$\frac{\partial^2 \tau}{\partial x^2} = \kappa^2 \frac{\partial \tau}{\partial t}, \tag{9.2}$$

the boundary relations

$$\begin{aligned} \tau(0, t) &= 0, \\ \tau(l, t) &= 0, \qquad t > 0, \end{aligned} \tag{9.3}$$

and the initial condition

$$\tau(x, 0) = f(x), \quad 0 < x < l. \tag{9.4}$$

Fourier's method of attack upon a problem such as this is one that is still widely current in practice, namely the method of the "separation of variables." Let us, to begin with, seek a function $\tau_\nu(x, t)$ to fulfill the equations (9.2) and (9.3), while being of the form of a product of a function of the single variable $x$ by a function of the single variable $t$, namely

$$\tau_\nu(x, t) = \phi_\nu(x)\psi_\nu(t). \tag{9.5}$$

Upon substituting this form into the equations in question, it is found that these are satisfied if the function $\phi_\nu(x)$ fulfills the system of relations

$$(9.6) \quad \begin{aligned} &\text{(i)} & \frac{d^2}{dx^2}\phi_\nu(x) + \lambda_\nu \kappa^2 \phi_\nu(x) &= 0, \\ &\text{(ii)} & \phi_\nu(0) &= 0, \\ &\text{(iii)} & \phi_\nu(l) &= 0, \end{aligned}$$

with $\lambda_\nu$ designating a constant, provided that with this same constant the function $\psi_\nu(t)$ fulfills the equation

$$\frac{d}{dt}\psi_\nu(t) + \lambda_\nu \psi_\nu(t) = 0. \tag{9.7}$$

As a solution of the ordinary differential equation (9.6i) the function $\phi_\nu(x)$ must familiarly be of the form

$$\phi_\nu(x) = b_\nu \sin \sqrt{\lambda_\nu}\,\kappa x + \gamma_\nu),$$

in which $b_\nu$ and $\gamma_\nu$ may be any constants. It will fulfill the condition (9.6ii) if $\gamma_\nu = 0$, and then also the condition (9.6iii) if $\sqrt{\lambda_\nu}$ is any multiple of the constant $\pi/\kappa l$, namely, when

$$\lambda_\nu = \left(\frac{\nu\pi}{\kappa l}\right)^2, \tag{9.8}$$

with $\nu$ an integer. The associated solution of the equation (9.7) is then clearly

$$\psi_\nu(t) = e^{-(\nu^2\pi^2/\kappa^2 l^2)t}.$$

Through the formula (9.5) it has thus been found that the equations (9.2) and (9.3) are satisfied by the function

$$\tau_\nu(x, t) = b_\nu e^{-(\nu^2\pi^2/\kappa^2 l^2)t} \sin\frac{\nu\pi x}{l}, \tag{9.9}$$

in fact by each of the infinite set of functions obtained from this formula by setting $\nu = 1, 2, 3, \cdots$.

Now it is a characteristic property of linear equations or of any system of such, that the sum of any set of solutions is itself a solution. One is motivated thus to infer from the set (9.9) a formal solution having the structure

$$\tau(x,t) = \sum_{\nu=1}^{\infty} b_\nu e^{-(\nu^2\pi^2/\kappa^2 l^2)t} \sin\frac{\nu\pi x}{l}\,. \tag{9.10}$$

Aside from questions of convergence which are clearly to be raised in this connection, a primary matter still to be dealt with is the fulfillment of the condition (9.4) whatever the function $f(x)$ may be. Upon substituting the value $t=0$ into the formula (9.10), this is seen to devolve into the relation

$$\sum_{\nu=1}^{\infty} b_\nu \sin\frac{\nu\pi x}{l} = f(x), \qquad 0 < x < l. \tag{9.11}$$

The question of the representability of any function $f(x)$ in a series of sines is thus clearly brought into issue.

We propose to review in the following chapters Fourier's mode of coping with this problem. There are a number of reasons why a consideration of this may be regarded as worthwhile and of interest. It is, to begin with, ingenious and skillful. Aside from that it is a notable exemplar of work in the spirit of mathematics in the eighteenth century. Although this theory of Fourier's was actually created in the next century, and was crowned by the prize of the Academy of Paris in 1811, its disregard for rigor was even then outmoded and seemed, in fact, to run counter to the better standards of which Fourier himself was conscious. It is a formalism—no more—a play upon symbols in accordance with accepted rules but without much or any regard for content or significance. As such it has, of course, no place in the mathematics of our time. Fourier's work has had the profoundest effect both upon the development of pure mathematical concepts and upon the extension of the range of mathematical applications to the sciences and technology. These deserts, however, sprung in the main from Fourier's interpretations and not from his manipulations. It was, no doubt, partially because of his very disregard for rigor that he was able to take conceptual steps which were inherently impossible to men of more critical genius.

## CHAPTER 10

**Fourier's formal solution of his problem.** [11] In the study of a representation (9.11) it comes to a mere matter of the choice of a unit of measurement to identify the length $l$ with the value $\pi$. We shall suppose this to have been done since an appreciable formal simplification results from it. To summarize the problem at issue, then, it is that of determining from any given function $f(x)$ a set of constants $b_\nu$ such that

$$\sum_{\nu=1}^{\infty} b_\nu \sin \nu x = f(x), \tag{10.1}$$

for $0<x<\pi$.

Fourier began his considerations of this relation by substituting in it for each sine function its power series equivalent, namely

$$\sin \nu x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\nu^{2n-1}}{(2n-1)!} x^{2n-1}.$$

Upon interchanging the order of the summations, an operation which was at that time generally resorted to without question, the relation (10.1) was made to appear in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}\left(\sum_{\nu=1}^{\infty} \nu^{2n-1} b_\nu\right) x^{2n-1}.$$

The function $f(x)$ has thus been related to a series in powers of $x$, and since such a series must, in fact, be its MacLaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{[k]}(0) x^k, \tag{10.2}$$

in which $f^{[k]}(x)$ stands for the $k$th derivation of $f(x)$, it was to be concluded by a comparison of the coefficients of like powers of $x$ that $f^{[k]}(0)$ is zero whenever $k$ is even; and that otherwise

$$\sum_{\nu=1}^{\infty} \nu^{2n-1} b_\nu = (-1)^{n-1} f^{[2n-1]}(0), \qquad n = 1, 2, 3, \cdots. \tag{10.3}$$

The effect of this consideration has thus been to throw the constants $b_\nu$ into the rôle of the infinitely many unknowns in a system of infinitely many linear equations.

To deal with a system of this type Fourier had to invent his own method. He chose to base this upon the use of a chain of ordinary algebraic systems

$$\sum_{\nu=1}^{r} \nu^{2n-1} \beta_\nu(r) = \phi_n(r), \qquad n = 1, 2, \cdots, r, \tag{10.4}$$

of arbitrarily large degree $r$, the relevancy of these to the infinite system (10.3) to be assured by an adoption of the relations

$$\phi_n(\infty) = (-1)^{n-1} f^{[2n-1]}(0), \qquad n = 1, 2, 3, \cdots. \tag{10.5}$$

Inasmuch as the solution $\beta_\nu(r)$, $\nu = 1, 2, \cdots, r$, of the system (10.4) is unique, whatever the degree $r$ may be, it was tacitly assumed that the solution of the infinite system could be inferred from those of the finite ones in the manner

$$\beta_\nu(\infty) = b_\nu, \qquad \nu = 1, 2, 3, \cdots. \tag{10.6}$$

In the system (10.4) let the $n$th equation be multiplied by $r^2$ and let the next following equation then be subtracted from it. If this is done for each value of $n$ from 1 to $(r-1)$ the result is the system of equations

$$\sum_{\nu=1}^{r-1} \nu^{2n-1}\{[r^2 - \nu^2]\beta_\nu(r)\} = r^2\phi_n(r) - \phi_{n+1}(r), \qquad n = 1, 2, \cdots, (r-1). \tag{10.7}$$

This suggests imposing upon the members $\phi_n(r)$ the relations

$$r^2\phi_n(r) - \phi_{n+1}(r) = \phi_n(r-1), \qquad n = 1, 2, \cdots, (r-1), \tag{10.8}$$

for, since the coefficients of the system (10.7) will then be precisely those of the system given by the equations (10.4) when $r$ is replaced by $(r-1)$, it may be inferred that the unknowns $(r^2-\nu^2)\beta_\nu(r)$ in the one case and $\beta_\nu(r-1)$ in the other case, are, in fact, the same, namely that

$$\beta_\nu(r) = \frac{\beta_\nu(r-1)}{r^2 - \nu^2}.$$

By suitable iterations of this relation it is evidently found that

$$\beta_\nu(r) = \frac{\beta_\nu(\nu)}{\prod_{n=\nu+1}^{r} (n^2 - \nu^2)}, \qquad r > \nu, \tag{10.9}$$

whence it follows by a simple formal step from the finite $r$ to the infinite, that

$$b_\nu = \frac{\beta_\nu(\nu)}{\prod_{n=\nu+1}^{\infty} (n^2 - \nu^2)}. \tag{10.10}$$

To any modern investigator this conclusion could, of course, be only meaningless, for the infinite product involved in it is manifestly divergent. Nor is the source of this unfortunate result difficult to trace. It lies, namely, in the naive adoption of the relations (10.8) in the face of the fact that these are quite inconsistent with the previously adopted assignments (10.5). It requires no keen critical faculty to observe at once that under the formula (10.9) the sequence of values $\beta_\nu(r)$ inevitably converges to zero with $1/r$ and hence that no conclu-

sion other than that of the vanishing of each coefficient $b_\nu$ would be logically admissible. Fourier had no intention whatsoever of drawing that conclusion, and hence proceeded undismayed with the analysis of his formula. In so doing he found, quite naturally, that the divergence it already involved could be formally compensated for only by the introduction of still other divergencies.

Let the determinant $D(x)$ be defined by the formula

$$D(x) = \begin{vmatrix} 1 & 2 & \cdots (\nu - 1) & x \\ 1^3 & 2^3 & \cdots (\nu - 1)^3 & x^3 \\ 1^5 & 2^5 & \cdots (\nu - 1)^5 & x^5 \\ \cdot & \cdot & \cdots & \cdot \\ 1^{2\nu-1} & 2^{2\nu-1} & \cdots (\nu - 1)^{2\nu-1} & x^{2\nu-1} \end{vmatrix}, \tag{10.11}$$

and let the cofactor of its element in the $i$th row and $j$th column be denoted by $D_{i,j}$. Then since the determinant of the system (10.4) when $r = \nu$ is precisely $D(\nu)$, it is seen at once that by Cramer's rule

$$\beta_\nu(\nu) = \frac{\sum_{n=1}^{\nu} D_{n,\nu}\phi_n(\nu)}{D(\nu)}. \tag{10.12}$$

This formula can be made much more explicit. The determinant $D(x)$ is, in the first place, found to admit (*cf.* appendix V) of the evaluation

$$D(x) = (-1)^{\nu-1} D_{\nu,\nu} x \prod_{n=1}^{\nu-1} (n^2 - x^2). \tag{10.13}$$

Upon expanding the left-hand member of this equality by the elements of its last column, and by agreeing to define the coefficients $c_n(r)$ for any value of $r$ not less than $\nu$ by the relations

$$\prod_{n=1, n\neq\nu}^{r} (n^2 - x^2) = \sum_{n=1}^{r} c_n(r) x^{2n-2}, \tag{10.14}$$

it evidently follows further that

$$\sum_{n=1}^{\nu} D_{n,\nu} x^{2n-1} = (-1)^{\nu-1} D_{\nu,\nu} \sum_{n=1}^{\nu} c_n(\nu) x^{2n-1}.$$

Now in any identical equation between polynomials or power series, the coefficients of like powers of $x$ in the two members of the equation must be the same. It will be seen at once that because of this the substitution of any quantity whatsoever in the place of any specific power of $x$ will not destroy the equality. In the equation above, therefore, the replacement of $x^{2n-1}$ for each value of $n$ by the respective quantity $\phi_n(\nu)$ is legitimate. It leads to the result

$$\sum_{n=1}^{\nu} D_{n,\nu}\phi_n(\nu) = (-1)^{\nu-1}D_{\nu,\nu}\sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu),$$

and this, together with the value of $D(\nu)$ that is obtained from the relation (10.13), yields upon substitution into the equation (10.12) the formula

$$(10.15) \qquad \beta_\nu(\nu) = \frac{\sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu)}{\nu\prod_{n=1}^{\nu-1}(n^2 - \nu^2)}.$$

Let it be observed now that the removal of the factor $(r^2-x^2)$ from the left-hand member of the equation (10.14) has the mere effect of reducing the index $r$ to $(r-1)$. From that relation it is thus seen that

$$\sum_{n=1}^{r} c_n(r)x^{2n-2} = (r^2 - x^2)\sum_{n=1}^{r-1} c_n(r-1)x^{2n-2},$$

namely that

$$\sum_{n=1}^{r} c_n(r)x^{2n-2} = \sum_{n=1}^{r-1} c_n(r-1)\left[r^2x^{2n-2} - x^{2n}\right].$$

If in this each power $x^{2j}$ is replaced by the respective value $\phi_{j+1}(r)$, it follows because of the relation (10.8) that

$$\sum_{n=1}^{r} c_n(r)\phi_n(r) = \sum_{n=1}^{r-1} c_n(r-1)\phi_n(r-1).$$

The sum on the left of this equality is thus independent of $r$, and this having been established it is a simple formal step to write

$$(10.16) \qquad \sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu) = \sum_{n=1}^{\infty} c_n(\infty)\phi_n(\infty).$$

It was familiar in Fourier's time (*c.f.* appendix VI) that

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x},$$

a relation which in a formal sense, if in no other, yields the formula

$$\prod_{n=1, n\neq\nu}^{\infty} (n^2 - x^2) = \left(\prod_{n=1, n\neq\nu}^{\infty} n^2\right)\left(1 - \frac{x^2}{\nu^2}\right)^{-1} \frac{\sin \pi x}{\pi x}.$$

Upon replacing the left-hand member of this by the equivalent series (10.14) and substituting on the right the series equivalents

$$\left(1 - \frac{x^2}{\nu^2}\right)^{-1} = \sum_{q=0}^{\infty} \frac{x^{2q}}{\nu^{2q}},$$

$$\frac{\sin \pi x}{\pi x} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\pi^{2k-2}}{(2k-1)!} x^{2k-2},$$

it becomes

$$\sum_{n=1}^{\infty} c_n(\infty) x^{2n-2} = \left(\prod_{n=1,n\neq\nu}^{\infty} n^2\right) \sum_{q=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\pi^{2k-2}}{\nu^{2q}(2k-1)!} x^{2q+2k-2}.$$

The procedure of replacing each power $x^{2j-2}$ by $\phi_j(\infty)$ leads from this to the equation

$$\sum_{n=1}^{\infty} c_n(\infty)\phi_n(\infty) = \left(\prod_{n=1,n\neq\nu}^{\infty} n^2\right) \sum_{q=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\pi^{2k-2}}{\nu^{2q}(2k-1)!} \phi_{q+k}(\infty),$$

or, by virtue of the relations (10.5) and (10.16), to the relation

$$\sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu) = \left(\prod_{n=1,n\neq\nu}^{\infty} n^2\right) \sum_{q=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{q}\pi^{2k-2}}{\nu^{2q}(2k-1)!} f^{[2q+2k-1]}(0).$$

With this result the formula (10.15) for $\beta_\nu(\nu)$ becomes wholly explicit, and through it the formula (10.10) assumes the form

$$(10.17) \qquad b_\nu = \frac{\prod\limits_{n=1,n\neq\nu}^{\infty} n^2}{\nu \prod\limits_{n=1,n\neq\nu}^{\infty} (n^2 - \nu^2)} \sum_{q=0}^{\infty} \frac{(-1)^q}{\nu^{2q}} \sum_{k=1}^{\infty} \frac{f^{[2q+2k-1]}(0)}{(2k-1)!} \pi^{2k-2}.$$

Inasmuch as all quantities in the right-hand member of this are to be regarded as known when $f(x)$ is known, this result amounts, at least in a formal sense, to an evaluation of the constant $b_\nu$.

## CHAPTER 11

**The reduction and interpretation of the solution.** [12] The result (10.17), although it formally accomplishes the task originally set, namely the construction of a formula through which the coefficients $b_\nu$ are expressed in terms of the given function $f(x)$, will nevertheless hardly be found completely satisfying. For the purposes of practical calculation, namely, some reduction of its intricacy would clearly be imperative, and Fourier, realizing this, turned his attention to its simplification. His effectiveness in achieving this is eloquent commentary upon his skill in analytical manipulations.

Consider, to begin with, the product of the values $(n^2 - \nu^2)$ which the formula contains. The first $(\nu - 1)$ factors of this are negative and have as their product

$$(-1)^{\nu-1}\prod_{n=1}^{\nu-1}(\nu-n)\prod_{n=1}^{\nu-1}(\nu+n).$$

The product of the remaining (positive) factors may be written in the form

$$\prod_{n=\nu+1}^{\infty}(n-\nu)\prod_{n=\nu+1}^{\infty}(n+\nu),$$

and thus, by simple changes of the index in each of the partial products, the entire expression may be made to appear as

$$(-1)^{\nu-1}\prod_{n=1}^{\nu-1}n\prod_{n=\nu+1}^{2\nu-1}n\prod_{n=1}^{\infty}n\prod_{n=2\nu+1}^{\infty}n.$$

Since in this each natural integer except $\nu$ and $2\nu$ occurs twice, the excepted ones occurring just once, an alternative form for the product is evidently

$$\frac{(-1)^{\nu-1}}{2\nu^2}\prod_{n=1}^{\infty}n^2.$$

It has been found thus that formally

$$\prod_{n=1,n\neq\nu}^{\infty}(n^2-\nu^2)=\frac{(-1)^{\nu-1}}{2}\prod_{n=1,n\neq\nu}^{\infty}n^2. \tag{11.1}$$

Consider now the formula

$$f(x)=\sum_{n=1}^{\infty}\frac{f^{[2n-1]}(0)}{(2n-1)!}x^{2n-1}, \tag{11.2}$$

which is the equivalent of the relation (10.2) by virtue of the fact that in the latter each coefficient $f^{[k]}(0)$ with an even index $k$ is zero. A $2q$-fold term by term differentiation leads from this to the companion formula

$$f^{[2q]}(x)=\sum_{n=q+1}^{\infty}\frac{f^{[2n-1]}(0)}{(2n-2q-1)!}x^{2n-2q-1},$$

and if in this $x$ is given the value $\pi$ and the index of summation is suitably changed, the result is the equation

$$\sum_{k=1}^{\infty}\frac{f^{[2k-2q-1]}(0)}{(2k-1)!}\pi^{2k-1}=f^{[2q]}(\pi). \tag{11.3}$$

The substitution of the evaluations (11.1) and (11.3) into the formula (10.17) causes a reduction of this latter to the form

$$b_\nu=\frac{2}{\nu}(-1)^{\nu-1}\sum_{q=0}^{\infty}\frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q}\pi}. \tag{11.4}$$

The infinite series which still appears in this result suggests the formal definition of a function $u(x)$ by the relation

$$u(x) = \sum_{q=0}^{\infty} \frac{(-1)^q f^{[2q]}(x)}{\nu^{2q+1}}.$$

Upon differentiating this twice term by term and thereupon adjusting the index of summation, it is found that

$$u''(x) = -\nu^2 \sum_{q=1}^{\infty} \frac{(-1)^q f^{[2q]}(x)}{\nu^{2q+1}},$$

and hence that

$$u''(x) + \nu^2 u(x) = \nu f(x).$$

This is a differential equation of an elementary type. Its reduced equation is solved by the function $(c_1 \sin \nu x + c_2 \cos \nu x)$ and from this fact the method of "variation of parameters" [13] leads readily to the conclusion that the equation itself has a solution

$$U(x) = \int_0^x f(s) \sin \nu(x - s) ds.$$

This is, of course, verifiable at once by direct substitution into the differential equation. Upon setting $x = \pi$ it is thus found that

$$U(\pi) = (-1)^{\nu-1} \int_0^\pi f(s) \sin \nu s ds. \tag{11.5}$$

Let the integral in this formula now be integrated by parts $2n$ times in succession, the trigonometric factor being each time the one to be integrated. Since, as may be seen from the formula (11.2), every even ordered derivative of $f(s)$ is zero at $s=0$, while the function $\sin \nu s$ vanishes at both $s=0$ and $s=\pi$, the result of these integrations is the relation

$$U(\pi) = \sum_{q=0}^{n} \frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q+1}} + \frac{(-1)^{n+\nu}}{\nu^{2q+2}} \int_0^\pi f^{[2n+2]}(s) \sin \nu s ds.$$

By the step from the finite $n$ to the infinite the formal relation

$$U(\pi) = \sum_{q=0}^{\infty} \frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q+1}},$$

may be drawn therefrom, and this together with the formula (11.5) yields the equality

$$\frac{1}{\nu} \sum_{q=0}^{\infty} \frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q}} = (-1)^{\nu-1} \int_0^\pi f(s) \sin \nu s ds.$$

By virtue of this the formula (11.4) is now once more and finally reduced to the form

$$b_\nu = \frac{2}{\pi}\int_0^\pi f(s)\sin \nu s ds. \tag{11.6}$$

This is, of course, the familiar formula in virtue of which the coefficients $b_\nu$ are generally known as those of Fourier. It is the formula (1.4) for the special case in which $l=\pi$.

Elegant though the conclusion (11.6) unquestionably is, the verdict of any critical appraisal of Fourier's accomplishment to the point of its derivation must inevitably be profoundly disappointing. As to the result, in the first place, that was not new. It had been contained in the mathematical literature for over a decade—to be precise, since the publication of the memoir of Euler that was discussed in chapter 8. While, to be sure, Euler's results applied only to functions of a certain class, that is no less true of Fourier's, since his deductions were based upon such material restrictions as the representability of the function $f(x)$ in power series of the form (11.2) that converge when $x=\pi$.

Nor could any advantage be claimed by Fourier in the matter of method. On the contrary—and even leaving aside the important fact that by its employment of divergent processes it divested itself of all rigorous validity—the method of Fourier suffers in almost every respect by comparison with that of Euler. The device of referring the problem to a system of linear equations, ingenious though it is, is nevertheless quite foreign to the nature of the problem. The trigonometric functions are conspicuously endowed with many peculiar properties and fulfill a great many characteristic interrelationships. Of this important fact Fourier's approach in no way avails itself, while Euler's, by contrast, exploits it to the utmost. In cutting directly to the heart of the matter Euler thus attained his result more perspicuously and incomparably more cheaply. In this respect the superiority is all his.

With the priority and preference in manipulative matters thus denied him, Fourier's claim to renown must be based upon other grounds, and these are, namely, those of interpretation. Approaching the formula (11.6) afresh, without regard for the manner of its derivation, it was observed by him that through it each coefficient $b_\nu$ admits of interpretation as the area between the abscissas $x=0$ and $x=\pi$, and under the graph

$$y = \frac{2}{\pi} f(x)\sin \nu x. \tag{11.7}$$

Such an area is evidently conceivable, and retains its clear-cut significance, in association with functions $f(x)$ that are in a very general sense quite arbitrary. Certainly these functions need not be assumed to be continuous or expansible by any simple analytical formulas. They might be graphically defined and

might well represent distributions of functional values that are extremely erratic. On the basis of such considerations Fourier concluded that *any* and *every* function $f(x)$ had associated with it a set of constants $b_\nu$.

From this fact alone it would not follow, of course, that with such coefficients the representation of the function $f(x)$ by a series (10.1) would result. As has been seen in chapters 1 and 5, the masters of the eighteenth century had rejected such a possibility as manifestly absurd. In this matter, however, Fourier was willing to disregard opinions and precedents, however well established, and to look further for himself. From calculations of the coefficients $b_\nu$ with small indices $\nu$ in the cases of a great variety of functions $f(x)$, and from subsequent plottings of the respective initial segments of the resulting trigonometric series, he came to convictions upon two salient points, namely: (i) that the series (10.1) *always* represents the function over the interval $0<x<\pi$, and (ii) that in general this representation does not persist for values of $x$ outside that interval.

Fourier's announcement of these facts was quite generally met with incredulity. Even the mass of his substantiating evidence won, in many cases, only grudging and reluctant acceptance. The implications behind the new assertions were too revolutionary to be easily assimilated. They called for no less than a fundamental revision of many concepts that were wholly traditional, some of them lying at the very basis of mathematical analysis. On the other hand Fourier's new theory did now finally vindicate the half century old reasoning of Daniel Bernoulli by which he had convinced himself, if no others, that any curve from which a taut elastic string could spring into vibration could be represented by a trigonometric series.

## CHAPTER 12

**The Dirichlet integrals.** Once Fourier had deduced the formula (11.6), he, like Euler before him, observed that in a schematic way the result is recoverable in a most direct and simple manner by the mere expedient of multiplying the relation (10.1) through by $\sin \mu x$ with any natural integer $\mu$, and then integrating term by term over the interval $(0, \pi)$. The infinite series reduces under this process to a single term, because of the evaluations

$$\int_0^\pi \sin \nu x \sin \mu x dx = 0, \qquad \text{for } \nu \neq \mu, \tag{12.1}$$

and the formula for $b_\mu$ thus emerges. The procedure is obviously adaptable also to the case of a cosine representation

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \tag{12.2}$$

the evaluations

$$\int_0^\pi \cos \nu x \cos \mu x dx = 0, \qquad \text{for } \nu \neq \mu, \tag{12.3}$$

leading in this case to the formulas

$$a_\nu = \frac{2}{\pi} \int_0^\pi f(s) \cos \nu s ds, \qquad \nu = 1, 2, 3, \cdots . \tag{12.4}$$

An extension of these results to the case of a function that is given over the larger interval $(-\pi, \pi)$ is of importance and is easily deduced. Through the relation

$$f(x) = f_0(x) + f_e(x),$$

with

$$f_0(x) \equiv \tfrac{1}{2}[f(x) - f(-x)],$$
$$f_e(x) \equiv \tfrac{1}{2}[f(x) + f(-x)],$$

the function $f(x)$ is expressed as the sum of two components of which the first one is an odd function and the second one even. Now the relations (10.1) and (12.2) for these respective functions, namely

$$f_0(x) = \sum_{\nu=1}^{\infty} b_\nu \sin \nu x,$$

$$f_e(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x,$$

obviously remain quite unchanged if $x$ is replaced by $-x$. Any validity they may have over the interval $(0, \pi)$ therefore implies the same over the larger interval $(-\pi, \pi)$. It follows at once that for the originally given function the representation in question is of the form

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} [a_\nu \cos \nu x + b_\nu \sin \nu x]. \tag{12.5}$$

The formulas for the coefficients in this are, moreover, easily freed from reference to the components $f_0(x)$ and $f_e(x)$, and may thus be brought to expression directly in terms of the function $f(x)$ itself. Since

$$\int_0^\pi f_0(s) \sin \nu s ds = \frac{1}{2} \int_{-\pi}^\pi f(s) \sin \nu s ds,$$

as may easily be verified, and also

$$\int_0^\pi f_e(s) \cos \nu s ds = \frac{1}{2} \int_{-\pi}^\pi f(s) \cos \nu s ds,$$

the formulas in question are, namely, seen to be

$$
(12.6)\qquad
\begin{aligned}
a_\nu &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos \nu s ds,\\
b_\nu &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin \nu s ds.
\end{aligned}
$$

In the instance that the interval $(-\pi, \pi)$ which has been taken to be basic is replaced by the more general interval $(-l, l)$, the representation (12.5), (12.6), is correspondingly replaced by that given in the formulas (1.1), (1.2).

Let the sum of the first $2N+1$ terms of the series (12.5) be designated by $S_N(x)$, thus

$$
(12.7)\qquad S_N(x) = \frac{a_0}{2} + \sum_{\nu=1}^{N} [a_\nu \cos \nu x + b_\nu \sin \nu x].
$$

The substitution of the values of the coefficients (12.6) into this gives it the aspect

$$
S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left[\frac{1}{2} + \sum_{\nu=1}^{N} \cos \nu(s-x) ds\right],
$$

and this may be contracted by the use of elementary trigonometric relations (*cf.* appendix VII) into the wholly compact form

$$
(12.8)\qquad S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin\left[(N+\frac{1}{2})(s-x)\right]}{\sin\left[\frac{1}{2}(s-x)\right]}\,ds.
$$

An equivalent manner of writing this, and one that has some analytic advantages, is

$$
(12.9)\qquad S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin\left[(N+\frac{1}{2})(s-x)\right]}{(s-x)}\,ds + \int_{-\pi}^{\pi} F(s)\Psi(s, x, N) ds,
$$

with the symbols $F(s)$ and $\Psi(s, x, N)$ having the significance

$$
\begin{aligned}
F(s) &\equiv \frac{f(s)}{2\pi}\left\{\frac{1}{\sin\left[\frac{1}{2}(s-x)\right]} - \frac{1}{\frac{1}{2}(s-x)}\right\},\\
\Psi(s, x, N) &\equiv \sin\left[(N+\tfrac{1}{2})(s-x)\right].
\end{aligned}
$$

From the latter of these formulas it is easily inferred that

$$
\left|\Psi(s, x, N)\right| \leqq 1, \quad \text{and}
$$

$$
\left|\int_{\alpha}^{\beta} \Psi(s, x, N) ds\right| \leqq \frac{4}{2N+1}, \quad \text{for } -\pi \leqq \alpha < \beta \leqq \pi.
$$

The function $\Psi(s, x, N)$ thus possesses the properties:

(i) that it is bounded uniformly as to $N$ and $s$;

(ii) that its integral over any sub-interval of the range $(-\pi, \pi)$ converges to zero with $1/N$ uniformly as to the sub-interval.

These properties are sufficient to insure [14] the relation

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} F(s)\Psi(s, x, N)ds = 0, \tag{12.10}$$

whenever the function $F(s)$ is integrable (in the sense of Lebesgue) over the interval $(-\pi, \pi)$.

Now from the definition of the function $F(s)$ it may readily be seen that integrability is assured to it by that of the function $f(s)$ provided the point $x$ is in the interior of the interval $(-\pi, \pi)$. Thus for every integrable function $f(x)$ the final integral in the relation (12.9) converges to zero, and that relation thus implies that

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \frac{\sin [(N + \frac{1}{2})(s - x)]}{(s - x)} ds, \tag{12.11}$$

whenever the right-hand limit involved in this exists. This permits us at once the conclusion: that any function $f(x)$ which is integrable and which is furthermore such that for it

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \frac{\sin [(N + \frac{1}{2})(s - x)]}{(s - x)} ds = f(x), \tag{12.12}$$

is a function which is representable by a trigonometric series in the manner (12.5), (12.6).

Although Fourier made calculations upon many specific cases, he gave no general proof of his ultimate assertion of the trigonometric representability of *arbitrary* functions. Indeed no such proof could have been given, since in the omission of all qualifications upon the functions the assertion is too broad to be valid. The first proof that was both satisfactory in the matter of rigor and ample in the matter of generality was given by Dirichlet in the year 1829. In the manner that has been indicated above, this proof was based by Dirichlet upon an establishment of the relation (12.12). The integrals involved in that relation and in (12.8) are accordingly known generally as "Dirichlet integrals." Dirichlet's proof in its original form, or as it has been improved and refined, is to be found at many places in the mathematical literature [15]. We shall, therefore, go no further into it here but shall draw this part of the discussion to its close. In the following part the primary subject of study is to be a generalization of the entire theory in which the representations of functions in trigonometric terms sink to the status of special cases.

# PART II

## CHAPTER 13

**The differential boundary problem.** Among the most powerful of mathematical means for the formulation of natural laws are the linear differential equations of either the partial or the ordinary type. Varied and diverse as physical phenomena certainly are, they nevertheless submit quite generally to description by such equations. The flow of heat in material bodies and the vibrations of elastic strings under tension are instances of this that have already been noted.

By its very nature as the formulation of a more or less general law, any particular differential equation applies, of course, to the entire category of manifestations which the law itself governs. To single out for description any individual phenomenon from such a category, therefore requires that some auxiliary means beyond the equation itself be resorted to. This ordinarily takes on the form of a set of one or more restricting relations that are expressive of the characterizing initial or boundary conditions. The differential equation together with such relations is commonly designated as a *differential system.* It is also said to define a *differential boundary problem.* The system consisting of the equations (4.6) and (4.7), for instance, thus defines a partial differential boundary problem, namely the one which is descriptive of a certain stretched string vibrating with specified end points and initial position of release. The equations (9.2), (9.3) and (9.4), define a similar problem, one which describes a linear flow of heat from specific initial temperatures and under certain boundary conditions.

The method that was employed in the reduction of the boundary problem of chapter 9 was designated there as that of the "separation of variables." Its immediate effect was to refer the partial differential system to an *ordinary* system (9.6), this latter having the peculiarity of involving an unspecified constant or parameter, which was designated by $\lambda_\nu$. Following the solution of this ordinary boundary problem at characteristic values of this parameter, the theory led in a natural way to the further problem of representing an arbitrary function in terms of the respective solutions. This method was in no way especially designed for the problem of chapter 9. It is on the contrary one that is highly flexible and of very wide applicability. In the particular instance there considered the ordinary differential equation which characterized the problem was one whose solutions were trigonometric functions, and it was because of that, that the representation of the function $f(x)$ took the form (9.11), namely that of the Fourier theory. This feature was special, to the extent that it would not even have maintained for the equation (9.2) if the coefficient $\kappa^2$ involved in it had been dependent upon the coördinate $x$, rather than constant. In the following a discussion is to be framed which is free from such peculiar specializations.

Let the variable $x$ be real, with the range

$$a \leqq x \leqq b, \tag{13.1}$$

and on this interval let $p(x)$, $q(x)$, and $r(x)$, be differentiable functions. The symbol $L(\phi, \lambda)$ is to designate the differential expression

$$L(\phi, \lambda) \equiv \phi'' + p(x)\phi' + [q(x)\lambda + r(x)]\phi. \tag{13.2}$$

In this $\lambda$ is to play the rôle of a parameter, the range of which is to be the entire complex plane. The differential equation

$$L(\phi, \lambda) = 0, \tag{13.3}$$

is, then, one that is regular, in the sense that it has no singular points upon the interval (13.1). As an equation of the second order it will, of course, not generally be explicitly solvable. Certain facts concerning its solutions are, however, familiar. Of these the following ones will be especially relevant to the discussion proposed [16].

(i) The equation admits of solutions $\phi(x, \lambda)$ that have continuous second derivatives as to $x$, and that are analytic in $\lambda$ over the entire complex $\lambda$ plane.

(ii) There is a pair of such solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$, that are linearly independent as functions of $x$ for all values of $\lambda$.

(iii) The Wronskian $\Omega(x, \lambda)$ of this pair, namely the determinant

$$\Omega(x, \lambda) \equiv \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ \phi_1'(x, \lambda) & \phi_2'(x, \lambda) \end{vmatrix}, \tag{13.4}$$

is subject to the relation

$$\Omega(x, \lambda) = \Omega(a, \lambda)e^{-\int_a^x p(x)\,dx}, \tag{13.5}$$

with $\Omega(a, \lambda)$ an analytic function of $\lambda$ that is different from zero for all $\lambda$.

(iv) The general solution of the equation (13.3) is expressible in terms of the pair $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ by a form

$$h_2\phi_1(x, \lambda) + h_1\phi_2(x, \lambda), \tag{13.6}$$

in which the coefficients $h_1$, $h_2$ are constants as to $x$, though they may be functions of $\lambda$.

For any equation (13.3) there are known to be infinitely many pairs of solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$, that have the properties enumerated. Of these pairs any one serves in every way as well as any other, and the choice that is made is accordingly immaterial. For the sake of avoiding gratuitous complications, however, it will be supposed throughout the discussion that when a choice of such a pair in the instance of any specific equation has been made, it will be consistently adhered to. To that extent, then, the designations $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ will be understood to apply not to random but to specific solutions.

Let the symbols $\beta_{i,j}, \gamma_{i,j}, i=1, 2; j=1, 2, 3, 4;$ now denote any constants which are such that when

$$\alpha_{i,j}(\lambda) = \beta_{i,j}\lambda + \gamma_{i,j}, \tag{13.7}$$

then the matrix

$$\left\| \begin{array}{cccc} \alpha_{1,1}(\lambda) & \alpha_{1,2}(\lambda) & \alpha_{1,3}(\lambda) & \alpha_{1,4}(\lambda) \\ \alpha_{2,1}(\lambda) & \alpha_{2,2}(\lambda) & \alpha_{2,3}(\lambda) & \alpha_{2,4}(\lambda) \end{array} \right\| \tag{13.8}$$

is of the rank 2 for every value of $\lambda$. In terms of the values of any function $\theta$ and its derivative $\theta'$ at $x = a$ and $x = b$, the forms

$$A_i(\theta, \lambda) \equiv \alpha_{i,1}(\lambda)\theta'(a) + \alpha_{i,2}(\lambda)\theta(a) + \alpha_{i,3}(\lambda)\theta'(b) + \alpha_{i,4}(\lambda)\theta(b), \quad i = 1, 2, \tag{13.9}$$

are then always linearly independent. Under these circumstances the differential system

$$\begin{aligned} L(u, \lambda) &= 0, \\ A_1(u, \lambda) &= 0, \\ A_2(u, \lambda) &= 0, \end{aligned} \tag{13.10}$$

defines an ordinary boundary problem. It is this problem which is to be central to our discussion. It will be seen at once to include as a special case the problem (9.6), and to do that even if in the latter the coefficient $\kappa^2$ varies with $x$. Many other boundary problems that stem from physical origins are also included, as will upon occasion be seen in the following.

## CHAPTER 14

**The characteristic values and solutions.** Since any solution $u(x)$ of the differential system (13.10) must in particular solve the differential equation (13.3), it must have the form

$$u(x) = h_2\phi_1(x, \lambda) + h_1\phi_2(x, \lambda). \tag{14.1}$$

Except for the trivial solution $u(x) \equiv 0$, which we shall herewith specifically and permanently rule out of this discussion, the values $h_1$, $h_2$ will not both be zero. The substitution of this form into the boundary relations of the system give to the latter the aspect

$$\begin{aligned} h_2A_{1,1}(\lambda) + h_1A_{1,2}(\lambda) &= 0, \\ h_2A_{2,1}(\lambda) + h_1A_{2,2}(\lambda) &= 0, \end{aligned} \tag{14.2}$$

in which the abbreviations

$$A_{i,j}(\lambda) \equiv A_i(\phi_j(x, \lambda), \lambda), \qquad i, j = 1, 2, \tag{14.3}$$

have been resorted to.

The equations (14.2) constitute an algebraic system in which the values $h_1$, $h_2$, function as the unknowns. Since this system is homogeneous its non-trivial solvability is contingent upon the vanishing of its determinant $\Delta(\lambda)$, where

$$\Delta(\lambda) \equiv \begin{vmatrix} A_{1,1}(\lambda) & A_{1,2}(\lambda) \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) \end{vmatrix}. \tag{14.4}$$

A proper solution of the boundary problem thus exists if and only if $\lambda$ is a root of the so-called *characteristic equation*

$$\Delta(\lambda) = 0. \tag{14.5}$$

These roots are called the *characteristic values* (*Eigenwerte*) of the boundary problem. The multiplicity with which such a value occurs as a root of the equation (14.5) is also designated to be its *multiplicity* as a characteristic value. If the root is one at which the elements of the determinant (14.4) do not all vanish, namely at which the rank of the determinant is 1, it is said to be a characteristic value of the *index* 1. On the other hand a value at which the elements do all vanish is said to be of the index 2. It is not difficult to see that in this latter case its multiplicity must also be at least 2, and hence that in every case

$$(\text{The index}) \leqq (\text{The multiplicity}). \tag{14.6}$$

There is material advantage to be gained by formally regarding a value whose index is 2 as being, in fact, two coincident characteristic values. We shall herewith, once for all, adopt this convention.

The determinant $\Delta(\lambda)$ is an analytic function for all values of $\lambda$. The number of its zeros in any finite region of the $\lambda$ plane is, therefore, finite. Thus, in particular, only a finite number of these zeros fulfill a relation $|\lambda| < N$, whatever the constant $N$ may be, a fact from which it follows that they—the characteristic values—may be sequentially ordered in a succession of non-decreasing absolute value. With the assignment of subscripts in such a succession, the characteristic values thus follow each other in the array

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \cdots. \tag{14.7}$$

In this, then,

$$|\lambda_r| \leqq |\lambda_{r+1}|, \tag{14.8}$$

for every subscript $r$. Each characteristic value of the index 2 occurs in the array twice, occupying, as we may and shall assume, two consecutive positions. Each characteristic value of the index 1 occurs just once.

Consider now any characteristic value $\lambda_r$ that is of the index 1. The pairs of values $(h_1, h_2) = (h_1^{(r)}, h_2^{(r)})$ which satisfy the system (14.2), namely for which

$$\begin{aligned} h_2^{(r)} A_{1,1}(\lambda_r) + h_1^{(r)} A_{1,2}(\lambda_r) &= 0, \\ h_2^{(r)} A_{2,1}(\lambda_r) + h_1^{(r)} A_{2,2}(\lambda_r) &= 0, \end{aligned} \tag{14.9}$$

have in this case members that stand in a fixed ratio to each other. Any such pair yields through the formula

$$u_r(x) = h_2^{(r)} \phi_1(x, \lambda_r) + h_1^{(r)} \phi_2(x, \lambda_r), \tag{14.10}$$

an associated solution for the boundary problem, and the solution so obtainable from any other pair is a mere multiple of this. The function $u_r(x)$ is known as a *characteristic solution* (*Eigenfunktion*) of the boundary problem.

If now, on the other hand, $\lambda_r$ and $\lambda_{r+1}$ are the designations of a characteristic value of the index 2, then at this value the system (14.9) is vacuous since each quantity $A_{i,j}(\lambda_r)$ is zero. There is, therefore, no restriction upon the choice of $h_1$, $h_2$, and hence in particular two pairs that are not multiples of each other may be taken. These lead through the formula (14.1) to two linearly independent characteristic solutions $u_r(x)$, $u_{r+1}(x)$, which we may regard as associated with the values $\lambda_r$ and $\lambda_{r+1}$ respectively. Thus in every instance each symbol $\lambda_r$ has associated with it a function $u_r(x)$.

There exist boundary problems that have only a finite number of characteristic values or even none at all. There also exists, however, a large class of such problems for which the characteristic values are infinitely numerous. It is only to problems of this latter category that the continuing discussion will, in all its phases, be relevant.

## CHAPTER 15

**The adjoint boundary problem.** With the coefficient functions $p(x)$, $q(x)$, and $r(x)$, that occur in the expression (13.2) let $M(\phi, \lambda)$ be defined by the formula

$$M(\phi, \lambda) \equiv \phi'' - (p\phi)' + [q(x)\lambda + r(x)]\phi. \tag{15.1}$$

This differential form is said to be *adjoint to* the form $L(\phi, \lambda)$. It is customary also to refer to it as *the adjoint of* $L(\phi, \lambda)$. If it is completely written out, thus

$$\phi'' - p(x)\phi' + [q(x)\lambda + r(x) - p'(x)]\phi,$$

and its adjoint is in turn constructed, this latter is found to be again the form $L(\phi, \lambda)$. The relationship of being adjoint is thus a reciprocal one, either of two forms so associated being the adjoint of the other. For the two adjoint forms the equality

$$\psi L(\phi, \lambda) - \phi M(\psi, \lambda) = \frac{d}{dx} Q(\phi, \psi, x), \tag{15.2}$$

with

$$Q(\phi, \psi, x) \equiv \phi'(x)\psi(x) - \phi(x)\psi'(x) + p(x)\phi(x)\psi(x), \tag{15.3}$$

is easily verified for any two suitably differentiable functions $\phi$ and $\psi$. The relation is thus an identity. It is generally known as the "Lagrange identity," and is the source of many important analytical formulas.

The notion of the adjoint relationship is extensible in the most direct and immediate way to differential equations such as the equation (13.3). This latter and the equation

$$M(\psi, \lambda) = 0, \tag{15.4}$$

are thus likewise said to be adjoint. As the discussion proceeds there will be ample illustration of the manner in which the differential equations of an adjoint pair, or their respective solutions, interplay in the development of a theory. Even here it may be observed that the solubility of either equation implies that of the other, since the solutions of either are simply expressible in terms of those of the other. Thus if the solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ of the equation (13.3) and their Wronskian (13.4) are used to construct the functions $\psi_1(x, \lambda)$, $\psi_2(x, \lambda)$ by the formulas

$$\psi_1(x, \lambda) \equiv \frac{-\phi_2(x, \lambda)}{\Omega(x, \lambda)}, \qquad \psi_2(x, \lambda) \equiv \frac{\phi_1(x, \lambda)}{\Omega(x, \lambda)}, \tag{15.5}$$

these latter are found by direct substitution to satisfy the equation (15.4). They are similarly seen to fulfill the relations

$$Q(\phi_i, \psi_j, x) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \tag{15.6}$$

and to be linearly independent. In terms of them the general solution of the equation (15.4) is accordingly expressible in the form

$$v(x, \lambda) = k_2\psi_1(x, \lambda) + k_1\psi_2(x, \lambda), \tag{15.7}$$

with coefficients $k_1$, $k_2$ that are free from $x$. The relations (15.6) when applied to the form (15.7) yield at once the evaluations

$$\begin{aligned} Q(\phi_1, v, x) &= k_2, \\ Q(\phi_2, v, x) &= k_1. \end{aligned} \tag{15.8}$$

The extension of the notion of the adjoint relationship to the complete boundary problem (13.10), although it is not immediate is nevertheless possible, and may in fact be made in several formally different but essentially equivalent ways. We shall do this in the following manner. Consider the differential system

$$\begin{aligned} M(v, \lambda) &= 0, \\ v(a) &= \mu_2\alpha_{1,1}(\lambda) + \mu_1\alpha_{2,1}(\lambda), \\ -v'(a) + p(a)v(a) &= \mu_2\alpha_{1,2}(\lambda) + \mu_1\alpha_{2,2}(\lambda), \\ -v(b) &= \mu_2\alpha_{1,3}(\lambda) + \mu_1\alpha_{2,3}(\lambda), \\ v'(b) - p(b)v(b) &= \mu_2\alpha_{1,4}(\lambda) + \mu_1\alpha_{2,4}(\lambda), \end{aligned} \tag{15.9}$$

in which there occurs besides the function $v(x)$ also a pair of "parameters" $\mu_1$, $\mu_2$ that are independent of $x$. The coefficients $\alpha_{i,j}(\lambda)$ are to be those which were defined by the formulas (13.7). We shall show that this system in fact defines a boundary problem which is essentially of the type (13.10), and that the param-

eters $\mu_1$, $\mu_2$ may be looked upon as standing for certain specific linear forms in the values $v'(a)$, $v(a)$, $v'(b)$ and $v(b)$, with coefficients that are functions of $\lambda$.

The condition that the set of boundary relations of the system (15.9) be consistent in the "unknowns" $\mu_1$, $\mu_2$, is that the matrix

$$\left\|\begin{array}{rcc} v(a) & \alpha_{1,1} & \alpha_{2,1} \\ -v'(a)+p(a)v(a) & \alpha_{1,2} & \alpha_{2,2} \\ -v(b) & \alpha_{1,3} & \alpha_{2,3} \\ v'(b)-p(b)v(b) & \alpha_{1,4} & \alpha_{2,4} \end{array}\right\|$$

be of the rank 2. Now by hypothesis there is, for every $\lambda$, some two rowed determinant from the last two columns of this matrix that is not zero. This occurs as a minor in two of the three rowed determinants of the matrix. The results of setting these latter equal to zero are two equations in the quantities $v'(a)$, $v(a)$, $v'(b)$, $v(b)$. That these equations are independent follows at once from the fact that each contains one of the four quantities which the other does not contain. Thus the differential equation of the system (15.9) is seen to have imposed upon it two linear boundary conditions. A boundary problem is thus defined. The boundary relations of this problem evidently have coefficients that are polynomials in $\lambda$. To this significant extent the problem is accordingly similar in form to the problem (13.10). It is true that the coefficients of the boundary relations of the system (13.10) were taken to be linear polynomials, whereas those of the newly found system may be quadratic. That, however, is not truly important, for the assumption of the coefficients of the problem (13.10) to be of the first degree in $\lambda$ was motivated only by the desire for simplicity, and is in no way essential. Finally some pair of the equations (15.9) is always solvable for $\mu_1$ and $\mu_2$. By that solution $\mu_1$ and $\mu_2$ are expressed as linear forms in the values $v'(a)$, $v(a)$, $v'(b)$, $v(b)$ as was asserted above to be possible.

It is a matter to be observed that if in the equations (15.9) the parameters $\mu_1$, $\mu_2$ were to be both zero, it would follow that $v'(a)$ and $v(a)$ would also necessarily vanish. Only the trivial solution of the differential equation $M(v, \lambda)$ conforms to these values. Since this solution is to be ruled out of the discussion, it is evident that the simultaneous vanishing of $\mu_1$ and $\mu_2$ is likewise to be barred. It is to be understood henceforth, therefore, that of the value pair $\mu_1$, $\mu_2$ at least one member is in every case different from zero.

If in any differential system the differential equation is replaced by its general solution, and the boundary relations are replaced by independent linear combinations of them, the content of the system clearly remains unchanged. In consonance with this let the first two of the boundary relations of the set (15.9) be multiplied respectively by the factors $\phi_j'(a, \lambda)$ and $\phi_j(a, \lambda)$ with $j=1, 2$, and let them then be added. The left-hand members thus obtained are found to be $Q(\phi_j, v, a)$, and thus by the relations (15.8) the resulting equalities are

$$k_{3-j} = \mu_2 A_{1,j}^{(a)}(\lambda) + \mu_1 A_{2,j}^{(a)}(\lambda), \qquad j = 1, 2, \tag{15.10}$$

with

$$A_{i,j}^{(a)}(\lambda) \equiv \alpha_{i,1}(\lambda)\phi_j'(a, \lambda) + \alpha_{i,2}(\lambda)\phi_j(a, \lambda), \qquad i, j = 1, 2. \tag{15.11}$$

In a similar way the last two of the relations (15.9) may be combined with the multipliers $-\phi_j'(b, \lambda)$ and $-\phi_j(b, \lambda)$ to assume the forms

$$k_{3-j} = \mu_2[A_{1,j}^{(a)}(\lambda) - A_{1,j}(\lambda)] + \mu_1[A_{2,j}^{(a)}(\lambda) - A_{2,j}(\lambda)], \qquad j = 1, 2. \tag{15.12}$$

In content the equations (15.7), (15.10) and (15.12) are thus equivalent to the set (15.9).

Consider now the case in which $\lambda$ is any characteristic value of the boundary problem (15.9), namely a value for which a set of elements $v(x)$, $\mu_1$, $\mu_2$, fulfilling the equations (15.9) exists. From the fulfillment of the equations (15.10) and (15.12) it follows then at once that

$$\begin{aligned} \mu_2 A_{1,1}(\lambda) + \mu_1 A_{2,1}(\lambda) &= 0, \\ \mu_2 A_{1,2}(\lambda) + \mu_1 A_{2,2}(\lambda) &= 0. \end{aligned} \tag{15.13}$$

Since $\mu_1$ and $\mu_2$ are not both zero, the determinant of this system must vanish. This determinant is, however, precisely $\Delta(\lambda)$, as that was defined by the formula (14.4). Thus $\lambda$ must be a root of the equation (14.5), namely a characteristic value of the boundary problem (13.10). Every characteristic value of the adjoint boundary problem is thus also a characteristic value of the given one.

The converse of this may also be established. Thus let $\lambda_n$ be any value from the set (14.7). With $\lambda$ at this value the system (15.13) is non-trivially satisfiable by values $\mu_1^{(n)}$, $\mu_2^{(n)}$, which, of course, fulfill the relations

$$\begin{aligned} \mu_2^{(n)} A_{1,1}(\lambda_n) + \mu_1^{(n)} A_{2,1}(\lambda_n) &= 0, \\ \mu_2^{(n)} A_{1,2}(\lambda_n) + \mu_1^{(n)} A_{2,2}(\lambda_n) &= 0. \end{aligned} \tag{15.14}$$

Let $v_n(x)$ be the solution of the first three equations of the set (15.9) with $\lambda = \lambda_n$ and $\mu_j = \mu_j^{(n)}$, $j = 1, 2$. The familiar "existence theorem" for a linear ordinary differential equation [17] gives assurance of both the existence and the uniqueness of this function. Now with the values at hand the equations (15.10) are fulfilled. Because of this, however, and with the equations (15.14), the relations (15.12) are seen to be also fulfilled. This means that the entire system (15.9) admits this solution, namely that $\lambda_n$ is also a characteristic value of the boundary problem (15.9). We may go even somewhat further. If the index of $\lambda_n$ relative to the boundary problem (13.10) is 1, the system (15.14) determines the values $\mu_1^{(n)}$, $\mu_2^{(n)}$, except for a common multiplicative factor which remains arbitrary. From the manner in which $v_n(x)$ is determined it is then seen that this function is also fixed except for a constant multiplier. If, on the other hand, $\lambda_n$ is of the index 2 relative to the boundary problem (13.10), the system (15.14) admits of solution by two linearly independent pairs of values $\mu_1$, $\mu_2$. Each of these leads in the manner described to an associated solution $v(x)$, and the two of these so ob-

tained are also linearly independent. It has thus been shown that adjoint boundary problems have the same characteristic values, and, moreover, that each such value has the same index relative to each of the two problems.

For the characteristic solutions $v_n(x)$ of the boundary problem (15.9) the formula (15.7) yields the form

$$v_n(x) = k_2^{(n)}\psi_1(x, \lambda_n) + k_1^{(n)}\psi_2(x, \lambda_n). \tag{15.15}$$

The coefficients in this, as they may be drawn from the equations (15.10), have the evaluations

$$\begin{aligned} k_2^{(n)} &= \mu_2^{(n)} A_{1,1}^{(a)}(\lambda_n) + \mu_1^{(n)} A_{2,1}^{(a)}(\lambda_n), \\ k_1^{(n)} &= \mu_2^{(n)} A_{1,2}^{(a)}(\lambda_n) + \mu_1^{(n)} A_{2,2}^{(a)}(\lambda_n). \end{aligned} \tag{15.16}$$

In general the boundary problems (13.10) and (15.9) are distinct. In a restricted class of cases, however, they may be effectively the same. Boundary problems of this class are said to be *self-adjoint.* Many familiar physical phenomena admit of mathematical formulations in terms of self-adjoint differential systems.

## CHAPTER 16

**Generalized orthogonality.** Of the methods for the determination of coefficients in a trigonometric representation, that of Euler (Chapter 8) was seen to be by all odds the simpler one. It is more direct and very much shorter than Fourier's (Chapters 10, 11). And this advantage was recognized in Chapter 11 to be attributable essentially to the fact that from the very start it exploits strategically the peculiar properties of the functions in terms of which the representation is made. These properties are particularly those which enter into the so-called *orthogonality* of the trigonometric functions, namely, those which come to their most familiar expression, at least in part, in the relations (12.1), (12.3), (8.10) *etc.* A directive influence upon the present discussion is evidently to be discerned in this fact. The rôle that is filled by the trigonometric functions in the classical theory is to be assigned in this generalization to the characteristic solutions of the boundary problem (13.10). The discovery of interrelations between these solutions, and especially of such as reduce to orthogonality under suitable specialization, therefore appears as a significant issue at this turn.

In terms of the coefficients $h_1^{(r)}$, $h_2^{(r)}$, with which the relation (14.10) maintains, let the function $U_r(x, \lambda)$ be defined by the formula

$$U_r(x, \lambda) \equiv h_2^{(r)}\phi_1(x, \lambda) + h_1^{(r)}\phi_2(x, \lambda). \tag{16.1}$$

This is evidently a solution of the differential equation (13.3), specifically one which becomes a characteristic solution of the boundary problem (13.10) when $\lambda$ is given the value $\lambda_r$. If, then, as usual, $v_n(x)$ is used to designate the $n$th characteristic solution of the adjoint problem, it is clear that

$$L(U_r, \lambda) = 0,$$
$$M(v_n, \lambda) = (\lambda - \lambda_n)q(x)v_n(x).$$

In virtue of these relations the identity (15.2), with $U_r(x, \lambda)$ and $v_n(x)$ in the place of $\phi$ and $\psi$, yields, upon integration, the equation

$$(16.2) \qquad (\lambda - \lambda_n)\int_a^b q(x)U_r(x, \lambda)v_n(x)dx = Q(U_r, v_n, a) - Q(U_r, v_n, b).$$

Now when $\mu_1^{(n)}$, $\mu_2^{(n)}$ and $\lambda_n$ stand in the place of $\mu_1$, $\mu_2$ and $\lambda$, the boundary relations of the system (15.9) are fulfilled by the function $v_n(x)$, while their right-hand members may be written as

$$\mu_2^{(n)}[\alpha_{1,j}(\lambda) - (\lambda - \lambda_n)\beta_{1,j}] + \mu_1^{(n)}[\alpha_{2,j}(\lambda) - (\lambda - \lambda_n)\beta_{2,j}], \qquad j = 1, 2, 3, 4.$$

Upon multiplying them respectively by $U_r'(a, \lambda)$, $U_r(a, \lambda)$, $U_r'(b, \lambda)$, $U_r(b, \lambda)$, and then adding them, it is accordingly found that

$$\begin{aligned} Q(U_r, v_n, a) - Q(U_r, v_n, b) &= \mu_2^{(n)}A_1(U_r, \lambda) + \mu_1^{(n)}A_2(U_r, \lambda) \\ &\quad - (\lambda - \lambda_n)[\mu_2^{(n)}B_1(U_r) + \mu_1^{(n)}B_2(U_r)], \end{aligned}$$

the symbols $B_i$ having been introduced here in the sense

$$(16.3) \qquad B_i(\phi) \equiv \beta_{i,1}\phi'(a) + \beta_{i,2}\phi(a) + \beta_{i,3}\phi'(b) + \beta_{i,4}\phi(b), \qquad i = 1, 2.$$

The equation (16.2) therefore assumes, after a division by $(\lambda - \lambda_n)$, the form

$$(16.4) \qquad \int_a^b q(x)U_r(x, \lambda)v_n(x)dx + \mu_2^{(n)}B_1(U_r) + \mu_1^{(n)}B_2(U_r) = \Phi_{n,r}(\lambda),$$

in which the right-hand member is explicitly given by the formula

$$(16.5) \qquad \Phi_{n,r}(\lambda) \equiv \frac{\mu_2^{(n)}A_1(U_r, \lambda) + \mu_1^{(n)}A_2(U_r, \lambda)}{\lambda - \lambda_n}, \qquad \text{if } \lambda \neq \lambda_n.$$

We are to be specifically concerned with the form of the relation (16.4) and hence with the value of the expression (16.5) when $\lambda = \lambda_r$.

From the relations (16.1) it is to be seen at once that

$$(16.6) \qquad A_j(U_r, \lambda) = h_2^{(r)}A_{j,1}(\lambda) + h_1^{(r)}A_{j,2}(\lambda), \qquad j = 1, 2.$$

The equations (14.9) thus assure the relations

$$(16.7) \qquad A_j(U_r, \lambda_r) = 0, \qquad j = 1, 2,$$

and from these it follows that the expression (16.5) vanishes at $\lambda_r$, whenever $\lambda_r \neq \lambda_n$. For its evaluation at $\lambda = \lambda_n$ we may observe that by virtue of the formulas (16.6) the numerator of the expression (16.5) may be written out explicitly as

$$\overset{(r)}{h_2}\left[\overset{(n)}{\mu_2}A_{1,1}(\lambda) + \overset{(n)}{\mu_1}A_{2,1}(\lambda)\right] + \overset{(r)}{h_1}\left[\overset{(n)}{\mu_2}A_{1,2}(\lambda) + \overset{(n)}{\mu_1}A_{2,2}(\lambda)\right],$$

and that the relations (15.14) thereupon show it to reduce to zero at $\lambda_n$. At this value of $\lambda$ the expression (16.5) is, therefore, indeterminate. To fulfill the relation (16.4) it must, however, be continuous. The value to be assigned it at $\lambda_n$ is, therefore, that which is obtainable by an application of the familiar "l'Hospital's rule," namely, with the use of a superscribing dot to denote a derivative with respect to $\lambda$, thus

$$\dot{F}(x, \lambda_n) \equiv \left[\frac{\partial}{\partial\lambda}F(x, \lambda)\right]_{\lambda=\lambda_n}, \tag{16.8}$$

the value

$$\Phi_{n,r}(\lambda_n) = \overset{(n)}{\mu_2}\dot{A}_1(U_r, \lambda_n) + \overset{(n)}{\mu_1}\dot{A}_2(U_r, \lambda_n). \tag{16.9}$$

For the further analysis of this formula we must distinguish between the case in which the index of $\lambda_n$ is 1, and that in which it is 2.

If $\lambda_n$ is of the index 1 we are still concerned with the formula (16.9) only in the case that $r=n$. In this instance the relation

$$A_{\sigma,\tau}(\lambda_n) \neq 0, \tag{16.10}$$

is, moreover, fulfilled for some choice of the subscripts $\sigma$, $\tau$, and this implies through the equations (14.9) and (15.14) that

$$\overset{(n)}{h_\tau} \neq 0, \qquad \overset{(n)}{\mu_\sigma} \neq 0. \tag{16.11}$$

Now the relations (14.4) and (16.1) assure the equation

$$\begin{vmatrix} A_1(U_n, \lambda) & A_2(U_n, \lambda) \\ A_{1,\tau}(\lambda) & A_{2,\tau}(\lambda) \end{vmatrix} = (-1)^\tau \overset{(n)}{h_\tau}\Delta(\lambda),$$

and a differentiation of this, together with the evaluations (16.7), shows that

$$\begin{vmatrix} \dot{A}_1(U_n, \lambda_n) & \dot{A}_2(U_n, \lambda_n) \\ A_{1,\tau}(\lambda_n) & A_{2,\tau}(\lambda_n) \end{vmatrix} = (-1)^\tau \overset{(n)}{h_\tau}\dot{\Delta}(\lambda_n).$$

The determinant on the left of this equality is, however, that of the system which is comprised of the equation (16.9) with $r=n$, and the equation

$$0 = \overset{(n)}{\mu_2}A_{1,\tau}(\lambda_n) + \overset{(n)}{\mu_1}A_{2,\tau}(\lambda_n),$$

which is one of the pair (15.14). The eliminant of the coefficient of $A_{\sigma,\tau}(\lambda_n)$ in this system is, therefore, found to be the relation

$$\Phi_{n,n}(\lambda_n) = (-1)^{\sigma+\tau}\frac{\overset{(n)}{h_\tau}\overset{(n)}{\mu_\sigma}}{A_{\sigma,\tau}(\lambda_n)}\dot{\Delta}(\lambda_n). \tag{16.12}$$

The right-hand member of this is different from zero or equal to zero according as the multiplicity of $\lambda_n$ as a root of the characteristic equation (14.5) is equal to or greater than 1. It has been shown thus for the case at hand, namely when the index of $\lambda_n$ is 1, that the relations

$$\Phi_{n,r}(\lambda_r)\begin{cases}= 0, & \text{if } r \neq n,\\ \neq 0, & \text{if } r = n,\end{cases} \tag{16.13}$$

maintain if and only if the multiplicity of $\lambda_n$ and its index are equal.

Consider now the case of a characteristic value $\lambda_n$ of the index 2, and let its designations in the array (14.7) be $\lambda_m$ and $\lambda_{m+1}$. The set of equations obtained from the relation (16.9) when $n$ and $r$ are given values from the pair $m$, $m+1$, yield the determinant relation

$$\begin{vmatrix} \Phi_{m,m}(\lambda_n) & \Phi_{m+1,m}(\lambda_n) \\ \Phi_{m,m+1}(\lambda_n) & \Phi_{m+1,m+1}(\lambda_n) \end{vmatrix} = \begin{vmatrix} \dot{A}_1(U_m, \lambda_n) & \dot{A}_2(U_m, \lambda_n) \\ \dot{A}_1(U_{m+1}, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n) \end{vmatrix} \cdot \begin{vmatrix} \overset{(m)}{\mu_2} & \overset{(m+1)}{\mu_2} \\ \overset{(m)}{\mu_1} & \overset{(m+1)}{\mu_1} \end{vmatrix}.$$

Now the formulas (16.6) assure the equation

$$\begin{vmatrix} A_1(U_m, \lambda) & A_2(U_m, \lambda) \\ A_1(U_{m+1}, \lambda) & A_2(U_{m+1}, \lambda) \end{vmatrix} = \begin{vmatrix} \overset{(m)}{h_2} & \overset{(m+1)}{h_2} \\ \overset{(m)}{h_1} & \overset{(m+1)}{h_1} \end{vmatrix} \Delta(\lambda),$$

and a two-fold differentiation of this together with the evaluation (16.7) leads to the equality

$$2\begin{vmatrix} \dot{A}_1(U_m, \lambda_n) & \dot{A}_2(U_m, \lambda_n) \\ \dot{A}_1(U_{m+1}, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n) \end{vmatrix} = \begin{vmatrix} \overset{(m)}{h_2} & \overset{(m+1)}{h_2} \\ \overset{(m)}{h_1} & \overset{(m+1)}{h_1} \end{vmatrix} \ddot{\Delta}(\lambda_n).$$

The combination of this result with that above evidently leads to the relation

$$\begin{vmatrix} \Phi_{m,m}(\lambda_n) & \Phi_{m+1,m}(\lambda_n) \\ \Phi_{m,m+1}(\lambda_n) & \Phi_{m+1,m+1}(\lambda_n) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \overset{(m)}{\mu_2} & \overset{(m+1)}{\mu_2} \\ \overset{(m)}{\mu_1} & \overset{(m+1)}{\mu_1} \end{vmatrix} \cdot \begin{vmatrix} \overset{(m)}{h_2} & \overset{(m+1)}{h_2} \\ \overset{(m)}{h_1} & \overset{(m+1)}{h_1} \end{vmatrix} \ddot{\Delta}(\lambda_n). \tag{16.14}$$

If the multiplicity of $\lambda_n$ is greater than 2 the right-hand member of the equation (16.14) is zero. It is clear that in that case the equation is contradicted by the relations (16.13) and that these latter, therefore, do not maintain. If, on the other hand, the multiplicity of $\lambda_n$ is 2, namely the same as its index, no such contradiction is involved. If in this case the parameter pairs $\mu_1^m$, $\mu_2^m$ and $\mu_1^{m+1}$, $\mu_2^{m+1}$, which have thus far been left unspecified except that they be linearly independent, are now specified to fulfill the relations

$$\begin{aligned} \overset{(m)}{\mu_2}\dot{A}_1(U_{m+1}, \lambda_n) + \overset{(m)}{\mu_1}\dot{A}_2(U_{m+1}, \lambda_n) &= 0, \\ \overset{(m+1)}{\mu_2}\dot{A}_1(U_m, \lambda_n) + \overset{(m+1)}{\mu_1}\dot{A}_2(U_m, \lambda_n) &= 0, \end{aligned} \tag{16.15}$$

it follows at once through the equations (16.9) that $\Phi_{m,m+1}(\lambda_n) = \Phi_{m+1,m}(\lambda_n) = 0$,

and then through the equation (16.14) that $\Phi_{n,n}(\lambda_n)\neq 0$, for $n=m$, $m+1$. The relations (16.13), which by virtue of the equation (16.4) take the form

$$(16.16)\quad \int_a^b q(x)u_r(x)v_n(x)dx + \overset{(n)}{\mu_2}B_1(u_r) + \overset{(n)}{\mu_1}B_2(u_r)\begin{cases} = 0, & \text{if } r\neq n,\\ \neq 0, & \text{if } r = n,\end{cases}$$

thus maintain whenever the characteristic value $\lambda_n$ is one whose index and multiplicity are equal, and do not maintain in any other case. These relations are the ones that were sought. They are, namely, those which are expressive of the generalized orthogonality which subsists among the characteristic solutions of two boundary problems that are adjoint.

## CHAPTER 17

**The formal representation of an arbitrary function.** In the instance of any boundary problem which admits of infinitely many characteristic solutions, the physical context in connection with which the problem arises leads in a natural way to the question of the representability of an arbitrary function in terms of these solutions in the manner

$$(17.1)\qquad f(x) = \sum_{r=0}^{\infty} c_r u_r(x).$$

This has already been observed in several instances of the Fourier theory, which is, of course, exemplary of the more general case. As in the trigonometric case, the crux of the formal problem thrown up in this way devolves upon a determination of the coefficients $c_r$. We shall consider this matter now, not in a rigorous way, but formally. The relation (17.1) will, therefore, be taken to be amenable to all such operations as shall be made upon it, and no consideration will be given to matters of convergence. The deductions will, therefore, of course, be devoid of all power of proof. Their purpose is purely an exploratory one.

To begin with, let the symbols $f_i$, $i=1, 2$ be used as abbreviations for the expressions $B_i(f)$, as these latter are obtainable from the relations (16.3). The equation (17.1) thus has associated with it the pair of auxiliary relations

$$(17.2)\qquad f_i = \sum_{r=0}^{\infty} c_r B_i(u_r), \qquad i = 1, 2,$$

and in terms of the elements $f(x)$, $f_1$ and $f_2$ the formulas

$$(17.3)\qquad I_n(f) \equiv \int_a^b q(x)v_n(x)f(x)dx + \overset{(n)}{\mu_2}f_1 + \overset{(n)}{\mu_1}f_2, \qquad n = 0, 1, 2, \cdots,$$

may be taken to define their left-hand members. Upon substituting into these formulas the infinite series evaluations (17.1) and (17.2), interchanging the order of the integrations and summations, and collecting the terms in any coefficient $c_r$, it is found that alternatively

$$I_n(f) = \sum_{r=0}^{\infty} c_r \Phi_{n,r}(\lambda_r).$$

The symbols $\Phi_{n,r}(\lambda_r)$ in this stand, as heretofore, for the left-hand members of the relations (16.16). Since these latter vanish whenever $r \neq n$, the equation reduces to the simple form

$$I_n(f) = c_n \Phi_{n,n}(\lambda_n). \tag{17.4}$$

Let it be assumed now as a hypothesis which is to cover the entire remaining portion of these deductions, that *the boundary problem* (13.10) *in question is one for which there are infinitely many characteristic values, and for which, moreover, each of these is of a multiplicity that is equal to its index.* By the relations (16.13) the equations (17.4) yield, then, for each $n$, the evaluation

$$c_n = \frac{I_n(f)}{\Phi_{n,n}(\lambda_n)}. \tag{17.5}$$

It will be evident even from the most casual review of the procedure described that it is limited, insofar as actual applicability is concerned, to the functions of a materially restricted class. The expressions $B_i(f)$ are, for instance, significant only for functions that are differentiable at $x=a$ and $x=b$, while other heavy restrictions are manifestly involved. Beyond that it is clear that the result can imply nothing of the representability of a function $f(x)$, since that representability was at the very outset assumed. A theory of representation must, accordingly, be approached differently.

Let it be supposed, therefore, that a function $f(x)$ which is arbitrary except that it possesses certain requisite properties of integrability, and a pair of constants $f_1, f_2$ are given. The constants are likewise to be regarded as arbitrary. In particular they need have no specific relation to the values of $f(x)$. From these elements $f(x)$, $f_1$, and $f_2$ the values $I_n(f)$ are constructible through the formulas (17.3). There is thus associated with them a sequence of constants (17.5), or, in other words, a series of the form (17.1). We shall indicate this association in the manner

$$f(x) \sim \sum_{n=0}^{\infty} \frac{I_n(f) u_n(x)}{\Phi_{n,n}(\lambda_n)}. \tag{17.6}$$

There is no implication at this stage that the series here written down is convergent, or, even should that be the case, that its value is $f(x)$. The continuing theory is to be shaped toward the investigation of those matters. As it stands, the series is to be approached afresh and with no regard for the manner in which it was deduced. Its convergence is to be studied. Any conditions, if such are found, under which the series does converge and to the value $f(x)$, will be conditions under which the symbol of association in the relation (17.6) may be replaced by one of equality and a theory of representation maintains. The direction of our further theoretical developments is thus forecast.

## CHAPTER 18

**Some examples.** Throughout the discussion thus far no references whatsoever to illustrative examples have been made. A stage has now been reached, however, at which the consideration of some explicit cases may be both interesting and instructive. We shall, therefore, interrupt the theoretical developments at this point to review some particular examples with the special purpose of illustrating concretely some of the salient features of the theory as it has thus far been deduced. It will be seen in connection with one of these cases how the present more general theory wholly includes that of Fourier.

Because of the breadth of the basis upon which the theory is built, it is a simple matter to construct in great variety examples that are subsumed under it. In the choice that is set forth below, however, an effort has been made to avoid in as large a measure as possible all such complications as are not germane to the matters essentially at issue. Such could be only distracting and not illuminating. For this reason, in particular, the boundary problems that are drawn upon have all been chosen to be such as involve differential equations that are explicitly solvable. The theory in no way requires that. For convenience in the notation the symbol $\rho$ has been introduced to stand for a square root of $\lambda$, thus

$$\rho^2 = \lambda. \tag{18.1}$$

This, however, is to be looked upon merely as an abbreviation. The effective parameter will continue to be $\lambda$ even when an expression is written in terms of $\rho$. It will generally be found on this account that a form that is indeterminate as to $\rho$ is quite specific in $\lambda$, as is the case with the form $\sin \alpha\rho/\rho$, which near $\lambda = 0$ is to be thought of as defined by the power series

$$\alpha - \frac{\alpha^3}{3!}\lambda + \frac{\alpha^5}{5!}\lambda^2 \cdots .$$

It may also be worth observing in conjunction with the notation defined through the relation (16.8), that

$$\dot{F} = \frac{1}{2\rho}\frac{\partial F}{\partial \rho}\,. \tag{18.2}$$

*Example 1. The boundary problem*

$$\begin{gathered} u'' - \frac{2}{x}u' - \left(\lambda - \frac{2}{x^2}\right)u = 0, \\ u\left(\frac{\pi}{2}\right) = 0, \\ u\left(\frac{3\pi}{2}\right) = 0. \end{gathered} \tag{18.3}$$

The fundamental interval at the ends of which the boundary relations of this problem apply, namely $\pi/2 \leqq x \leqq 3\pi/2$, is one upon which the differential equation has continuous coefficients and no singular points. Beyond that the coefficients $\alpha_{i,j}$ of the boundary relations are the elements of the matrix

$$\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and this, the matrix (13.8), is of the rank 2 for all values of $\lambda$. The boundary problem is thus of the type (13.10). Its differential equation admits as linearly independent solutions analytic in $\lambda$, the functions

$$\phi_1(x, \lambda) \equiv x[e^{\rho x} + e^{-\rho x}],$$

$$\phi_2(x, \lambda) \equiv \frac{x}{\rho}[e^{\rho x} - e^{-\rho x}],$$

and as it is formed from these the determinant (14.4) is

$$\Delta(\lambda) \equiv \begin{vmatrix} \frac{\pi}{2}[e^{\rho\pi/2} + e^{-\rho\pi/2}] & \frac{\pi}{2\rho}[e^{\rho\pi/2} - e^{-\rho\pi/2}] \\ \frac{3\pi}{2}[e^{3\rho\pi/2} + e^{-3\rho\pi/2}] & \frac{3\pi}{2\rho}[e^{3\rho\pi/2} - e^{-3\rho\pi/2}] \end{vmatrix}.$$

The evaluation

$$\Delta(\lambda) = \frac{3\pi^2}{2}\left[\frac{e^{\rho\pi} - e^{-\rho\pi}}{\rho}\right],$$

shows that its zeros occur at the points $\rho = \pm ni$, with $n = 1, 2, 3, \cdots$. At each of these points the determinant $\Delta(\lambda)$ is reduced to the rank 1 and $\dot{\Delta}(\lambda)$ is not zero. The characteristic values, which are infinitely many, are thus all of the index and multiplicity 1. They are to be arranged after the fashion (14.7) in the order

$$\lambda_n = -(n + 1)^2, \qquad n = 0, 1, 2, \cdots.$$

When $r$ is any even integer, say $r = 2n$, the coefficient of $h_2^{(2n)}$ in the first one of the equations (14.9) is zero. This is seen to determine $h_1^{(2n)}$ to be zero, while leaving $h_2^{(2n)}$ arbitrary (not zero). This latter coefficient may evidently be chosen, therefore, so that the respective characteristic solution, as it is given by the formula (14.10), is explicitly

$$u_{2n}(x) \equiv x \cos (2n + 1)x.$$

In a quite similar manner it is found that $h_2^{(2n-1)} = 0$, and that we may accordingly choose

$$u_{2n-1}(x) \equiv x \sin 2nx.$$

By the formulas (15.9) the differential system adjoint to (18.3) is

$$v'' + \frac{2}{x} v' - \lambda v = 0,$$

$$v\left(\frac{\pi}{2}\right) = 0,$$

$$-v'\left(\frac{\pi}{2}\right) - \frac{4}{\pi} v\left(\frac{\pi}{2}\right) = \mu_2,$$

$$-v\left(\frac{3\pi}{2}\right) = 0,$$

$$v'\left(\frac{3\pi}{2}\right) + \frac{4}{3\pi} v\left(\frac{3\pi}{2}\right) = \mu_1.$$

Of the equations in this, the first, the second, and the fourth, effectively define the adjoint boundary problem, while the remaining ones give equivalents of the parameters $\mu_1$ and $\mu_2$. The formulas (15.5) yield as solutions of the differential equation in this system the functions

$$\psi_1(x, \lambda) \equiv \frac{-1}{4\rho x} \left[e^{\rho x} - e^{-\rho x}\right],$$

$$\psi_2(x, \lambda) \equiv \frac{1}{4x} \left[e^{\rho x} + e^{-\rho x}\right].$$

The equations (15.16) show that for any $n$, $k_1^{2n-1} = 0$ and $k_2^{2n} = 0$. The characteristic solutions $v_n(x)$ may accordingly be taken to be

$$v_{2n-1}(x) \equiv \frac{\sin 2nx}{x},$$

$$v_{2n}(x) \equiv \frac{\cos (2n + 1)x}{x}.$$

The boundary problem (18.3) does not involve $\lambda$ in its boundary relations. Hence the constants $\beta_{i,j}$ that appear in the formulas (13.7) and (16.3) are all zero, and the forms (16.3) accordingly all vanish. The relation (16.4) thus shows that

$$\Phi_{n,r}(\lambda_r) = -\int_{\pi/2}^{3\pi/2} u_r(x) v_n(x) dx,$$

and the relations of orthogonality (16.16) therefore assume a familiar, purely trigonometrical form. With the special choice of constants $f_1 = 0$, $f_2 = 0$, the formulas (17.3) and (17.6) yield, in association with an arbitrary function $f(x)$, the series of characteristic solutions

$$f(x) \sim c_0 x \cos x + c_1 x \sin 2x + c_2 x \cos 3x + c_3 x \sin 4x + \cdots$$

with the coefficients

$$c_{2n-1} = \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} f(s)\,\frac{\sin 2ns}{s}\,ds,$$

$$c_{2n} = \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} f(s)\,\frac{\cos (2n+1)s}{s}\,ds.$$

*Example 2. The boundary problem*

$$\begin{aligned} u'' - \frac{1}{x}u' + \frac{16\pi^2x^2}{9}\lambda u &= 0,\\ u(1) &= 0,\\ 2u'(1) - u'(2) &= 0. \end{aligned} \tag{18.4}$$

This boundary problem is of the form (13.10) on the fundamental interval $1 \leqq x \leqq 2$. Its differential equation admits the solutions

$$\phi_1(x,\lambda) \equiv \cos\left(\frac{2\pi\rho}{3}x^2\right),$$

$$\phi_2(x,\lambda) \equiv \frac{1}{\rho}\sin\left(\frac{2\pi\rho}{3}x^2\right),$$

and as formed from these functions

$$\Delta(\lambda) \equiv \begin{vmatrix} \cos\left(\dfrac{2\pi\rho}{3}\right) & \dfrac{1}{\rho}\sin\left(\dfrac{2\pi\rho}{3}\right) \\ \dfrac{8\pi\rho}{3}\left[\sin\left(\dfrac{8\pi\rho}{3}\right) - \sin\left(\dfrac{2\pi\rho}{3}\right)\right] & \dfrac{8\pi}{3}\left[\cos\left(\dfrac{2\pi\rho}{3}\right) - \cos\left(\dfrac{8\pi\rho}{3}\right)\right] \end{vmatrix}.$$

The evaluation

$$\Delta(\lambda) = \frac{8\pi}{3}(1 - \cos 2\pi\rho),$$

shows that the zeros occur at the points $\rho = 0\ \pm 1,\ \pm 2, \cdots$. There are thus infinitely many characteristic values, and since at each of them the determinant is reduced to the rank 1 they are all of the index 1. We may accordingly write

$$\lambda_n = n^2, \qquad n = 0, 1, 2, \cdots.$$

But it is now seen at once that $\dot{\Delta}(\lambda_n) = 0$ for each $n$. The characteristic values are thus each of a multiplicity that is higher than its index and the boundary problem is therefore excluded by the hypothesis of chapter 17.

*Example 3. The boundary problem*

$$u'' + \lambda u = 0,$$
$$(18.5) \qquad u(-\pi) - u(\pi) = 0,$$
$$u'(-\pi) - u'(\pi) = 0.$$

On the interval $-\pi \leqq x \leqq \pi$ this boundary problem is of the form (13.10). We find for it

$$(18.6) \qquad \phi_1(x, \lambda) \equiv \cos \rho x, \qquad \phi_2(x, \lambda) \equiv \frac{\sin \rho x}{\rho},$$

and accordingly

$$(18.7) \qquad \Delta(\lambda) \equiv \begin{vmatrix} 0 & -\dfrac{2}{\rho}\sin \rho\pi \\ 2\rho \sin \rho\pi & 0 \end{vmatrix}.$$

Since from this

$$(18.8) \qquad \Delta(\lambda) = 4 \sin^2 \rho\pi,$$

its zeros occur for $\rho = 0, \pm 1, \pm 2, \cdots$. At the first of these the rank of the determinant (18.7) is 1, at all others it is 0. The first characteristic value $\lambda_0 = 0$ is thus of the index 1 and the remaining ones are of the index 2. They are accordingly to be arranged thus

$$(18.9) \qquad \lambda_0 = 0, \qquad \lambda_{2n-1} = \lambda_{2n} = n^2, \qquad n = 1, 2, 3, \cdots.$$

Each value is found to be of a multiplicity equal to its index. By the relations (14.9) it is shown that $h_1^{(0)} = 0$, so that the first characteristic solution is $u_0(x) \equiv 1$, or any multiple of this. For the other characteristic values the equations (14.9) are vacuous and the coefficients $h_1^{(r)}$ and $h_2^{(r)}$ accordingly arbitrary. We may therefore choose $u_{2n-1}(x)$ and $u_{2n}(x)$ as $\phi_1(x, \lambda_{2n-1})$ and $\phi_2(x, \lambda_{2n})$ respectively,

$$(18.10) \qquad \begin{aligned} u_{2n-1}(x) &\equiv \cos nx, \\ u_{2n}(x) &\equiv \sin nx. \end{aligned}$$

The boundary problem (18.5) is readily seen to be self-adjoint. It is, therefore, admissible to choose $v_n(x) \equiv u_n(x)$ for every $n$. Since the boundary relations are independent of $\lambda$ the forms $B_i(u_n)$ all vanish, and it is found accordingly that

$$\Phi_{n,r}(\lambda_r) = \int_{-\pi}^{\pi} u_r(x) u_n(x) dx.$$

The relations (16.13) are thus, in this instance, merely expressive of the familiar property of orthogonality of the sines and cosines of multiples of $x$. With the

choice of constants $f_1=0$, $f_2=0$, the formulas (17.3) and (17.5) define the coefficients $c_n$ to be those of Fourier, and the relation (17.6) is simply the association of a function $f(x)$ with its Fourier series.

## CHAPTER 19

**Another example.** Although the examples that have been cited in the chapter above were earmarked among themselves by some conspicuous qualitative dissimilarities, they fail, even when taken together, to illustrate adequately some of the more pronounced departures from classical theory which the generalization permits. The reason for this lies in the fact that the boundary problems that are basic to them all involve only such boundary relations as do not depend upon the parameter $\lambda$. In the following a problem of the excepted class is to be briefly analyzed.

*Example 4. The boundary problem*

$$\begin{aligned} u'' + 4\lambda u &= 0, \\ u(0) &= 0, \\ u'(1) + [\nu^2 + 1 - \lambda]u(1) &= 0, \end{aligned} \tag{19.1}$$

with $\nu$ standing for any real constant.

Such boundary problems as this present themselves in connection with a variety of physical investigations of which the following ones may be looked upon as in some measure typical. [18]

*Problem (i).* A right cylindrical solid with a cross-section of any shape and size, and with plane terminal faces at $x=0$ and $x=1$, has its lateral surface insulated against the passage of heat and has an initial distribution of temperatures depending only upon the longitudinal coördinate $x$. At the time $t=0$ this solid is placed in contact with a quantity of liquid at one of its terminal faces, and the liquid is thereupon kept well stirred to insure that the temperature is uniform throughout it at each instant. The temperature of the liquid and the temperature distribution in the solid at any subsequent time are to be calculated.

*Problem (ii).* A solid metal sphere with an initial distribution of temperatures that is symmetrical about its center, is cooled by being plunged into a mass of liquid. The liquid is kept well stirred. The temperatures of the liquid and those within the solid during the cooling are to be determined.

*Problem (iii).* A mass of material is uniformly distributed at the time $t=0$ throughout a jelly in a cylindrical container. The jelly is covered with a liquid that is kept well stirred. From the experimental measurements of the concentration of material in the liquid as a function of the time, the coefficient of diffusion of the material in the jelly is to be found.

The boundary problem (19.1) is of the form (13.10). Its differential equation admits as solutions the functions

$$\phi_1(x, \lambda) \equiv \cos 2\rho x, \tag{19.2}$$
$$\phi_2(x, \lambda) \equiv \frac{\sin 2\rho x}{\rho},$$

and the evaluation of the determinant

$$\Delta(\lambda) \equiv \begin{vmatrix} 1 & 0 \\ -2\rho \sin 2\rho + (\nu^2+1-\lambda) \cos 2\rho & 2 \cos 2\rho + (\nu^2+1-\lambda) \dfrac{\sin 2\rho}{\rho} \end{vmatrix}, \tag{19.3}$$

gives as the characteristic equation

$$2 \cos 2\rho + (\nu^2 + 1 - \rho^2) \frac{\sin 2\rho}{\rho} = 0. \tag{19.4a}$$

Alternative forms of this are

$$\cot 2\rho = \frac{\rho}{2} - \frac{\nu^2 + 1}{2\rho}, \tag{19.4b}$$

and

$$e^{4\rho i} = \frac{(\rho + i)^2 - \nu^2}{(\rho - i)^2 - \nu^2}\,. \tag{19.4c}$$

The characteristic values are thus the squares of the roots of a transcendental equation and it is readily seen that they do not admit of expression by any elementary formula. The essential facts concerning them are nevertheless deducible.

Thus if $\rho$ is any complex value with a positive imaginary part, the point $(\rho+i)$ in the complex plane is more distant than the point $(\rho-i)$ from any point on the axis of reals, and hence in particular from the points $\nu$ and $-\nu$. Hence

$$|(\rho + i) - \nu| > |(\rho - i) - \nu|,$$
$$|(\rho + i) + \nu| > |(\rho - i) + \nu|,$$

and from these relations it follows that the right-hand member of the equation (19.4c) is greater than 1 in absolute value. But for a value of $\rho$ such as this the left-hand member of that equation has an absolute value that is less than 1. This $\rho$ is, therefore, not one for which the equation is satisfied. A similar argument establishes that same fact for any value of $\rho$ that has a negative imaginary part. The roots of the equation must thus all be real.

Now that being so, it is observable from the equation in the form (19.4b), that its roots are the abscissas of the points common to the graphs

$$y = \cot 2\rho,$$

$$y = \frac{\rho}{2} - \frac{\nu^2 + 1}{2\rho},$$

in the Cartesian $(\rho, y)$ plane. These graphs are easily seen to intersect infinitely often and without being tangent to each other at any intersection point. From this it follows at once that there are infinitely many characteristic values and that they are each of the multiplicity 1. From the general relation (14.6), or from the fact that the determinant (19.3) has a constant non-vanishing element, it is to be seen that every characteristic value is of the index 1.

By the first one of the equations (14.9) it is shown that for each $r$, $h_2^{(r)} = 0$. The characteristic solutions may, therefore, be taken as the functions

$$u_n(x) \equiv \sin 2\sqrt{\lambda_n}\, x, \tag{19.5}$$

and since the boundary problem is self-adjoint we may also take $v_n(x) \equiv u_n(x)$. The formulas (16.3) and the boundary relations of the system (15.9) give in this case the evaluations

$$B_1(u_r) = 0, \qquad B_2(u_r) = -\, u_r(1),$$

$$\overset{(n)}{\mu_2} = -\, v_n'(0), \qquad \overset{(n)}{\mu_1} = -\, v_n(1).$$

It is thus found from the equations (16.4) that

$$\Phi_{n,r}(\lambda_r) = 4\int_0^1 u_r(x)u_n(x)dx + u_r(1)u_n(1).$$

Because of the simple form of the functions $u_n(x)$ the integration in this expression is explicitly possible, and the relations of generalized orthogonality (16.16) are therefore verifiable upon the basis of the equation (19.4a). The formulas (17.3) and (17.5) yield as the coefficients in the association

$$f(x) \sim \sum_{n=0}^{\infty} c_n \sin 2\sqrt{\lambda_n}\, x, \tag{19.6}$$

the values

$$c_n = \frac{4\int_0^1 f(s)u_n(s)ds - f_1u_n'(0) - f_2u_n(1)}{4\int_0^1 \overset{2}{u_n}(s)ds + \overset{2}{u_n}(1)}. \tag{19.7}$$

A scrutiny of the formulas (19.7) reveals one of the salient features in which this and the Fourier representations are effectively dissimilar. While the coefficients of the latter are wholly determined by the function $f(x)$, those given by the formulas (19.7) depend also upon the prescribed constants $f_1$ and $f_2$. The same function $f(x)$ may thus be associated here with many different representations. This is quite consonant with the nature of the physical problems from which boundary problems of the type (19.1) arise, as may easily be appreciated from a consideration of the problems (i) and (ii) that were cited above. In each of those cases the function $f(x)$ represents the initial temperatures in the solid. The subsequent temperatures therein depend, however, not only upon these but also upon the initial temperature of the fluid into which the solid is immersed or with which it is placed in contact. It is through the constants $f_1$ and $f_2$ that this initial fluid temperature comes to account. [19]

## CHAPTER 20

**The Green's function.** A noteworthy feature of the Fourier's series, and one which is almost invariably taken as the point of departure for studies of its convergence properties, is the fact that any initial segment of it may be explicitly summed and hence expressed by a compact formula. This was observed in chapter 12, the summation of the segment $S_N(x)$ as given by the relation (12.7) having been accomplished there by the formula (12.8). It is far from obvious that the advantages inherent in this are retainable in the generalization of the theory, for the derivation of the formula in question is directly and explicitly based upon relationships that are peculiarly trigonometric. We shall show that this may nevertheless be done, the key to the requisite deductions residing in a certain function, the so-called Green's function, which generally plays an important rôle in the theory of boundary problems.

Let $\lambda$ be taken and retained throughout the deductions of this chapter as *not* a characteristic value of the boundary problem (13.10). This problem then admits of no solution, which is to say that a function which fulfills its boundary relations cannot also satisfy the stipulations of its differential equation for all values of $x$ upon the fundamental interval. Now it is significant that the concession which must be made on the part of the differential equation is very slight, amounting to no more, in fact, than the partial relaxation of its stipulations at only a single point $x=s$ of the interval. The function which fulfills the specifications except at $x=s$ may even be required to be continuous there. The failure will occur in its first derivative, which is subject at this point to an ordinary discontinuity. When this discontinuity is of the proper sign and of the unit magnitude, matters which are adjustable without otherwise affecting the issues, the function is called the Green's function. Inasmuch as it depends upon the location of the point $s$ as well as upon the variables $x$ and $\lambda$, it is to be denoted by the symbol $G(x, s, \lambda)$. Its properties, by way of summary, are, then:

(i) As a function of $x$ it satisfies the differential equation of the system (13.10) over each of the intervals $a<x<s$, and $s<x<b$;

(ii) It is continuous in $x$ over the whole interval $(a, b)$, but its derivative is discontinuous at the point $x=s$ to accord with the prescription

$$\frac{\partial}{\partial x}G(x, s, \lambda)\Big]_{x=s+} - \frac{\partial}{\partial x}G(x, s, \lambda)\Big]_{x=s-} = 1; \tag{20.1}$$

(iii) It fulfills the boundary relations of the differential system (13.10).

By these properties the function $G(x, s, \lambda)$ is uniquely determined, as the following derivation of its form will incidentally show.

As a solution of the differential equation (13.3) the function is expressible upon each of the intervals $(a, s)$ and $(s, b)$ as a linear combination of the solutions $\phi_j(x, \lambda)$, $j=1, 2$, namely

$$G(x, s, \lambda) = \begin{cases}\gamma_{1,2}\phi_1(x, \lambda) + \gamma_{1,1}\phi_2(x, \lambda), & \text{for} \quad a \leqq x \leqq s, \\ \gamma_{2,2}\phi_1(x, \lambda) + \gamma_{2,1}\phi_2(x, \lambda), & \text{for} \quad s \leqq x \leqq b.\end{cases}$$

The specifications (ii) impose upon these forms the relations

$$\gamma_{2,2}\phi_1(s, \lambda) + \gamma_{2,1}\phi_2(s, \lambda) = \gamma_{1,2}\phi_1(s, \lambda) + \gamma_{1,1}\phi_2(s, \lambda),$$
$$\gamma_{2,2}\phi_1'(s, \lambda) + \gamma_{2,1}\phi_2'(s, \lambda) - \gamma_{1,2}\phi_1'(s, \lambda) - \gamma_{1,1}\phi_2'(s, \lambda) = 1,$$

in other words, a pair of equations which may be solved into the form

$$\gamma_{1,1} = \gamma_{2,1} - \frac{\phi_1(s, \lambda)}{\Omega(s, \lambda)},$$

$$\gamma_{1,2} = \gamma_{2,2} + \frac{\phi_2(s, \lambda)}{\Omega(s, \lambda)}.$$

Because of the formulas (15.5) these relations are alternatively

$$\gamma_{1,1} = \gamma_{2,1} - \psi_2(s, \lambda),$$
$$\gamma_{1,2} = \gamma_{2,2} - \psi_1(s, \lambda).$$

The formulas for $G(x, s, \lambda)$ are, therefore, more explicitly

$$G(x, s, \lambda) = \gamma_{2,2}\phi_1(x, \lambda) + \gamma_{2,1}\phi_2(x, \lambda) + g_1(x, s, \lambda), \tag{20.2}$$

in which the final term is that defined by the relations

$$g_1(x, s, \lambda) = \begin{cases}- \phi_1(x, \lambda)\psi_1(s, \lambda) - \phi_2(x, \lambda)\psi_2(s, \lambda), & \text{for} \quad a \leqq x \leqq s, \\ 0, & \text{for} \quad s \leqq x \leqq b.\end{cases} \tag{20.3}$$

The requirement (iii), that $G(x, s, \lambda)$ fulfill the boundary relations, comes to its expression in terms of the symbols (13.9), (14.3) and (15.11), in the pair of equations

$$\gamma_{2,2}A_{i,1}(\lambda) + \gamma_{2,1}A_{i,2}(\lambda) - A_{i,1}^{(a)}(\lambda)\psi_1(s, \lambda) - A_{i,2}^{(a)}(\lambda)\psi_2(s, \lambda) = 0, \qquad i = 1, 2.$$

Their solution determines for the coefficients $\gamma_{2,2}$ and $\gamma_{2,1}$ the evaluations

$$\gamma_{2,j} = \frac{(-1)^{j+1}}{\Delta(\lambda)} \begin{vmatrix} A_{1,j}(\lambda) & A_{1,1}^{(a)}(\lambda)\psi_1(s, \lambda) + A_{1,2}^{(a)}(\lambda)\psi_2(s, \lambda) \\ A_{2,j}(\lambda) & A_{2,1}^{(a)}(\lambda)\psi_1(s, \lambda) + A_{2,2}^{(a)}(\lambda)\psi_2(s, \lambda) \end{vmatrix}, \qquad j = 1, 2.$$

The substitution of these into the formula (20.2) leads to the result

$$G(x, s, \lambda) = g_1(x, s, \lambda) + \frac{g_2(x, s, \lambda)}{\Delta(\lambda)}, \tag{20.4}$$

in which

$$g_2(x, s, \lambda) = - \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ A_{1,1}(\lambda) & A_{1,2}(\lambda) & A_{1,1}^{(a)}(\lambda)\psi_1(s, \lambda) + A_{1,2}^{(a)}(\lambda)\psi_2(s, \lambda) \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) & A_{2,1}^{(a)}(\lambda)\psi_1(s, \lambda) + A_{2,2}^{(a)}(\lambda)\psi_2(s, \lambda) \end{vmatrix}. \tag{20.5}$$

The Green's function has thus been completely determined.

The method which has in this way been described in connection with the differential system (13.10), may now be applied equally well to the calculation of the Green's function $G_a(x, s, \lambda)$ of the adjoint system (15.9). If this is done it will be found that the two Green's functions are simply related by the equation

$$G_a(x, s, \lambda) \equiv G(s, x, \lambda). \tag{20.6}$$

The function (20.4) thus operates as the Green's function of the adjoint differential system when its second argument is taken to be the variable and the first one is fixed. The set of its properties enumerated above may thus be extended to include the following ones:

(iv) As a function of $s$ it satisfies the differential equation of the system (15.9) over each of the intervals $a<s<x$ and $x<s<b$;

(v) It is continuous in $s$ over the whole interval $(a, b)$, but its derivative as to $s$ is discontinuous at the point $s=x$ to accord with the prescription

$$\frac{\partial}{\partial s}G(x, s, \lambda)\Big]_{s=x+} - \frac{\partial}{\partial s}G(x, s, \lambda)\Big]_{s=x-} = 1; \tag{20.7}$$

(vi) As a function of $s$ it fulfills the boundary relations of the differential system (15.9), with appropriate determinations of $\mu_1(x, \lambda)$ and $\mu_2(x, \lambda)$.

It is on the whole simpler to verify these facts than to deduce them. By the formulas (20.4), (20.3) and (20.5), the function $G(x, s, \lambda)$ is evidently expressed as a linear combination of the solutions $\psi_j(s, \lambda)$, $j=1, 2$, on each of the intervals $(a, x)$ and $(x, b)$. The property (iv) is thus assured to it. The forms

$$\phi_1(x, \lambda)\psi_1(s, \lambda) + \phi_2(x, \lambda)\psi_2(s, \lambda),$$
$$\phi_1(x, \lambda)\psi_1'(s, \lambda) + \phi_2(x, \lambda)\psi_2'(s, \lambda),$$

are readily seen from the relations (15.5) and (13.5) to take on the values 0 and 1 respectively at the point $s=x$. By that the properties (v) are implied. Finally the formulas (15.5) and (15.11) may be drawn upon to supply the evaluations

$$A_{i,1}^{(a)}(\lambda)\psi_1(a, \lambda) + A_{i,2}^{(a)}(\lambda)\psi_2(a, \lambda) = \alpha_{i,1}(\lambda), \qquad i = 1, 2.$$

By virtue of them it is clear that

$$G(x, a, \lambda) = \frac{-1}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ A_{1,1}(\lambda) & A_{1,2}(\lambda) & \alpha_{1,1}(\lambda) \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) & \alpha_{2,1}(\lambda) \end{vmatrix}.$$

This, however, is simply the statement that

$$G(x, a, \lambda) = \mu_2(x, \lambda)\alpha_{1,1}(\lambda) + \mu_1(x, \lambda)\alpha_{2,1}(\lambda),$$

namely that the first boundary relation of the system (15.9) is fulfilled, with the parameter values

$$\mu_i(x, \lambda) = \frac{(-1)^i}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ A_{i,1}(\lambda) & A_{i,2}(\lambda) \end{vmatrix}, \qquad i = 1, 2. \tag{20.8}$$

By the same procedure it may equally well be shown that the remaining boundary relations are also fulfilled with the same values (20.8). Thus the property (vi) maintains.

Although it will not be relevant to the discussion which follows, a conspicuous property of the Green's function may still be mentioned. This is, namely, that the non-homogeneous differential system

$$L(w, \lambda) = F(x),$$
$$A_1(w, \lambda) = 0,$$
$$A_2(w, \lambda) = 0,$$

which is obviously related to the system (13.10), is solved, whatever the function $F(x)$ may be, by the formula

$$w(x, \lambda) = \int_a^b G(x, s, \lambda)F(s)ds.$$

## CHAPTER 21

**The residues of the Green's function.** If the requirement that $\lambda$ be fixed is now abandoned and this parameter is, on the contrary, regarded henceforth as a variable that is free to range over the complex plane, it is recognizable from the formulas (20.3) and (20.5) that the functions $g_1(x, s, \lambda)$ and $g_2(x, s, \lambda)$ are analytic in $\lambda$. The same is also true of the function $\Delta(\lambda)$. The formula (20.4) therefore shows that the Green's function is analytic except for poles, these latter occurring at and only at the zeros of $\Delta(\lambda)$, namely at the characteristic values. It has been assumed as a part of our general hypothesis that each characteristic value is of a multiplicity equal to its index. As a consequence of that it will be found that for each $n$ the product $(\lambda-\lambda_n)G(x, s, \lambda)$ approaches a finite limit when $\lambda\to\lambda_n$. This limit is known as the *residue* of $G(x, s, \lambda)$ at $\lambda_n$. We shall use the prefix "$\mathrm{res}_n$" to designate it, thus

$$\mathrm{res}_n\, G(x, s, \lambda) = \lim_{\lambda\to\lambda_n} (\lambda - \lambda_n)G(x, s, \lambda), \tag{21.1}$$

and shall show that certain residues closely related to these are significantly associated with the terms of the series in a representation (17.6).

Since the function $g_1(x, s, \lambda)$ is analytic in $\lambda$ when $x$ and $s$ have any specified values, this part of $G(x, s, \lambda)$ evidently makes no contribution to the right-hand member of the relation (21.1). It is, therefore, only the part $g_2(x, s, \lambda)/\Delta(\lambda)$ that needs consideration. Upon expanding the determinant (20.5) by the elements of its last column, therefore, it is found that the relation (21.1) may be more explicitly written as

$$\mathrm{res}_n\, G(x, s, \lambda) = \lim_{\lambda\to\lambda_n} (\lambda - \lambda_n)\sum_{i=1}^{2} E_i(x, \lambda)\left[A_{3-i,1}^{(a)}(\lambda)\psi_1(s, \lambda) + A_{3-i,2}^{(a)}(\lambda)\psi_2(s, \lambda)\right],$$

the functions $E_i(x, \lambda)$ being defined by the formulas

$$E_i(x, \lambda) = \frac{(-1)^i}{\Delta(\lambda)}\begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ A_{i,1}(\lambda) & A_{i,2}(\lambda) \end{vmatrix}, \qquad i = 1, 2. \tag{21.2}$$

From the alternative form

$$\begin{aligned}&\mathrm{res}_n\, G(x, s, \lambda) \\ &\quad = \sum_{i=1}^{2} \left\{\mathrm{res}_n\, E_i(x, \lambda)\right\}\left[A_{3-i,1}^{(a)}(\lambda_n)\psi_1(s, \lambda_n) + A_{3-i,2}^{(a)}(\lambda_n)\psi_2(s, \lambda_n)\right],\end{aligned} \tag{21.3}$$

it thus follows that the evaluation of the residues of $G(x, s, \lambda)$ may be made to depend upon the determination of residues of the functions (21.2). To make these determinations we must consider separately the case in which the index of $\lambda_n$ is 1 and that in which it is 2. In doing that it will be convenient to employ the symbol $\delta_{i,j}$ to stand for the "Kronecker delta," namely in the sense

$$\delta_{i,j} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Suppose then, to begin with, that $\lambda_n$ is of the index 1. Then

$$\lim_{\lambda \to \lambda_n} \frac{\Delta(\lambda)}{\lambda - \lambda_n} = \dot{\Delta}(\lambda_n),$$

and there is a pair of subscripts $\sigma$, $\tau$, for which the relations (16.10) and (16.11) maintain. It will be clear that we may write

$$(21.4) \qquad (\lambda - \lambda_n)E_i(x, \lambda) = \frac{-(\lambda - \lambda_n)}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ A_{1,1}(\lambda) & A_{1,2}(\lambda) & \delta_{3-i,1} \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) & \delta_{3-i,2} \end{vmatrix}, \quad i = 1, 2,$$

and that therefore

$$(21.5) \qquad \operatorname{res}_n E_i(x, \lambda) = \frac{-1}{\dot{\Delta}(\lambda_n)} \begin{vmatrix} \phi_1(x, \lambda_n) & \phi_2(x, \lambda_n) & 0 \\ A_{1,1}(\lambda_n) & A_{1,2}(\lambda_n) & \delta_{3-i,1} \\ A_{2,1}(\lambda_n) & A_{2,2}(\lambda_n) & \delta_{3-i,2} \end{vmatrix}.$$

This formula may be considerably modified. It will be seen that if it is multiplied on the right by the determinant

$$\begin{vmatrix} h_2^{(n)} & \delta_{1,\tau} & 0 \\ h_1^{(n)} & \delta_{2,\tau} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

and this operation is then compensated for by dividing out the value of this determinant, $(-1)^\tau h_\tau^{(n)}$, the resulting form is, by virtue of the relations (14.9) and (14.10),

$$\frac{-1}{(-1)^\tau h_\tau^{(n)} \dot{\Delta}(\lambda_n)} \begin{vmatrix} u_n(x) & \phi_\tau(x, \lambda_n) & 0 \\ 0 & A_{1,\tau}(\lambda_n) & \delta_{3-i,1} \\ 0 & A_{2,\tau}(\lambda_n) & \delta_{3-i,2} \end{vmatrix}.$$

The further multiplication of this on the left by the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \delta_{1,\sigma} & \delta_{2,\sigma} \\ 0 & \mu_2^{(n)} & \mu_1^{(n)} \end{vmatrix},$$

and the appropriate subsequent division gives it the form

$$\frac{1}{(-1)^{\sigma+\tau}h_\tau^{(n)}\mu_\sigma^{(n)}\dot{\Delta}(\lambda_n)}\begin{vmatrix} u_n(x) & \phi_\tau(x, \lambda_n) & 0 \\ 0 & A_{\sigma,\tau}(\lambda_n) & \delta_{3-i,\sigma} \\ 0 & 0 & \mu_i^{(n)} \end{vmatrix},$$

by virtue of the relations (15.14). The expansion of this reduces it, because of the equation (16.12), to the evaluation

$$\text{(21.6)} \qquad \operatorname{res}_n E_i(x, \lambda) = \frac{u_n(x)\mu_i^{(n)}}{\Phi_{n,n}(\lambda_n)}, \qquad i = 1, 2,$$

and the substitution of these results into the equation (21.3), together with an application of the relations (15.16) and (15.15), leads to the conclusion that

$$\text{(21.7)} \qquad \operatorname{res}_n G(x, s, \lambda) = \frac{u_n(x)v_n(s)}{\Phi_{n,n}(\lambda_n)}.$$

Finally with the formulas (21.6) and (21.7) at hand it is a simple matter to recognize that the value of

$$\text{(21.8)} \qquad \operatorname{res}_n \left\{\int_a^b q(s)G(x, s, \lambda)f(s)ds + E_1(x, \lambda)f_2 + E_2(x, \lambda)f_1\right\},$$

is precisely the term

$$\frac{I_n(f)u_n(x)}{\Phi_{n,n}(\lambda_n)},$$

of the series (17.6).

If $\lambda_n$ is of the index 2 and appears in the array (14.7) as $\lambda_m$ and $\lambda_{m+1}$, the relation (21.4) is more appropriately taken in the form

$$(\lambda - \lambda_n)E_i(x, \lambda) = \frac{-\begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ \dfrac{A_{1,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{1,2}(\lambda)}{\lambda - \lambda_n} & \delta_{3-i,1} \\ \dfrac{A_{2,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{2,2}(\lambda)}{\lambda - \lambda_n} & \delta_{3-i,2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \dfrac{A_{1,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{1,2}(\lambda)}{\lambda - \lambda_n} & 0 \\ \dfrac{A_{2,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{2,2}(\lambda)}{\lambda - \lambda_n} & 0 \end{vmatrix}}.$$

Since in this instance each element $A_{i,j}(\lambda)$ vanishes at $\lambda_n$, it is seen that

$$\lim_{\lambda \to \lambda_n} \frac{A_{i,j}(\lambda)}{\lambda - \lambda_n} = \dot{A}_{i,j}(\lambda_n),$$

and hence that

$$\operatorname{res}_n E_i(x, \lambda) = \frac{-\begin{vmatrix} \phi_1(x, \lambda_n) & \phi_2(x, \lambda_n) & 0 \\ \dot{A}_{1,1}(\lambda_n) & \dot{A}_{1,2}(\lambda_n) & \delta_{3-i,1} \\ \dot{A}_{2,1}(\lambda_n) & \dot{A}_{2,2}(\lambda_n) & \delta_{3-i,2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \dot{A}_{1,1}(\lambda_n) & \dot{A}_{1,2}(\lambda_n) & 0 \\ \dot{A}_{2,1}(\lambda_n) & \dot{A}_{2,2}(\lambda_n) & 0 \end{vmatrix}}.$$

Let each one of the determinants in this ratio be multiplied on the right by

$$\begin{vmatrix} h_2^{(m)} & h_2^{(m+1)} & 0 \\ h_1^{(m)} & h_1^{(m+1)} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Because of the relations (14.10) and (16.6) it then takes the form

$$\frac{-\begin{vmatrix} u_m(x) & u_{m+1}(x) & 0 \\ \dot{A}_1(U_m, \lambda_n) & \dot{A}_1(U_{m+1}, \lambda_n) & \delta_{3-i,1} \\ \dot{A}_2(U_m, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n) & \delta_{3-i,2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \dot{A}_1(U_m, \lambda_n) & \dot{A}_1(U_{m+1}, \lambda_n) & 0 \\ \dot{A}_2(U_m, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n) & 0 \end{vmatrix}},$$

and if this is again modified by multiplying each of the determinants on the left by the factor

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \mu_2^{(m+1)} & \mu_1^{(m+1)} \\ 0 & \mu_2^{(m)} & \mu_1^{(m)} \end{vmatrix},$$

the result, as a consequence of the relations (16.9) and (16.13), is

$$\frac{-\begin{vmatrix} u_m(x) & u_{m+1}(x) & 0 \\ 0 & \Phi_{m+1,m+1}(\lambda_n) & \mu_i^{(m+1)} \\ \Phi_{m,m}(\lambda_n) & 0 & \mu_i^{(m)} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & \Phi_{m+1,m+1}(\lambda_n) & 0 \\ \Phi_{m,m}(\lambda_n) & 0 & 0 \end{vmatrix}}.$$

Upon expansion this yields the formula

$$(21.9)\qquad \operatorname{res}_n E_i(x, \lambda) = \frac{u_m(x)\mu_i^{(m)}}{\Phi_{m,m}(\lambda_n)} + \frac{u_{m+1}(x)\mu_i^{(m+1)}}{\Phi_{m+1,m+1}(\lambda_n)}, \qquad i = 1, 2,$$

in accordance with which the relation (21.4) assumes the explicit form

$$(21.10)\qquad \operatorname{res}_n G(x, s, \lambda) = \frac{u_m(x)v_m(s)}{\Phi_{m,m}(\lambda_m)} + \frac{u_{m+1}(x)v_{m+1}(s)}{\Phi_{m+1,m+1}(\lambda_{m+1})}.$$

Thus when $\lambda_n$ is of the index 2 the residue (21.8) is the sum

$$\frac{I_m(f)u_m(x)}{\Phi_{m,m}(\lambda_m)} + \frac{I_{m+1}(f)u_{m+1}(x)}{\Phi_{m+1,m+1}(\lambda_{m+1})},$$

of the pair of terms which appear in the series (17.6) in association with $\lambda_n$.

From these conclusions it may now be seen that if the points representing the characteristic values are plotted in the complex plane, and if the first $(N+1)$ of them taken in the order of increasing distance from the origin are enclosed by a curve $C_N$, the segment $S_N(x)$ of the series (17.6) that is made up of terms that are associated with these characteristic values is given by the formula

$$S_N(x) = \sum_{n=0}^{N} \operatorname{res}_n \left\{\int_a^b q(s)G(x, s, \lambda)f(s)ds + E_1(x, \lambda)f_2 + E_2(x, \lambda)f_1\right\}.$$

Such a sum of residues is, however, familiarly expressible as a contour integral with respect to $\lambda$ over the curve $C_N$, namely as

$$(21.11)\quad S_N(x) = \frac{1}{2\pi i}\int_{C_N} \left\{\int_a^b q(s)G(x, s, \lambda)f(s)ds + E_1(x, \lambda)f_2 + E_2(x, \lambda)f_1\right\} d\lambda.$$

This is the formula in the general theory that stands in the place of the Fourier formula (12.8)

## CHAPTER 22

**The Fourier's series again.** The formula (21.11) is useful, like its specialized counterpart (12.8), as the natural medium through which an investigation of the associated representations of arbitrary functions may be made. As the contour of integration $C_N$ is successively taken to include more and more characteristic values the formula sums more and more terms of the series, and the approach of the integral to a limit reflects the convergence of the series. An identification of the function (if any) which the series represents thus becomes accessible through a study of the formula's integrand, more particularly through an analysis of the Green's function and the functions (21.2) when $|\lambda|$ is large.

The exposition of the complete analysis that would be requisite for the general case would at this point go well beyond the bounds which have been set for the scope of this discussion. The character of the functions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ as these depend upon $\lambda$ would need to be determined. In general, however, no explicit formulas for these functions are available, since the differential equation (13.3) is not ordinarily solvable. The difficulties which this circumstance interposes will be manifest, although, to be sure, they are not insurmountable. So-called asymptotic forms for the solutions of an equation of the type (13.3) are deducible by well established methods, and these are entirely adequate to the requirements. This theory of asymptotic solutions, however, we do not propose to go into here. Without it, a restriction of the discussion to considerably narrower confines than have hitherto been observed is in the nature of things inevitable. We shall yield to the necessity by limiting the exposition henceforward to the basis of an outright, albeit a wholly typical, specialization. In fact, therefore, the further investigation is to take the form of an extended analysis of the example 3 of chapter 18. It will be recalled that the theory of the boundary problem basic to that example is none other than the theory of Fourier's series.

For the boundary problem (18.5), and with the choice of constants $f_1 = 0$, $f_2 = 0$, the special form of the formula (21.11) to which the attention is to be given is

$$S_N(x) = \frac{1}{2\pi i}\int_{C_N}\int_{-\pi}^{\pi} G(x, s, \lambda) f(s)\, ds\, d\lambda. \tag{22.1}$$

The functions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ may be chosen as those that are given by the relations (18.6), and these lead through the formulas (15.11) and (15.5) to the evaluations

$$A_{1,1}^{(a)}(\lambda) = \cos \rho\pi, \qquad A_{1,2}^{(a)}(\lambda) = -\frac{\sin \rho\pi}{\rho},$$

$$A_{2,1}^{(a)}(\lambda) = \rho \sin \rho\pi, \qquad A_{2,2}^{(a)}(\lambda) = \cos \rho\pi,$$

and

$$\psi_1(s, \lambda) \equiv -\frac{1}{\rho}\sin \rho s, \qquad \psi_2(s, \lambda) \equiv \cos \rho s.$$

The formula (20.5) is thus explicitly

$$g_2(x, s, \lambda) = -\begin{vmatrix} \cos \rho x & \dfrac{1}{\rho}\sin \rho x & 0 \\ 0 & -\dfrac{2}{\rho}\sin \rho\pi & -\dfrac{1}{\rho}\sin \rho(\pi + s) \\ 2\rho \sin \rho\pi & 0 & \cos \rho(\pi + s) \end{vmatrix},$$

and this with the relations (20.3) and (20.4) yields the form

$$(22.2)\quad G(x, s, \lambda) = \frac{\cos \rho(\pi + s - x)}{2\rho \sin \rho\pi} - \begin{cases} \dfrac{\sin \rho(x - s)}{\rho}, & \text{when } -\pi \leqq x \leqq s, \\ 0, & \text{when } s \leqq x \leqq \pi. \end{cases}$$

In the complex $\lambda$-plane the characteristic values are located at the points $n^2$, with $n=0, 1, 2, 3, \cdots$. The contour $C_N$, which must enclose precisely the first $(N+1)$ of these, may, therefore, be chosen as the circle centered at the origin and of the radius $(N+\frac{1}{2})^2$. In the $\rho$-plane the semi-circle $\Gamma_N$ which is centered at $\rho=0$, which has the radius $(N+\frac{1}{2})$, and upon which $0 \leqq \arg \rho < \pi$, is an image of the circle $C_N$ under the mapping defined by the relation (18.1). The formula (22.1) with its integration expressed in terms of $\rho$ is, therefore,

$$(22.3)\qquad S_N(x) = \frac{1}{\pi i} \int_{\Gamma_N} \int_{-\pi}^{\pi} \rho G(x, s, \lambda) f(s) ds d\rho.$$

When the point $x$ at which the sum $S_N(x)$ is to be considered lies in the interior of the interval $(-\pi, \pi)$, it is advantageous to take the formula (22.2) in the equivalent form

$$(22.4)\qquad G(x, s, \lambda) = \frac{-i}{2\rho} e^{|s-x|\rho i} + \frac{e^{\pi\rho i} \cos \rho(s - x)}{2\rho \sin \rho\pi},$$

the symbol $|s-x|$ standing as usual for $(s-x)$ or $(x-s)$ according as $x \leqq s$ or $x > s$. The formula (22.3) then assumes the form

$$(22.5)\qquad S_N(x) = \frac{-1}{2\pi} \int_{\Gamma_N} \int_{-\pi}^{\pi} e^{|s-x|\rho i} f(s) ds d\rho + \int_{-\pi}^{\pi} \Psi(s, x, N) f(s) ds,$$

with the function $\Psi(s, x, N)$ defined by the formula

$$(22.6)\qquad \Psi(s, x, N) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{\pi\rho i} \cos \rho(s - x)}{\sin \rho\pi} d\rho.$$

Since this latter function is explicitly integrable as to $s$, it is found at once that for any choice of $(\alpha, \beta)$ as a sub-interval of the range $(-\pi, \pi)$

$$(22.7)\qquad \begin{aligned} \int_{\alpha}^{\beta} \Psi(s, x, N) ds = {} & \frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{\pi\rho i} \sin \rho(\beta - x)}{\rho \sin \rho\pi} d\rho \\ & - \frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{\pi\rho i} \sin \rho(\alpha - x)}{\rho \sin \rho\pi} d\rho. \end{aligned}$$

The integrals which appear in this formula closely resemble that in the formula (22.6) since the sine and cosine functions are effectively similar in structure. The analysis of the one, which is to be set forth, will therefore be found obvi-

ously and readily adaptable to the others also. In the event that the point $x$ is an end point of the interval $(-\pi, \pi)$ a somewhat different grouping of terms in the formulas (22.4) and (22.5) is advantageous. The analysis is, however, essentially similar to that which applies when $-\pi<x<\pi$, and that being so we shall content ourselves with the discussion of this latter case alone.

When it is expressed entirely in terms of exponentials the formula (22.6) may be made to appear in the form

$$\Psi(s, x, N) = \frac{-1}{2}\int_{\Gamma_N} \frac{e^{[2\pi+(s-x)]\rho i} + e^{[2\pi-(s-x)]\rho i}}{\pi(1 - e^{2\pi\rho i})}\, d\rho. \tag{22.8}$$

Even relatively crude appraisals of the functions which appear in the integrand of this yield some significant results. Under the resolution $\rho=\xi+i\eta$, with $\xi$ and $\eta$ real, the equation of the arc $\Gamma_N$ is

$$\xi^2 + \eta^2 = (N + \tfrac{1}{2})^2, \qquad \eta \geqq 0,$$

and from this it is readily seen that each one of its points is either one for which

$$N + \tfrac{1}{4} \leqq |\xi| \leqq N + \tfrac{1}{2},$$

or else one for which

$$\eta > \tfrac{1}{4}\sqrt{3}.$$

At every point of the first one of these categories $\cos 2\pi\xi<0$, and the value $|1-e^{2\pi\rho i}|$ which is explicitly

$$\left\{1 - e^{-2\pi\eta}\cos 2\pi\xi + e^{-4\pi\eta}\right\}^{1/2}, \tag{22.9}$$

thus clearly exceeds 1. At any point of the second category the value (22.9) exceeds $(1-e^{-\pi\sqrt{3}/4})$, and is thus *ipso facto* greater than $1/\pi$. The relation

$$\pi\left|1 - e^{2\pi\rho i}\right| > 1, \tag{22.10}$$

thus maintains over the whole arc $\Gamma_N$.

Consider now the values

$$\left|e^{[2\pi\pm(s-x)]\rho i}\right|, \tag{22.11}$$

with $s$ and $x$ both upon the interval $(-\pi, \pi)$ and $\rho$ still upon $\Gamma_N$. For all such $s$ and $x$ the relation

$$2\pi \pm (s - x) \geqq \pi - |x|,$$

is easily verified, and since the formula $\rho=(N+\frac{1}{2})e^{i\theta}$, in which $\theta$ stands for $\arg\rho$, shows that the real part of $\rho i$ is $-(N+\frac{1}{2})\sin\theta$, it is seen that the values (22.11) are both less than

$$e^{-[\pi-|x|](N+\frac{1}{2})\sin\theta}$$

Since this latter value is increased by the substitution of $\frac{1}{2}\theta$ for $\sin\theta$ when $0 \leqq \theta \leqq \pi/2$ and of $\frac{1}{2}(\pi-\theta)$, for $\sin\theta$ when $\pi/2 \leqq \theta \leqq \pi$, we may evidently draw from the formula (22.8) the relation

$$| \Psi(s, x, N) | < \int_0^{\pi/2} e^{-[\pi-|x|](N+\frac{1}{2})\frac{1}{2}\theta}(N+\tfrac{1}{2})d\theta$$
$$+ \int_{\pi/2}^{\pi} e^{-[\pi-|x|](N+\frac{1}{2})\frac{1}{2}(\pi-\theta)}(N+\tfrac{1}{2})d\theta.$$

By explicit integrations this reduces to the inequality

$$| \Psi(s, x, N) | < \frac{4}{\pi - |x|}. \tag{22.12}$$

The companion result

$$\left| \int_\alpha^\beta \Psi(s, x, N) ds \right| < \frac{8}{(\pi - |x|)(N+\frac{1}{2})}, \tag{22.13}$$

may be drawn by similar reasoning from the formula (22.7).

These conclusions are significant. In accordance with them the function $\Psi(s, x, N)$ is uniformly bounded as to $s$ and $N$, and is such that its integral as to $s$ converges to zero with $1/N$ uniformly as to the interval of integration. These properties, however, are precisely those which were invoked in chapter 12 as being sufficient to insure the relation (12.10) for an arbitrary integrable function $F(s)$. In the present instance, therefore, we may similarly conclude upon the basis of them that the final integral in the equation (22.5) converges to zero. Since the remaining integral in that equation may be evaluated thus

$$-\frac{1}{2\pi}\int_{\Gamma_N}\int_{-\pi}^{\pi} e^{|s-x|\rho i} f(s)ds d\rho = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\frac{\sin(N+\frac{1}{2})(s-x)}{s-x}ds,$$

it is clear that the equation (22.5) implies for every integrable function $f(x)$ the relation

$$\lim_{N\to\infty}\left[S_N(x) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\frac{\sin(N+\frac{1}{2})(s-x)}{s-x}ds\right] = 0.$$

This is the relation (12.11) already familiar, and through it the reference of the sum $S_N(x)$ to the Dirichlet integral has evidently again been accomplished. The method of its accomplishment here, however, by contrast with that of chapter 12, is one of very general applicability. In particular it is one which in no way depends upon the special trigonometric combination formulas. With adaptations which essentially concern only details, this method is adequate to the analysis of the general series (17.6), namely to the representations of arbitrary functions in terms of the characteristic solutions of any properly constructed boundary problem [20].

## REFERENCES

1. Van Vleck, E. B., The influence of Fourier's Series upon the development of Mathematics. Science, vol. 39, 1914, pp. 113–124.

2. Burkhardt, H., Entwicklungen nach oscillierenden Functionen und Integration der Differentialgleichungen der mathematischen Physik. Jahresbericht der Deutschen Mathematiker= Vereinigung. Leipzig. 1908, p. 25 and p. 3.

3. Burkhardt, H., *loc. cit.*, pp. 1–5.

4. Burkhardt, H., *loc. cit.*, p. 8.

5. Burkhardt, H., *loc. cit.*, pp. 10–13.

Cantor, M., Vorlesungen über Geschichte der Mathematik, vol. 3 (2nd ed.), Leipzig, p. 384 and pp. 900–902.

6. Riemann, B., Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe. Gesammelte Werke, Leipzig, 1892, pp. 227–231.

Burkhardt, H., *loc. cit.*, pp. 14–24.

Van Vleck, E. B., *loc. cit.*, pp. 114, 115.

Gibson, G. A. On the History of the Fourier Series. Proceedings of the Edinburgh Mathematical Society, vol. XI, 1893, pp. 139–142.

7. Burkhardt, H., *loc. cit.*, pp. 29–31.

8. Burkhardt, H., *loc. cit.*, pp. 27–43.

Riemann, B., *loc. cit.*, pp. 233–234.

9. Burkhardt, H., *loc. cit.*, pp. 70–71.

10. Fourier, Joseph, The Analytical Theory of Heat (trans. by A. Freeman), Cambridge, 1878, pp. 104–115.

11. Fourier, *loc. cit.*, p. 168.

12. Fourier, *loc cit.*, p. 184.

13. Ford, L. R., Differential Equations, New York, 1933, p. 75.

14. Hobson, E. W., The Theory of Functions of a Real Variable, vol. II, 2nd ed. Cambridge, 1926, pp. 422, 423.

15. Hobson, *loc. cit.*, p. 482. Knopp, K. Theorie und Anwendungen der Unendlichen Reihen, Berlin, 1922, p. 337.

16. Ford, *loc. cit.*, pp. 66–68.

17. Ince, E. L., Ordinary Differential Equations, London, 1927, p. 73.

18. Langer, R. E. A Problem in Diffusion or in the Flow of Heat for a Solid in Contact with a Fluid. The Tôhoku Mathematical Journal, vol. 35, 1932, pp. 260–275.

19. A more detailed discussion of these problems, and references to the literature are to be found in the paper cited as reference [18].

20. The reference [18] includes a proof of the convergence of the representations of functions in terms of the solutions of the boundary problem (19.1). Both interior and end points of the interval are there considered.

## APPENDICES

### I. The solution of the system of equations

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} + qu_k + u_{k-1} &= 0, \qquad k = 1, 2, \cdots, (n-1), \\ u_n &= 0. \end{aligned} \tag{I.1}$$

If, in terms of unspecified constants $\alpha$ and $\beta$, a solution of the system is sought in the form

$$u_k = \alpha^k - \beta^k, \tag{I.2}$$

it is found upon substitution into the equations to be requisite that

$$\alpha^{k-1}[\alpha^2 + q\alpha + 1] - \beta^{k-1}[\beta^2 + q\beta + 1] = 0, \qquad k = 1, 2, \cdots, (n-1). \tag{I.3}$$

These conditions are fulfilled if $\alpha$ and $\beta$ are roots of the equation

$$x^2 + qx + 1 = 0,$$

namely if

$$\alpha + \beta = -q, \qquad \alpha\beta = 1. \tag{I.4}$$

Since $\beta$ must thus be the reciprocal of $\alpha$, whereas the form (I.2) must vanish when $k = n$, it is seen to be necessary that

$$\alpha^n - \alpha^{-n} = 0,$$

namely that

$$\alpha = e^{\nu\pi i/n}, \qquad \beta = e^{-\nu\pi i/n},$$

with an integral value of $\nu$. For such an index $\nu$ the relations (I.4) and (I.2) show that $q$ and $u_k$ have the values

$$q_\nu = -(e^{\nu\pi i/n} + e^{-\nu\pi i/n}) = -2\cos\frac{\nu\pi}{n}, \tag{I.5}$$

and

$$u_{\nu,k} = (e^{k\nu\pi i/n} - e^{-k\nu\pi i/n}) = 2i\sin\frac{k\nu\pi}{n},$$

respectively. Since the system under consideration is homogeneous, any multiple of a solution is also such. Hence we may write

$$u_{\nu,k} = A_\nu \sin\frac{k\nu\pi}{n}, \qquad k = 0, 1, 2, \cdots, n, \tag{I.6}$$

with the coefficient $A_\nu$ arbitrary.

For the indices $\nu=1, 2, \cdots, (n-1)$, the characteristic values of $q$, as given by the formula (I.5), are distinct. There can be no others, for by inspection the determinant of the system of equations is a polynomial of the degree $(n-1)$ in $q$.

**II. A proof of the relation**

$$\sin \nu\xi = \sin \xi \cdot p_{\nu-1}(\cos \xi), \tag{II.1}$$

**with $p_{\nu-1}$ a polynomial of the degree $(\nu-1)$.**

If in Demoivre's formula

$$(\cos \nu\xi + i \sin \nu\xi) = (\cos \xi + i \sin \xi)^\nu,$$

the right-hand member is expanded by the binomial theorem, each resulting term that involves $i$ to an odd power also involves $\sin \xi$ to such a power. Upon equating the pure imaginary components on the two sides of the equation it is thus found that

$$\sin \nu\xi = \sin \xi \cdot Q(\cos \xi, \sin^2 \xi),$$

each term of $Q$ being of the degree $(\nu-1)$ in $\cos \xi$ and $\sin \xi$, and of even degree in $\sin \xi$. By the substitution of $(1-\cos^2\xi)$ for $\sin^2\xi$ the form (II.1) results.

**III. A deduction of the identity**

$$\cos^j x = \frac{1}{2^{j-1}} \sum_{\mu=0}^{[j/2]} \binom{j}{\mu} \cos (j - 2\mu)x. \tag{III.1}$$

If in the familiar equality

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \tag{III.2}$$

each member is raised to the $j$th power and the one on the right is then expanded by the binomial theorem, the result is the relation

$$\cos^j x = \frac{1}{2^j} \sum_{\mu=0}^{j} \binom{j}{\mu} e^{(j-2\mu)ix}, \tag{III.3}$$

in which the symbol $\binom{j}{\mu}$ designates the coefficient of $a^\mu$ in the expansion of $(1+a)^j$. Since $\binom{j}{j-\mu}=\binom{j}{\mu}$, the formula (III.3) may be written alternatively as

$$\cos^j x = \frac{1}{2^j} \sum_{\mu=0}^{[j/2]} \binom{j}{\mu} \left\{ e^{(j-2\mu)ix} + e^{-(j-2\mu)ix} \right\},$$

with $[j/2]$ denoting the largest integer not exceeding $j/2$. By invoking the relation (III.2) again this may be given the form (III.1).

## IV. A derivation of the evaluations

$$\text{(IV.1)} \qquad \sum_{\mu=1}^{n-1} \cos\frac{\mu s\pi}{n} = \begin{cases} n - \frac{1}{2}(1 + \cos s\pi), & \text{if} \quad s \equiv 0 \pmod{2n}, \\ \quad - \frac{1}{2}(1 + \cos s\pi), & \text{if} \quad s \not\equiv 0 \pmod{2n}. \end{cases}$$

The first one of these evaluations is obvious. For when $s \equiv 0 \pmod{2n}$ each term in the sum on the left has the value 1 while the right-hand member is $(n-1)$. If $s \not\equiv 0$ (mod. $2n$) the relation

$$\cos\frac{\mu s\pi}{n} = \frac{1}{2}\left(e^{\mu s\pi i/n} + e^{-\mu s\pi i/n}\right)$$

leads at once to the equation

$$\sum_{\mu=1}^{n-1} \cos\frac{\mu s\pi}{n} = \frac{1}{2} \sum_{\mu=-n+1,\mu\neq 0}^{n-1} e^{\mu s\pi i/n},$$

or, as it may equally well be written, to

$$\text{(IV.2)} \qquad \sum_{\mu=1}^{n-1} \cos\frac{\mu s\pi}{n} = \frac{1}{2}\left[-1 + \sum_{\mu=-n+1}^{n-1} \left(e^{s\pi i/n}\right)^{\mu}\right].$$

The sum in the right-hand member of this is a geometric progression. It is summed, therefore, by the formula

$$\sum_{\mu=-n+1}^{n-1} \left(e^{s\pi i/n}\right)^{\mu} = \frac{\left(e^{s\pi i/n}\right)^{-n+1} - \left(e^{s\pi i/n}\right)^{n}}{1 - \left(e^{s\pi i/n}\right)},$$

and since

$$\left(e^{s\pi i/n}\right)^{\pm n} = (-1)^{s} = \cos s\pi,$$

its value reduces to $-\cos s\pi$. The formula (IV.2) thus takes on the form of the second evaluation (IV.1).

## V. An evaluation of the determinant

$$\text{(V.1)} \qquad D(x) \equiv \begin{vmatrix} 1 & 2 & \cdots (\nu-1) & x \\ 1^3 & 2^3 & \cdots (\nu-1)^3 & x^3 \\ 1^5 & 2^5 & \cdots (\nu-1)^5 & x^5 \\ \cdots & \cdots & \cdots & \cdots \\ 1^{2\nu-1} & 2^{2\nu-1} & \cdots (\nu-1)^{2\nu-1} & x^{2\nu-1} \end{vmatrix}.$$

If in this determinant the elements of each row, beginning with the last one and extending in turn back to the second one, are modified by subtracting from them $x^2$ times the corresponding elements of the preceding row, the value of the determinant is unchanged. In its new form, however, all but the first ele-

ment in the last column are zeros, and hence, upon an expansion by the elements of this column, it is found that

$$D(x) = (-1)^{\nu-1}x \begin{vmatrix} 1(1^2 - x^2) & 2(2^2 - x^2) & \cdots & [\nu - 1]([\nu - 1]^2 - x^2) \\ 1^3(1^2 - x^2) & 2^3(2^2 - x^2) & \cdots & [\nu - 1]^3([\nu - 1]^2 - x^2) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ 1^{2\nu-3}(1^2 - x^2) & 2^{2\nu-3}(2^2 - x^2) & \cdots & [\nu - 1]^{2\nu-3}([\nu - 1]^2 - x^2) \end{vmatrix}.$$

As it now stands, the factor $(n^2-x^2)$ is common to the elements of the $n$th column for $n=1, 2, \cdots, (\nu-1)$. Upon factoring these from the determinant the evaluation

$$D(x) = (-1)^{\nu-1}x\left[\prod_{n=1}^{\nu-1}(n^2 - x^2)\right]D_{\nu,\nu}, \tag{V.2}$$

is obtained, $D_{\nu,\nu}$ designating the cofactor of the element in the $\nu$th row and $\nu$th column of the original form of $D(x)$.

**VI. A formal deduction of the relation**

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x}. \tag{VI.1}$$

The roots of the equation $z^{2k}=1$ are obviously $z=\pm 1$ and $z=e^{\pm n\pi i/k}$, for $n=1, 2, \cdots, (k-1)$. The factorization

$$z^{2k} - 1 = (z^2 - 1)\prod_{n=1}^{k-1}\left\{(z - e^{n\pi i/k})(z - e^{-n\pi i/k})\right\},$$

is, therefore, proper. It is, however, clearly equivalent to

$$z^{2k} - 1 = (z^2 - 1)\prod_{n=1}^{k-1}\left(z^2 + 1 - 2z\cos\frac{n\pi}{k}\right),$$

a relation from which it follows that

$$\frac{z^k - z^{-k}}{2i} = \left(\frac{z - z^{-1}}{2i}\right)\prod_{n=1}^{k-1}\left(z + z^{-1} - 2\cos\frac{n\pi}{k}\right).$$

Now if in this the quantity $e^{i\pi x/k}$ is substituted for $z$, the equation becomes

$$\sin \pi x = \sin\frac{\pi x}{k}\prod_{n=1}^{k-1}\left[2\cos\frac{\pi x}{k} - 2\cos\frac{\pi n}{k}\right],$$

or, since

$$\cos\frac{\pi x}{k} - \cos\frac{\pi n}{k} = 2\left(\sin^2\frac{\pi n}{2k} - \sin^2\frac{\pi x}{2k}\right),$$

$$\text{(VI.2)}\qquad \frac{\sin \pi x}{\pi x} = \frac{\sin\left(\dfrac{\pi x}{k}\right)}{\pi x}\left\{2^{2k-2}\prod_{n=1}^{k-1}\sin^2\frac{\pi n}{2k}\right\}\prod_{n=1}^{k-1}\left\{1-\left[\frac{\sin\dfrac{\pi x}{2k}}{\sin\dfrac{\pi n}{2k}}\right]^2\right\}.$$

This is an identity. Its limiting form as $x \to 0$ must, therefore, maintain, namely

$$1 = \frac{1}{k}\left\{2^{2k-2}\prod_{n=1}^{k-1}\sin^2\frac{\pi n}{2k}\right\},$$

an evaluation by virtue of which the relation (VI.2) may itself be reduced to the form

$$\frac{\sin \pi x}{\pi x} = \frac{\sin\left(\dfrac{\pi x}{k}\right)}{\left(\dfrac{\pi x}{k}\right)}\prod_{n=1}^{k-1}\left\{1-\left[\frac{\sin\dfrac{\pi x}{2k}}{\sin\dfrac{\pi n}{2k}}\right]^2\right\}.$$

This is valid for all $k$, and we may, therefore, permit this index to become infinite. The evaluation (VI.1) formally results.

**VII. Establishment of the formula**

$$\text{(VII.1)}\qquad 1+\sum_{\nu=1}^{n} 2\cos \nu\theta = \frac{\sin\left[\dfrac{2n+1}{2}\theta\right]}{\sin\left[\tfrac{1}{2}\theta\right]},\quad \text{for}\quad \theta \not\equiv 0 \pmod{2\pi}.$$

Upon substitution of the relations

$$2\cos\nu\theta = e^{\nu\theta i} + e^{-\nu\theta i},$$

into the left-hand member of the formula (VII.1), this latter is found to be expressible as

$$\sum_{\nu=-n}^{n} e^{\nu\theta i}.$$

This is a geometric progression whose sum, if $\theta \not\equiv 0 \pmod{2\pi}$ is

$$\frac{e^{-n\theta i} - e^{(n+1)\theta i}}{1-e^{\theta i}}.$$

If in this the numerator and the denominator are each divided by the factor $-2ie^{\theta i/2}$, the fraction assumes the form of the right-hand member of the formula (VII.1). This latter is, therefore, established.

THE MATHEMATICAL ASSOCIATION OF AMERICA

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 8


CONTENTS




OCTOBER 1947

# The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)



EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions and advertising in the MONTHLY, should be sent to the Secretary-Treasurer, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 20th of the month preceding that in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# A DISTINGUISHED CONTRIBUTOR TO THE MONTHLY

The name, Victor Thébault, is familiar to every reader of this MONTHLY. As a prolific and ingenious contributor to its pages, he has been the subject of many inquiries sent by readers to members of the editorial staff. This tripartite tribute to Monsieur Thébault has been written by three men who have observed his work through the years with great interest.—EDITOR.

## VICTOR THÉBAULT—THE MAN

COL. W. E. BYRNE, Virginia Military Institute

In 1932, Victor Thébault became a member of the Mathematical Association of America. It was not long before his name appeared frequently among the contributors to the problem department. Since then many readers of this MONTHLY have wondered about the author of so many interesting problems and discussions.

Victor Michel Jean-Marie Thébault was born on March 6, 1882, at Ambrières-le-Grand (Mayenne), France. His father was a weaver. At the local primary school his teacher noted his native ability and took steps to obtain for him a scholarship at the teacher's college in Laval (Mayenne), where he remained as a student from 1898 to 1901. After graduation he became a school teacher at Pré-en-Pail (Mayenne) (1902–1905) until he was called to be a professor at the technical school of Ernée (Mayenne). In 1909, he received his certificate of capacity for a professorship (scientific) in teachers' colleges; this resulted from winning first place in competitive examination.

The modest salary of a professor was not sufficient to care for his family, now blessed with six children. Therefore Monsieur Thébault reluctantly gave up teaching to become a factory superintendent at Ernée (1910–23). This was followed by another change, to the position of Chief Insurance Inspector at Le Mans (Sarthe) (1924–40). In 1940 he retired to live at "Le Paradis" in Tennie (Sarthe).

That the farewell to teaching of mathematics did not mean any loss of interest in that field is attested by the honor conferred upon him by the French government. In 1932 he was made an Officier de l'Instruction Publique upon the recommendation of M. d'Ocagne, Member of the Institute, with the citation: "Personally I hold him in high esteem for his outstanding talent as a mathematician as shown by the numerous ingenious contributions to what is called elementary geometry, an unending source of problems whose solution requires a quite special gift of invention."

In 1935, he was made a Chevalier de l'Ordre de la Couronne of Belgium because of his activity in connection with the Scientific Society of Bruxelles, its *Annales*, and *Mathesis*.

Not only has Monsieur Thébault continued his publications (which include 15 communications to the Paris Académie des Sciences, hundreds of memoirs and articles concerning the modern geometry of the triangle and tetrahedron and the theory of numbers, and more than 1000 original problems), but also he

has contributed articles to the new French *Intermédiaire des Recherches Mathématiques*, and put his own library and services at the disposition of other mathematicians. Quite typical of the man is a notice appearing in the June, 1946, *Mathesis*: "M. V. Thébault, desirous of giving a new proof of his interest in *Mathesis*, offers to foreign subscribers the means of facilitating currency operations by accepting payment of their subscriptions at his postal checking account 339-03, Rennes, France." Moreover he established in 1943 the Prix Victor Thébault to be awarded every two years by the Paris Académie des Sciences to the author, preferably a teacher of the primary or secondary system, of an original study or of an interesting work on geometry or number theory.

Those who have had an occasion, as I have had, to correspond with Monsieur Thébault, have been greatly impressed by his charm and tact as well as by his keen interest in mathematics.

## THÉBAULT—THE NUMBER THEORIST

E. P. STARKE, Rutgers University

M. Thébault's prolific output of theorems and problems about numbers is a source of constant admiration, little short of wonder, on the part of all those upon whom the higher arithmetic exerts its fascination. Since 1934 this MONTHLY and the *National Mathematics Magazine* have published upwards of a hundred thirty of these little gems. Few tools are needed to handle these problems beyond the simplest ideas of scales of notation, congruences, and occasionally the solution of a Pell equation, but there is also required a certain degree of analytical skill and considerable insight into numerical relations.

The character of a large majority of these problems may be judged from a few samples.

Determine the largest and smallest perfect squares which can be written with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, used once each in both cases. (E404. 1940, 48.)

In the scale of $B$ there is a perfect square of sixteen digits of the form $abcdefghabcdefgh$. What is the smallest possible value of $B$? (E442. 1940, 656.)

Find three positive integers $x$, $y$, $z$, such that the sum of the ratios $y/z+z/x+x/y$ is an integer. (E682. 1945, 395.)

In what system of numbers with the base less than 10,000 are there the greatest number of squares of four digits of the form $aabb=(cc)^2$? (4184. 1945, 582.)

At first sight it might seem that so many problems of the same general nature could not all be interesting. On the other hand, however, the fascination increases and the little differences become as exciting to the student of the higher arithmetic as the next cross-word puzzle can be to the cross-word puzzle devotee.

No apology is needed for the expenditure of time and effort on these interesting little items. However, it seems appropriate to quote M. Thébault's own modest statement of his feelings on this point (*Mathésis* 1943, supplement).

"Some mathematicians exhibit a tendency, not altogether free from a certain disdain, to see in such problems only insignificant trifles. Trifles, if you please, but the solution of which often demands no less penetration of mind, ingenuity, and subtle artifice than many questions of allegedly deeper significance. Moreover the study of an elementary proposition sometimes necessitates an effort which is far from negligible, which constitutes an excellent intellectual exercise, and which leads to something worthwhile indeed.

"It is a fact that not all the great masters of the science have professed similar disdain for mathematical recreations. [Here M. Thébault gives quotations from Euler, Jacobi, and others.]

"This discipline by which we personally have been inspired has incited us to publish in a number of mathematical reviews a great many arithmetical curiosities in the form of notes or proposed problems which invited the reader to pause there with interest if not with delight. We have had the satisfaction of hearing it said that this modest contribution to the life of these journals, and to that extent to the mathematical growth of their readers, has been one of some service."

## THÉBAULT—THE GEOMETER

N. A. Court, University of Oklahoma

When Victor Thébault appeared on the mathematical scene during the first decade of the present century, the geometry of the triangle had already left behind its "period of mushroom growth." It was a well established and extensive body of doctrine which commanded the loyalty of a considerable number of enthusiastic workers.

During the ensuing years the keen imagination of M. Thébault found in this domain a fertile field for the development of his powers. Practically every part of this discipline has attracted his attention, from the most modest to the most abstruse. He dealt with equilateral triangles as well as with conics and cubics associated with the triangle, with quadrilaterals and polygons, with the various remarkable points of the triangle, with the orthopole, the isopole, and so on.

An eloquent comment upon the quality and originality that characterized Thébault's work from the very start is the fact that J. L. Coolidge included in his renowned and much quoted *Treatise on the Circle and the Sphere* (1916) a proposition of Thébault (p. 113) that came to his notice after the corresponding chapter of the book had already been completed.

Some elements associated with the triangle have had a particular appeal for M. Thébault, and he returned to them time and again. Among them is Feuerbach's theorem. He considered Feuerbach's points in the very first article he contributed to the *Nouvelles Annales de Mathêmatiques* (series 4, vol. 10, 1910, pp. 271–281). Subsequently he devoted half a dozen papers to this topic, at various times, in which he exhibited a wealth of properties of these points. As late as 1935, he published a paper *Sur les points de Feuerbach* in the *Annales*

*de la Société scientifique de Bruxelles*, series A, vol. 55, pp. 39, in which he proves the following curious proposition: In a triangle $(T)$ the four triangles having for vertices the four Feuerbach points of $(T)$ are similar to the four triangles having for vertices the feet of the bisectors of $(T)$.

As a side line, Thébault studied the "Shoemaker's knife" in several papers which appeared in various periodicals (*l'Enseignement Mathématique*, *Bulletin de la Société mathématique de France*, and so on).

The many and lasting contributions that M. Thébault made to the geometry of the triangle could not keep his inquiring mind from becoming attracted by the much less developed geometry of the tetrahedron which held out greater promise, even if it offered greater difficulty. His first paper dealing with the tetrahedron appeared in *Mathesis* in 1922. Of the many articles he has published since on this subject, articles in which he obtained a variety of important properties, the most important are those in which he associated with the tetrahedron spheres analogous to well-known circles associated with the triangle.

In 1932, he introduced the Longchamps sphere of the tetrahedron. In 1935 he added the Lucas spheres. Then came in rapid succession the Adams sphere, the Tucker sphere, the second Lemoine sphere. The adding of these spheres to the elements connected with the tetrahedron is undoubtedly a milestone in the development of the theory of the tetrahedron and promises to stimulate further advances. It is remarkable that this fruitful activity of M. Thébault took place during the sinister occupation of France by the Nazi hordes.

An outstanding feature of the prolific activities of Thébault is his interest in problems. There is hardly a periodical published anywhere with a problem department to which he has not contributed. The total number of questions proposed by him is counted by the hundreds and his record in this respect has few rivals. During the last decade or so American readers enjoyed the benefit of his contributions in this line both in the MONTHLY and in the *National Mathematics Magazine*. The originality and the variety of these questions have been the object of general admiration.

The quality and quantity of Thébault's geometrical contributions have earned for him a distinguished and enduring place among the most notable continuators of the work of Brocard, Lemoine, and Neuberg, the founders of the geometry of the triangle and the tetrahedron.

# CONCERNING THE EULER LINE OF A TRIANGLE*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. **Notation.** Consider a triangle $ABC$ inscribed in a circle $(O)$, of center $O$ and radius $R$. Let $G$ be the centroid, $H$ the orthocenter, $N$ the center of the nine-point circle $(N)$; $A_1$, $B_1$, $C_1$, the midpoints of the sides $BC=a$, $CA=b$, $AB=c$; $A'$, $B'$, $C'$, the feet of the altitudes on these sides; $S$ the area of the triangle. Unless otherwise stated we limit our discussion to the case of a triangle $ABC$ with acute angles, not all three equal, leaving the examination of the other cases to the reader.

2. **A homothetic transform of the orthic triangle.** The centers $O_a$, $O_b$, $O_c$ of the circumcircles of the triangles $BOC$, $COA$, $AOB$, of radii $R_a$, $R_b$, $R_c$, coincide with the intersections of the perpendicular bisectors of $BC$, $CA$, $AB$ and those of $OA$, $OB$, $OC$, and

$$R_a = R/2 \cos A, \qquad R_b = R/2 \cos B, \qquad R_c = R/2 \cos C.$$

Since these circles intersect in pairs in $A$, $B$, $C$ and meet at the circumcenter $O$ of triangle $ABC$, $O$ is the incenter of triangle $O_aO_bO_c$.† Triangle $O_aO_bO_c$ is directly homothetic to the orthic triangle $A'B'C'$. Since $O$ and $H$ are corresponding points of $O_aO_bO_c$ and $A'B'C'$, the lines $O_aA'$, $O_bB'$, $O_cC'$ are concurrent in a point $P$ of the Euler line of $ABC$, and

$$(1) \qquad k = PH/PO = r'/\tfrac{1}{2}R = 2R \cos A \cos B \cos C/\tfrac{1}{2}R = 4 \cos A \cos B \cos C,$$

$r' = 2R \cos A \cos B \cos C$ where $R/2$ is the inradius of triangles $A'B'C'$ and $O_aO_bO_c$. Thus we have

THEOREM 1. *Triangle $O_aO_bO_c$ is the transform, under the homothety $(P, k)$, of the orthic triangle $A'B'C'$, where $P$ is the point which divides the segment $HO$ in the ratio*

$$k = 4 \cos A \cos B \cos C.$$

The normal coördinates of $P$ with respect to $ABC$ are $\cos 2A/\cos A$, $\cos 2B/\cos B$, $\cos 2C/\cos C$. It is not difficult to verify that $P$ is the center of similitude of triangles $O_aO_bO_c$ and $A'B'C'$.

3. **The twin point of the circumcenter of the reference triangle.** The circles $(O_a')$, $(O_b')$, $(O_c')$, (with centers $O_a'$, $O_b'$, $O_c'$), symmetric to the circles $(O_a)$, $(O_b)$, $(O_c)$, with respect to $BC$, $CA$, $AB$, intersect in the twin point [D1] $Q$ of $O$. These circles intersect in pairs at $A$, $B$, $C$, so that $O$ and $Q$ are the foci of a conic $\Sigma$ inscribed in the triangle $O_a'O_b'O_c'$. The axis of $\Sigma$ along $OQ$ has a length equal to $R$. The locus of points symmetric to $Q$ with respect to the tangent lines

* Translated from the French by Col. W. E. Byrne.

† If $A$ is a right angle, $O_a$ is rejected to infinity in a direction perpendicular to $BC$ and the midpoint $O$ of $BC$ is equidistant from the lines $O_bO_c$, $O_cO_a$, $O_aO_b$. If $A$ is obtuse, $O$ is the excenter of the triangle $O_aO_bO_c$ opposite vertex $O_a$.

of $\Sigma$ is the circle $(O)$. The points of contact of $\Sigma$ with the sides of the triangle $O_a'O_b'O_c'$ are the points of intersection of the lines $OA$, $OB$, $OC$ with the corresponding sides. The segment $OQ$ is a diameter of the hyperbola of Jerabek [D2] of triangle $ABC$. The midpoint $\omega$ of $OQ$ is the orthopole of the Euler line $OH$. Hence $\omega$ is on the circle $(N)$ and $HQ=R$. It follows that $H$ lies on the circle of center $Q$ and radius $R$, the locus of points symmetric to $O$ with respect to the tangent lines of $\Sigma$. Furthermore, the circle of center $H$ and radius $R$ contains $Q$ and the transforms $(A_2, B_2, C_2)$ of $(A_1, B_1, C_1)$ and $(A'', B'', C'')$ of $(A', B', C')$ under the homothety $(O, 2)$. If we give to the distances from $\omega$ to the points $A'$, $B'$, $C'$, which are proportional to $b^2-c^2$, $c^2-a^2$, $a^2-b^2$, [1], the signs of these differences, we have [2]

$$\overline{\omega A'} + \overline{\omega B'} + \overline{\omega C'} = 0$$

and, by the homothetic transformation $(O, 2)$,

$$\overline{QA''} + \overline{QB''} + \overline{QC''} = 0.$$

This gives us

THEOREM 2. *The point $Q$ coincides with the symmetric with respect to $N$ of the Feuerbach point of the tangential triangle of $ABC$* [3] *and also with the Feuerbach point of the anticomplimentary triangle* [D3] *of triangle $A''B''C''$ (anticomplimentation with respect to $A''B''C''$).* [4].

**4. A relation between triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$.** The point $Q'$, symmetrical to $Q$ with respect to $N$, coincides with the point of intersection of the lines symmetric to $OH$ with respect to the sides of $ABC$. [5]. The point $Q$ is therefore the point of intersection of the lines $\Delta_a$, $\Delta_b$, $\Delta_c$, symmetric to $OH$ with respect to the sides of $A_2B_2C_2$. The lines $QA$, $QB$, $QC$ coincide also with $\Delta_a$, $\Delta_b$, $\Delta_c$, and as they are perpendicular to $O_b'O_c'$, $O_c'O_a'$, $O_a'O_b'$, the triangles $O_a'O_b'O_c'$ and $O_aO_bO_c$ are inversely similar. Let $H_a$, $H_a'$ be the feet of the altitudes $O_aH_a$, $O_a'H_a'$ of triangles $O_aO_bO_c$, $O_a'O_b'O_c'$; $O_a''$ be symmetric to $O_a'$ with respect to $O_b'O_c'$; $A_1'$ be the midpoint of $O_aO_a''$. Since triangle $O_a''O_b'O_c'$ is symmetric to triangle $O_a'O_b'O_c'$, triangles $O_aO_bO_c$, $O_a''O_b'O_c'$, $A_1'B_1C_1$ are directly similar and the midpoint $H_a''$ of $H_aH_a'$ is the foot of the altitude $A_1'H_a''$ of triangle $A_1'B_1C_1$. As the line $A_1'H_a'$ is parallel to $O_aO_a'$, it is perpendicular to $B_1C_1$, and consequently it coincides with $A_1'H_a''$. Hence $H_a'H_a$ is parallel to $O_aO_a'$ and, if we denote by $H'$ the intersection of $O_aH_a$ and $O_a'H_a'$, we have

$$O_aH'/H'H_a = O_a'H'/H'H_a'.$$

Thus the point $H'$, the intersection of any two corresponding altitudes, is the double point of the inversely conformal collineation carrying $O_aO_bO_c$ into $O_a'O_b'O_c'$; $H'$ is the common orthocenter of triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$. We state this as

THEOREM 3. *The triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$ are inversely similar, the double point being their common orthocenter.*

Note 1. If $H''$ is the orthocenter of triangle $O_a''O_b'O_c'$, the midpoint $H_a'$ of $H'H''$ coincides with the orthocenter of triangle $A_1'B_1C_1$. Hence if we construct on the sides of the medial triangle $A_1B_1C_1$ of $ABC$ triangles directly similar to triangle $O_aO_bO_c$ $(O_a'O_b'O_c')$, their orthocenters coincide with the feet of the altitudes of the triangle $O_aO_bO_c$ $(O_a'O_b'O_c')$.

Note 2. The properties discussed in this section also hold when the center $O$ of the circle $(O)$ is replaced by an arbitrary point of the plane of the reference triangle.

5. **A relation between the orthic triangle and triangle $O_a'O_b'O_c'$.** In virtue of (1), the area of triangle $O_aO_bO_c$ has for its value*

$$\overline{O_aO_bO_c} = S/8 \cos A \cos B \cos C.$$

In magnitude and sign [6] we have the relation

$$\overline{O_aO_bO_c} + \overline{O_a'O_b'O_c'} = \overline{ABC} = S$$

so that

$$\overline{O_a'O_b'O_c'} = S(1 - 1/8 \cos A \cos B \cos C),$$

and

$$k'' = O_bO_c/O_b'O_c' = [1/(1 - 8 \cos A \cos B \cos C)]^{1/2} = R/OH. \tag{2}$$

Furthermore

$$\begin{aligned} O_bO_c &= R \sin A/2 \cos B \cos C, & O_b'O_c' &= OH \sin A/2 \cos B \cos C,\\ O_cO_a &= R \sin B/2 \cos C \cos A, & O_c'O_a' &= OH \sin B/2 \cos C \cos A,\\ O_aO_b &= R \sin C/2 \cos A \cos B, & O_a'O_b' &= OH \sin C/2 \cos A \cos B, \end{aligned}$$

from which we obtain the ratio of similitude

$$k''' = O_b'O_c'/B'C' = OH/4R \cos A \cos B \cos C$$

of triangles $O_a'O_b'O_c'$ and $A'B'C'$. If we orthogonally project the orthocenter $H$ on $OO_a'$, simple calculations give

$$HO_a' = OH/2 \cos A, \qquad HO_b' = OH/2 \cos B, \qquad HO_c' = OH/2 \cos C.$$

From this we obtain the ratios

$$HO_a'/HA' = OH/4R \cos A \cos B \cos C = HO_b'/HB' = HO_c'/HC',$$

which show that $H$ coincides with the double point of the inversely conformal

* The symbol $\overline{ABC}$ is used to denote the signed area of the oriented triangle $ABC$.

collineation carrying $O_a'O_b'O_c'$ into $A'B'C'$. As $H$ is the incenter of triangle $A'B'C'$ and also the incenter of triangle $O_a'O_b'O_c'$, we find, by (2) that the radius of the inscribed circle of $O_a'O_b'O_c'$ is

$$\rho' = \frac{R}{2}(OH/R) = OH/2.$$

Summarizing we have

THEOREM 4. *Triangles $O_a'O_b'O_c'$ and $A'B'C'$ are inversely similar, the double point being the orthocenter $H$ of the reference triangle, which coincides with the center of the inscribed circle, of radius $OH/2$, of the triangle $O_a'O_b'O_c'$.*

It may be verified that $O_a'O_b'O_c'$ and $A'B'C'$ (or $O_aO_bO_c$) are both metaparallel [D4] and orthologic [D5].

6. **The isogonal conjugate of the twin point $Q$ of the circumcenter of the reference triangle.** As the points $O$ and $Q$ are diametrically opposite on the Jerabek hyperbola, the center of which coincides with the orthopole of $OH$, the isogonal conjugate $Q''$ of $Q$ is on $OH$. The points $H$, $Q''$ divide harmonically the diameter of the circle $(O)$ determined by $OH$, and $OH\cdot OQ''=R^2$, so that $OQ''=R/OH$. The pedal triangle $A'''B'''C'''$ of $Q''$ with respect to the triangle $ABC$ is inversely similar to $A'B'C'$ and inversely homothetic to triangle $O_a'O_b'O_c'$, since the sides of these two triangles are perpendicular to $AQ$, $BQ$, $CQ$. Since $O$ and $Q''$ are corresponding points in triangles $O_a'O_b'O_c'$ and $A'''B'''C'''$, their homothetic center $T$ is on $OH$ and the homothetic ratio is equal to

$$O_b'O_c'/B'''C''' = OH^2/2Rr' = 1/4 \cos A \cos B \cos C - 2.$$

We also have

$$\overline{A'B'C'}/\overline{A'''B'''C'''} = -OH^2/R^2;$$

so that

$$B'C'/B'''C''' = OH/R = O_b'O_c'/O_bO_c,$$

and finally the area relations, in magnitude and sign,

$$\overline{A'B'C'}\cdot\overline{O_aO_bO_c} = \overline{A'''B'''C'''}\cdot\overline{O_a'O_b'O_c'} = \overline{ABC}^2/4.$$

Furthermore, the center $h$ of the incircle $(h)$ of triangle $A'''B'''C'''$ is on $OH$, and

$$TH/Th = O_b'O_c'/B'''C''' = OH^2/2Rr' = OH^2/4R^2 \cos A \cos B \cos C.$$

It follows that

$$TO/TQ'' = OH^2/2Rr'$$

and that

$$hQ'' = 2Rr'/OH = 2\rho''',$$

$\rho'''$ being the radius of circle $(h)$.

THEOREM 5. *In a triangle $ABC$, the incenter $h$ of the pedal triangle of the isogonal conjugate $Q''$ of the point $Q$, on the hyperbola of Jerabek, diametrically opposite to the circumcenter $O$ of triangle $ABC$, is on the Euler line and the radius of $(h)$ is $hQ''/2$.*

Note that between the inradii of triangles $O_aO_bO_c$, $A'B'C'$, $O_a'O_b'O_c'$ and $A'''B'''C'''$ there is the relation

$$Rr' = 2\rho'\rho'''.$$

Also the incircles $(H)$ and $(h)$ of triangles $O_a'O_b'O_c'$ and $A'''B'''C'''$ subtend at the points $O$ and $Q''$, respectively, angles of 60°.

7. **Isogonal and harmonic conjugates.** The line $AN$ meets the perpendicular bisector $\Delta$ of $BC$ at $A_2$. Also

$$OA_2 = AH = 2R \cos A,$$

so that

$$OA_2 \cdot OO_a = R^2 = OV^2 = OV'^2,$$

where $V$, $V'$ are the points of intersection of $\Delta$ and the circle $(O)$. The lines $AA_2$ and $AO_a$ are harmonic conjugates with respect to the bisectors $AV$, $AV'$ of the angle $A$ of triangle $ABC$. They are isogonal conjugates with respect to the angle $(AO, AH)$ as well as with respect to the angle $(AB, AC)$. The lines $BB_2$ and $BO_b$, $CC_2$ and $CO_c$ have similar properties. Furthermore, triangles $ABC$ and $O_aO_bO_c$ are orthologic, the orthologic centers [D5] being coincident at $O$.

THEOREM 6. *Triangles $ABC$ and $O_aO_bO_c$ are both orthologic and in perspective. The center of perspectivity is the isogonal conjugate $N'$ of the center $N$ of the nine-point circle of triangle $ABC$. The orthologic centers coincide with the circumcenter $O$ of triangle $ABC$, and the axis of perspectivity is normal to $ON'$.*

We note that $N'$ is on the Jerabek hyperbola of triangle $ABC$ and, by virtue of the Weill-Aybar theorem [6],

$$N'O/N'N = 2R/OH.$$

Theorem 6 is a special case of the following

THEOREM 7. *Given a triangle $ABC$, two isogonal conjugate points $Q$, $Q'$, and an arbitrary point $P$ of the plane of triangle $ABC$, the harmonic conjugates of the lines $AP$, $BP$, $CP$ with respect to the pairs $AQ$ and $AQ'$, $BQ$ and $BQ'$, $CQ$ and $CQ'$, intersect in the isogonal conjugate $P'$ of $P$.*

8. **A special triangle.** Let us consider the triangle $ABC$ for which the nine-

point center $N$ is on the circumcircle $(O)$. For this special triangle

$$OH = 2R$$

and, consequently,

$$a^2 + b^2 + c^2 = 5R^2.$$

Some properties of this triangle have already been published [7]. Those which follow may be new.

1. The lines $AO_a$, $BO_b$, $CO_c$, joining the vertices of $ABC$ to the circumcenters of triangles $BOC$, $COA$, $AOB$, are parallel. They are perpendicular to the straight line on which are situated the three points obtained by taking the symmetrics of $A$, $B$, $C$ with respect to $BC$, $CA$, $AB$.

2. The circumcircle $(\Omega)$ of the tangential triangle of $ABC$ coincides with the circle described on $GH$ as a diameter.

3. The circumcircle of triangle $O_aO_bO_c$, of center $G$ and radius $R/3$, is tangent to $(O)$ at $N$.

4. The excircle (of center $H$) of triangle $O_a'O_b'O_c'$ is equal to the circumcircle $(O)$ of $ABC$. Triangle $O_a'O_b'O_c'$ is inversely congruent to the tangential triangle of $ABC$.

## Definitions

$D1$. Twin point. With respect to a reference triangle $ABC$, two points $P^*$ and $P$ are said to be twin points if $P$ is the isogonal conjugate (with respect to $ABC$) of a point $M$ and $P^*$ is the isogonal conjugate (with respect to $ABC$) of the inverse of $M$ with respect to the circumcircle of $ABC$.

In the present paper, if we take as $M$ the orthocenter $H$, $P$ will coincide with the circumcenter $O$ of $ABC$ and $P^*$ will coincide with the point $Q$ defined by Monsieur Thébault.

Also see Gallaty, *The Modern Geometry of the Triangle*, 2nd edition, page 62.

$D2$. Hyperbola of Jerabek. The hyperbola of Jerabek of a triangle $ABC$ is the transform by isogonal conjugates with respect to $ABC$ of the Euler line of triangle $ABC$.

$D3$. Anticomplimentary point. With respect to a triangle $ABC$ of centroid $G$, the point $P$ anticomplimentary to a point $P$ of the plane of $ABC$ is defined by

$$\overrightarrow{GP'} = -2\overrightarrow{GP}.$$

$D4$. Metaparallel triangles. Two triangles $A_1A_2A_3$, $B_1B_2B_3$ are said to be metaparallel if the lines from $A_1$ parallel to $B_2B_3$, from $A_2$ parallel to $B_3B_1$, and from $A_3$ parallel to $B_1B_2$ are concurrent. Then the parallels to $A_2A_3$ drawn through $B_1, \cdots$, will also be concurrent.

$D5$. Orthologic triangles. Two triangles $A_1A_2A_3$, $B_1B_2B_3$, are said to be orthologic if the perpendiculars drawn from $A_1$ to $B_2B_3$, $A_2$ to $B_3B_1$, $A_3$ to $B_1B_2$ are concurrent in a point $M$. Then the perpendiculars from $B_1$ to $A_2A_3, \cdots$, will also be concurrent in a point $M'$. The points $M$ and $M'$ are called orthologic centers.

**Bibliography**

1. V. Thébault, Ann. Soc. Scient. Bruxelles, 6. 52, serie A, 1932, p. 289.
2. V. Thébault, Journal de Vuibert, t. 34, 1920, p. 174.
3. J. R. Musselman, Mathesis, t. 51, 1937, p. 161, and t. 53, 1939, p. 39.
4. V. Thébault, Mathesis, 1932 (Supplément, p. 4).
5. P. Delens, Mathesis, t. 51, 1937, p. 265.
6. M. Weill, Nouvelles Annales, 1880, p. 253, and Gallaty, The Modern Geometry of the Triangle, 2nd edition, p. 79.
7. Davis, Educational Times, Reprints, 1915, p. 306, question 17487, and Beard, Educational Times, Reprints, 1915, p. 355, question 18082. Youngman, Educational Times, Reprints, 1915, p. 238, question 18027. R. Goormaghtigh, Intermédiaire des Mathématiciens, 1918, pp. 68, 134.

# A NAVIGATION COMPUTER

O. E. BROWN AND J. J. NASSAU, Case Institute of Technology

1. **Introduction.** This paper presents a mathematical description of a computer of the circular slide rule type prepared for the graphical solution of the spherical triangle, particularly as applied to the astronomical triangle. The computer is rectangular and on each side a transparent rotor is attached with the pivot point at the lower left hand corner. On each rotor appear two one-parameter families of curves, one family consisting of constant altitude curves and the other of curves of constant azimuth angle. Through each transparent rotor appears the *grid* made up of many dots. These dots define two sets of curves, one for constant meridian angle (*i.e.*, local hour angle with a range from 0° to 180°) and the other set for constant declination. The meridian angles represented are integral degrees while the declinations used are those of 44 navigation stars and a few integers under 30°.

Figure 1 shows the grid, Figure 2 the rotor, both with some intermediate curves and dots removed to permit the necessary reduction in size.

The finished computer has the following properties:

1. At one setting of a rotor both the altitude and azimuth angle are obtained.

2. The errors in altitude are rarely over four minutes of arc and their average without regard to sign is under two minutes.

3. Although, from the map projection point of view, each grid and rotor is of a size to represent $\frac{1}{8}$ of the celestial sphere, the one grid and two rotors are sufficient to solve any spherical triangle.

4. The declinations of each of 44 navigation stars are plotted permanently on the grid.

5. The entire device is about 11 by 12 inches and less than $\frac{1}{4}$ of an inch thick.

2. **A description of the mathematical principles of the computer.**

*The Astronomical Triangle.* The fundamental problem of finding an ob-

server's position by celestial observation is that of solving the *PZS* spherical triangle for *h* and *z*, given *d*, *L* and *t*. The vertices of this triangle are *Z*, the observer's zenith, *P*, the celestial pole above the horizon, and *S*, the heavenly body observed and hereafter referred to as the *star*. The arcs *SP*, *PZ*, and *ZS* are respectively the complements of the star's declination, the observer's assumed latitude, and the star's altitude. Given the quantities *d*, *L*, and *t*, the

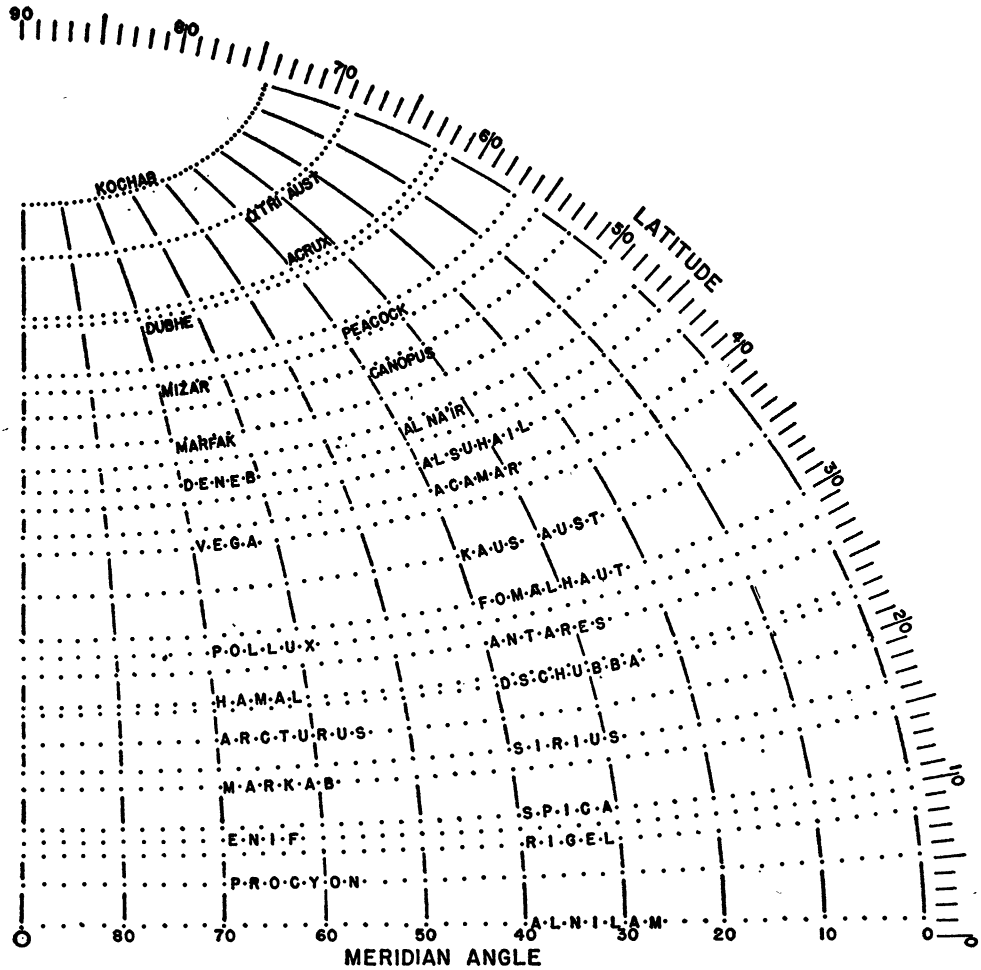



Fig. 1

quantities *h* and *z* are determined. They may be computed in any numerical case by using the formulas of spherical trigonometry or they may be found for given integral values in such tables as H.O. 214,* and H.O. 218.

* U.S. Hydrographic Office.

*Graphical Solution.* The required quantity, $h$, is expressed in terms of the known quantities $d$, $L$, and $t$ by the law of cosines as

(1) $$\sin h = \sin d \sin L + \cos d \cos t \cos L.$$

By the substitution

(2) $$\sin d = y, \qquad \cos d \cos t = x,$$

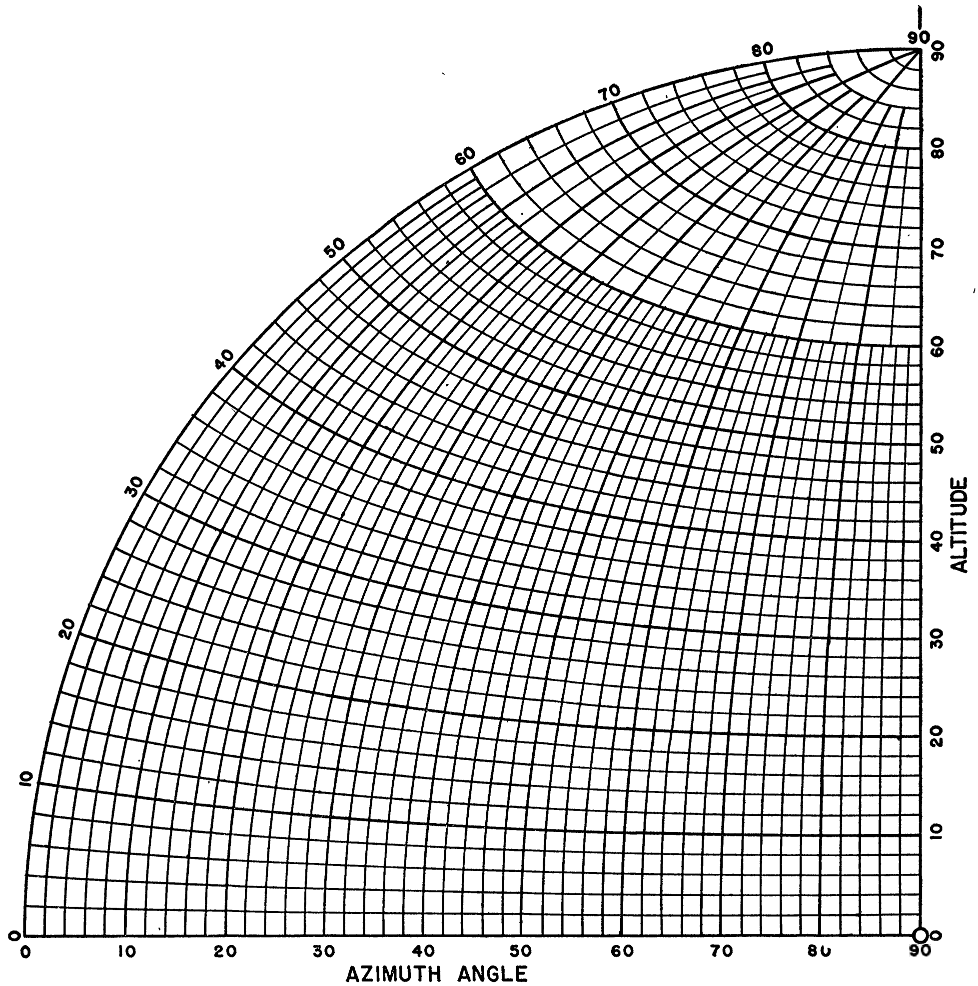



Fig. 2

Equation (1) becomes

(3) $$x \cos L + y \sin L - \sin h = 0,$$

which is the equation of a straight line with $L$ as normal angle and $\sin h$ as normal

intercept. Equations (2) are used to define a coördinate system. It is apparent that the "$d$ = constant curves" are straight lines parallel to the $x$ axis while the "$t$ = constant curves" are ellipses with common major axis. The figure, thus constructed, ultimately becomes the grid. These two families of curves are transformed to improve the scale and are replaced by dots at their points of intersection. Integer declinations are also replaced by the actual declinations of the 44 stars selected.

Applying the law of cosines again to the same triangle, we obtain:

$$\sin d = \sin h \sin L + \cos h \cos z \cos L, \tag{4}$$

which, after the substitution,

$$y' = \sin h, \qquad x' = \cos h \cos z \tag{5}$$

becomes

$$x' \cos L + y' \sin L - y = 0. \tag{6}$$

With $L$ and $y$ constant this equation defines another straight line with normal intercept $y$ and the same normal angle $L$ as that of the previous line. Obviously the coördinate system defined by Equations (5) is exactly similar to that defined by Equations (2), except that now the straight lines represent constant values of $h$ while the ellipses correspond to constant values of $z$. As in the previous case, the figure thus obtained is transformed to improve the scale and becomes the rotor of the computer.

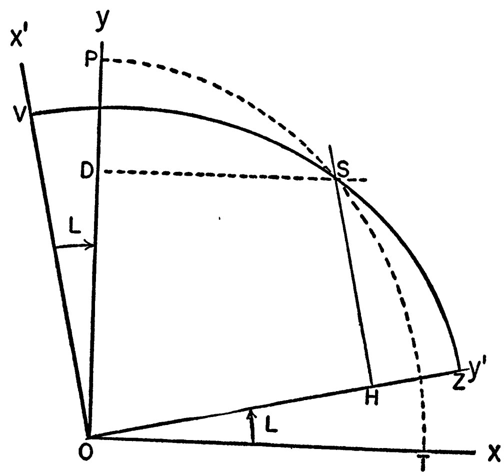



Fig. 3

Since equations (3) and (6) have the same normal angle, and the normal intercept of each is the ordinate of the other, the two may be superimposed as shown in Figure 3. The dotted ellipse $PST$ represents one of the hour angle curves of the grid and corresponds to an assumed value of $t$. The dotted line $SD$ is one of

the grid lines of constant declination to be marked on the finished computer by the name of the corresponding star. The straight line $SH$ perpendicular to $OY'$, and the ellipse $ZSV$ represent, respectively, constant altitude and constant azimuth curves on the rotor. The angle $L$ of the figure is the angle through which the rotor has been turned and corresponds to the assumed latitude of the problem. This is essentially the underlying principle of the computer. The star $S$ is located on the grid from its given coördinates $(t, d)$ on the $XOY$-system, the rotor is turned through the assumed latitude $L$ and the coördinates $(z, h)$ of the star are read on the $X'OY'$-system.

*The Transformation.* As the reader will observe, the lines and ellipses described previously in connection with the $XOY$-system make up an orthographic projection of the hour circles and parallels of the celestial sphere. The corresponding curves of the rotor represent the "parallels" of constant altitude and the vertical circles through $Z$. Computers of the same fundamental nature as the one described here have been built using the orthographic projection as well as some using a stereographic projection. These two projections have some advantage in preparation because of the lines, ellipses, and circles which make up their coördinate systems. The orthographic projection exhibits great radial crowding around the outside while the stereographic projection has its greatest crowding at the center. A suitable projection in our case is obtained by transforming the orthographic projection by means of the polar coördinate equations

$$\theta' = \theta, \qquad r' = \text{arc sin } r.$$

The resulting figure is known as an *azimuthal equidistant* projection about the point $O$. All distances from $O$ are true while the circumferential scale varies from the radial scale to $\pi/2$ times the radial scale.

*Reduction to One Quadrant.* Any quadrant of the grid figure is like any other quadrant except possibly for a reflection, and the same is true of the rotor figure. Thus one quadrant of grid figure and two quadrants of rotor figure are sufficient to represent any configuration. One quadrant of the grid figure is printed in duplicate on both sides of an opaque sheet and one transparent rotor quadrant is mounted over each.

*Reference Lines of the Grid and Rotor.* The grid carries rows of dots corresponding to the declination of the 44 navigation stars and a few additional rows for integer declinations under 30°. The dots on each row are spaced one degree apart, thus forming the family of hour circles. Each dot, therefore, represents the position of a star of known $d$ and $t$, taken to the nearest integer. The rotor carries the set of equal altitude curves one degree apart and azimuth lines two degrees apart. The latitude scale is marked on the grid at the limiting circumference of the rotor (Figure 1).

**3. Uses.** The computer is primarily a navigation instrument for securing the necessary quantities $(h, z)$ to establish the line of position. The value of $t$ is obtained in the usual way with the assumed longitude adjusted so that $t$ is equal to the nearest integer. Knowing the value of $t$ and the coördinates of the star

observed, the corresponding dot of the grid is located and circled with a pencil. The rotor is turned to the assumed latitude and the value of $h$ is read to the nearest minute by eye interpolation between the constant altitude curves. The value of $z$ is obtained likewise and read to the nearest degree. In observing a body in the solar system the required dot is penciled in at its proper position on the grid and $h$ and $z$ obtained as before. The surface of the grid is matte-finished to permit marking and erasing.

The above procedure conforms to the standard method for determining the line of position. The computer may also be used for identifying a star from its approximate altitude and azimuth.

4. **Accuracy.** The device is amazingly accurate. The navigator wants the azimuth only to the nearest degree but desires the altitude to the nearest few minutes. H.O. 218 gives the altitude to the nearest minute of arc while H.O. 214 gives the altitude to the nearest tenth of a minute. Bearing in mind that one minute of arc amounts to one nautical mile and the additional fact that the sextant reading, with which the computed altitude is to be compared, is probably in error by a few minutes it appears that this computer is sufficiently accurate for position fixing in the air and in surface craft if an error of a very few miles can be tolerated. The computer was used in long over-ocean flying with results essentially as stated above.

## THE JEEP PROBLEM: A MORE GENERAL SOLUTION

C. G. PHIPPS, University of Florida

1. **Introduction.** In this MONTHLY, January, 1947, N. J. Fine discusses the problem of advancing a single jeep a given distance beyond a base of supply. He finds where the dumps must be established in order that the amount of gasoline used is a minimum. This problem is one of a more general class whose solution I wish to discuss in detail here.

Now a solution which yields the minimum amount of gasoline for a given distance is also a solution which yields the maximum distance for a given initial amount, and conversely. It so happens that a solution for the latter conditions appears much easier to obtain. This is therefore the type of problem to be treated first.

Before formulating the various problems in detail, it is necessary to define the special terms that will be used. A group of jeeps traveling together will be referred to as a *caravan.* A *station* is defined as a point where it is necessary to establish a dump, or where the number of jeeps in the caravan is changed. The *home station* is the fixed base from which all jeeps originally set out. A *stage* is defined as the distance between two successive stations. In general, the dif-

ferent stages will be of different lengths. And lastly, the letter $d$ will be used throughout to indicate the number of miles a jeep can travel on a full load of gasoline. Consequently, all distances will be given in miles.

There are possible many final dispositions of the jeeps in a caravan. Only two will be considered here: (1) a jeep may be abandoned at a station without its returning any distance, or (2) a jeep may be returned the whole distance to the home station. Other dispositions may be treated by the general methods presently to be outlined.

**2. Special problems.** Suppose $m$ jeeps fully loaded with gasoline set out from the home station. Their object is to advance one of their number to the greatest possible distance away from the home station. Three cases will be considered: (*a*) none return, (*b*) all return, and (*c*) all but one return to the home station.

(*a*) *None return.* It is readily apparent that at some point gasoline must be transferred to the other jeeps and the empty ones abandoned. Otherwise the caravan would advance one of their number no farther than one jeep could advance alone. Where that transfer is made and how many are left at each station are the points to be determined.

When the caravan has advanced a distance of $d/m$ miles, exactly one load of gasoline has been consumed. If one jeep is emptied of its load, the other $m-1$ can proceed fully loaded; the empty one can be abandoned on the spot.

If a jeep is abandoned before this distance is reached, the other jeeps can not carry all the gasoline on hand. They could proceed fully loaded as above but from a point closer to the home station. Ultimately then the farthest advance would be less than if the station were established at a distance of $d/m$.

On the other hand, if the now empty jeep were to be driven farther than this point, it must consume gasoline carried by the other jeeps. This procedure would correspondingly reduce the supply of gasoline available for the others. With a smaller supply, they would ultimately reach a shorter distance.

Therefore the most advantageous length for the first stage is $d/m$, at the end of which the first jeep is abandoned. Likewise, for the remaining $m-1$ jeeps, the second stage would be $d/(m-1)$, and so on. The last stage would be $d$ since it would be traversed by a single jeep fully loaded. The total distance this last jeep has advanced would be $d$ times the series

$$1 + 1/2 + 1/3 + \cdots + 1/m.$$

It has been tacitly assumed above that $m$ jeeps partly loaded would still consume more gasoline than $m-1$ jeeps fully loaded. To simplify the solutions in the problems which follow, it is necessary in all of them to make the stronger assumption that a jeep consumes gasoline at exactly the same rate whether lightly or fully loaded.

(*b*) *All jeeps return.* In this case, it would require just as much gasoline for the return trip as for the outward trip. The caravan would advance until half a load had been used up; the other half load would be put into a dump for use on the return trip. The single jeep now empty could take enough from the

dump to return at once or it could wait until the others returned. The same amount of gasoline would be used either way.

Since half as much gasoline would be available for the outward trip, each stage would be half as long as before. Hence the greatest advance would be $d$ times the series

$$1/2 + 1/4 + 1/6 + \cdots + 1/2m.$$

Now it would be possible to establish dumps between the stations indicated above but it would be unnecessary. There would be no saving in doing so unless a jeep used less gasoline the lighter its load, contrary to our assumption. Hereafter, therefore, we shall be concerned with only the minimum number of stations needed for the solution.

(*c*) *All jeeps but one return.* Over the first stage, there will be $m$ trips outward bound and $m-1$ return. If the first station is established at a distance $d/(2m-1)$, this traffic will consume exactly one load of gasoline. To prove that this is the proper distance we argue as before. If the first station is established closer, not all the gasoline will be consumed. If it is established at a greater distance, gasoline will be consumed unnecessarily.

By repeating the argument, the second stage would be $d/(2m-3)$ in length, and so on. The last stage would again be $d$. Hence the total distance advanced by the last jeep would be $d$ times the series

$$1 + 1/3 + 1/5 + \cdots + 1/(2m - 1).$$

(*d*) *Equivalence for one jeep.* So far we have considered a caravan of jeeps. The problem as treated by Mr. Fine involved only one jeep. We wish now to establish the equivalence between the travel of one jeep and that of $m$ jeeps.

A single jeep can return to the home station or it can remain at the point of its greatest advance. Assume the former. Supplied with $m$ loads of gasoline, it must shuttle on the first stage between the home station and the first dump. Each time, it departs from the home station fully loaded, deposits in a dump all but enough to return to the home station, and reaches there with an empty tank for its next load. Thus the jeep makes $2m$ trips over the first stage. The consumption of gasoline is exactly the same as if $m$ different jeeps transported the same amount over the first stage. This is the same as case (*b*). Consequently, the minimum number of stations and their location needed to advance a single jeep to a maximum distance, and return, are identical in number and location with the stations of case (*b*).

If the jeep remains at the far end of its advance, the first stage is traversed $2m-1$ times. It should now be obvious that this situation corresponds to case (*c*).

**3. General principles of solution.** Several general principles governing the solution of this type of problem should now either be evident or else be easily established. These principles apply with equal force to the special problems above.

(I) The first of these is that, if the travel on any stage is begun with fully loaded jeeps, the length of this stage is such that all the travel along it, both outward and return, will consume exactly one load of gasoline.

Let $m$ be the number of fully loaded jeeps (or trips of one jeep) which are outward bound. Let $k$ be the number of jeeps which return, $0 \leqq k \leqq m$. Since there will be $m+k$ trips along this stage, the principle requires that the length of the stage be $d/(m+k)$. The proof goes the same as before: a shorter stage will not use all the gasoline; a longer stage will require too much.

It follows from this principle that, if there is to be a minimum number of stations, a stage does not end until the supply of gasoline to be transported beyond that point is an integral number of loads.

(II) Thus, if a fractional part of a jeep-load is to be consumed, it must be used on the first stage of the journey.

Or, thought of in another way, it will require an extra jeep to carry the fraction of a load. Naturally, this extra jeep will be sent back, or abandoned, as soon as possible; that is, as soon as the fractional load is used up.

(III) It is to be observed in passing that the number of terms in the series is equal to the number of loads of gasoline.

A fractional load will add an irregular term at the end. Other general principles will be developed in the section which follows.

4. **The inverse problem.** The inverse problem is that in which the distance is given and the number of stations and the amount of gasoline are each to be a minimum. Since the location of the stations for the greatest advance on a given supply of gasoline is the same as the location of the stations for going a given distance on a minimum amount, there remains only to fit the terms of the proper series to the distance to be covered in order to determine the number of loads needed.

(*a*) When the manner by which the jeeps are advanced leads us to a divergent series, as in the problems above, any distance from the home station can be reached. By Principle III, the number of loads of gasoline needed is exactly equal to the number of the terms of the series required to equal the distance. If the given distance is such that it is more than the sum of $m$ terms of the series but less than $m+1$ terms, a fractional load must be used on the first short stage (Principle II).

(*b*) The problem of reaching a certain point has an interesting variation when it is required to arrive at the point with a given amount of gasoline.

Since the method of transporting gasoline as outlined in Section 3 enables one to reach the greatest distance on a given supply, it is the most economical method for transporting to that point the amount on hand at any moment. This amount can be shown by a table or a broken line graph or other means (not including the amount needed for the scheduled return trips). We can then select the place where the amount available, above that needed for return, is equal to the amount it is required to deliver. From that point measure back-

wards the required distance and count the number of stations included. This is the number of loads required to transport the given amount to the given distance. A fraction of an interval will require a fraction of a load for the first stage. The problem can be complicated by varying the disposition of the jeeps or the number that are to remain with the delivered gasoline.

The result above can be summarized into a general principle.

(IV) The amount of gasoline delivered unconsumed must be saved from the final stage, or stages, of the journey.

5. **Conclusions.** The conclusions reached by Mr. Fine hold here whenever the corresponding series is harmonic. The locations of the stations are the same whether the distance traveled is to be a maximum or the amount of gasoline used is to be a minimum. In addition, the number of stations established can be made a minimum, in which case their locations are unique.

The number of variations upon these problems is almost endless. One could have rendezvous points where jeeps are to assemble. One could consider the delivery of a certain number of jeeps to another supply station by having caravans meet halfway. Still another variation would be to have tank-trucks accompany the jeeps. Most of such problems can be worked by the general principles developed here.

The application of these problems is found, as Mr. Fine suggested, in polar regions or other places where there is no local supply. It is to be noted that the first solution (Section 3, Problem $(a)$) is exactly the one adopted for a space rocket with a multiple charge.

## MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

### SUBSERIES OF A MONOTONE DIVERGENT SERIES

R. W. HAMMING, Murray Hill, New Jersey

1. **Introduction.** There is a class of theorems in analysis in which the proof depends on showing that some quantity is greater than the sum of a series which is either known or proved to be divergent. In studying such theorems it is natural to ask to what extent they can be weakened by taking fewer terms. If the series is merely divergent, then at most a finite number may be neglected. If, however, the series is also monotone, then more terms may be neglected. The following theorem shows how many must be kept in order to still be able to prove theorems of this class.

THEOREM. *Let $I(n)$ be an increasing sequence of distinct positive integers, and let $\sum d_n$ be a monotone divergent series. A necessary and sufficient condition for the divergence of the subseries $\sum d_{I(n)}$ is that $I(n)/n$ be bounded.*

In the paper originally submitted the condition lim inf $I(n)/n < \infty$ was derived as an application of another theorem. For the result that the condition lim sup $I(n)/n < \infty$ is necessary, the author is indebted to R. P. Agnew.

2. **Proof of the theorem.** The proof of the sufficiency of the condition $I(n)/n < M$ is left to the reader.

To prove the necessity of the condition, we assume that

$$\limsup I(n)/n = \infty \tag{1}$$

and exhibit a monotone divergent series $\sum d_n$ for which the subseries $\sum d_{I(n)}$ converges. Let $n_0 = I(n_0) = 0$, and choose a positive integer $n_1$ such that the formula

$$[I(n_{k+1}) - I(n_k)]/[n_{k+1} - n_k] > 2^k \tag{2}$$

holds when $k=0$. Then choose, in order, integers $n_2, n_3, \cdots$, such that both

$$n_{k+1} - n_k > n_k - n_{k-1} \tag{3}$$

and (2) hold for each $k=1, 2, 3, \cdots$. For each $k=0, 1, 2, \cdots$; choose constants $B(\nu)$ such that

$$2 \geqq B[I(n_k) + 1] \geqq B[I(n_k) + 2] \geqq \cdots \geqq B[I(n_{k+1})] \geqq 1.$$

For each $k=0, 1, 2, \cdots$, let

$$d_\nu = B(\nu)/2^k[n_{k+1} - n_k], \qquad I(n_k) < \nu \leqq I(n_{k+1}). \tag{4}$$

Then $\sum d_n$ is a monotone decreasing series with $d_n \to 0$. But for each $k=1, 2, 3, \cdots$, we have, where the summations cover the range

$$I(n_k) < \nu \leqq I(n_{k+1}),$$

$$\begin{aligned}\sum d_\nu = \sum B(\nu)/2^k(n_{k+1} - n_k) &> \sum 1/2^k(n_{k+1} - n_k) \\ = [I(n_{k+1}) - I(n_k)]/[2^k(n_{k+1} - n_k)] &> 1,\end{aligned}$$

and therefore $\sum d_n$ diverges. Finally, for each $k=1, 2, 3, \cdots$, we have, where the summations cover the range,

$$n_k < \nu \leqq n_{k+1},$$

$$\sum d_{I(\nu)} = \sum B[I(\nu)]/2^k(n_{k+1} - n_k) < \sum 2/2^k(n_{k+1} - n_k) = 1/2^{k-1},$$

and therefore $\sum d_{I(n)}$ converges. Thus the series $\sum d_n$ has the required properties and the theorem is proved.

## ONE MORE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

N. C. ANKENY, Stanford University

Many proofs of the Fundamental Theorem of Algebra, including various proofs based on the theory of analytic functions of a complex variable, are known. The following proof is a little different from the usual ones that use the Calculus of Residues.

To start with, we assume that the polynomial $P(z)$ is of degree $N \geqq 2$ and has real coefficients. We consider the integral

$$\int_{\Gamma} \frac{dz}{P(z)}, \tag{1}$$

where $\Gamma$ denotes the boundary of a semi-circle: $\Gamma$ consists of a straight line from $-R$ to $R$, where $R$ is a large positive number, and of a semi-circle in the upper half plane, of radius $R$ and center 0, connecting $R$ to $-R$. When $R$ tends to infinity, the integral over the circular arc approaches 0 by the usual argument, since the degree of $P(z)$ is at least 2.

Suppose now, in contradiction to the theorem we desire to prove, that $P(z)$ has no roots. Then $1/P(z)$ is everywhere regular and the value of (1) is 0. It follows that

$$\int_{-\infty}^{\infty} \frac{dx}{P(x)} = 0, \tag{2}$$

the integral being extended along the real axis. If, however, $P(x)$ has no roots, it keeps the same sign for all real values. This contradicts (2); and so it is proved that $P(z)$ must have roots.

If $Q(z)$ is any polynomial, let $\overline{Q}(z)$ denote the polynomial whose coefficients are conjugate complex to those of $Q(z)$ and apply the foregoing proof to $P(z) = Q(z)\overline{Q}(z)$.

## CLASSROOM NOTES

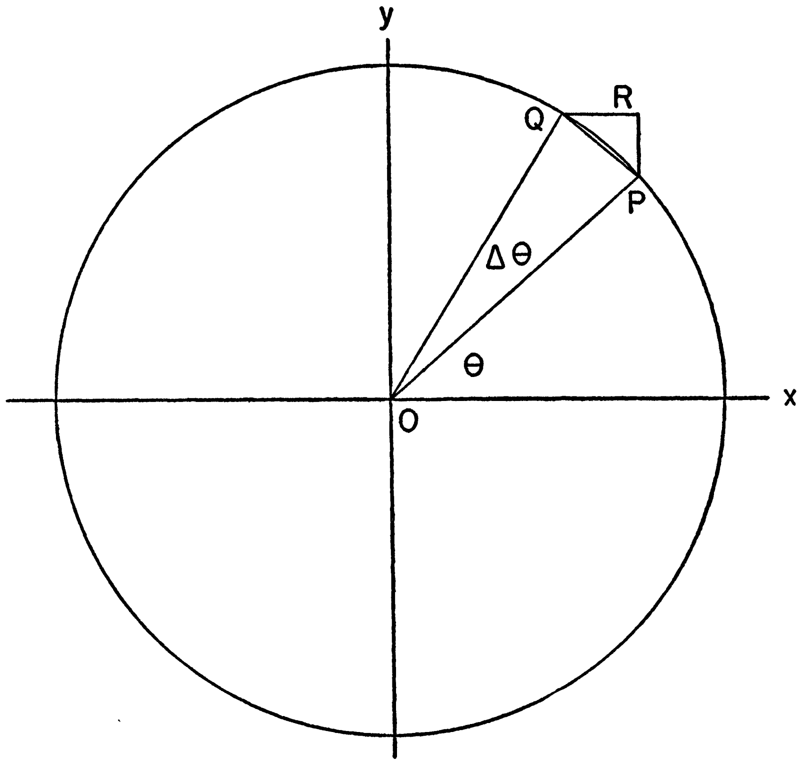
EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.*

### DIFFERENTIATION OF THE TRIGONOMETRIC FUNCTIONS

C. I. LUBIN, University of Cincinnati

The discussion of the differentiation of the common trigonometric functions can be based on relations which do not involve a particular trigonometric function the way the usual procedure involves the sine. This unifies the development and gives it a symmetry which might be a pedagogic advantage. On the other hand, this procedure makes use of such ideas as implicit differentiation, and possibly demands a somewhat more 'sophisticated' approach than the usual treatment.




The basic relation is obtained by using the circle

$$x^2 + y^2 = a^2 \tag{1}$$

and introducing the angle, $\theta$, made with the $x$-axis by a radius $OP$ drawn to the

point $P(x, y)$ on the circumference. (See figure.) Then

$$(2) \qquad \begin{aligned} x &= a \cos \theta \\ y &= a \sin \theta. \end{aligned}$$

Let the angle $\theta$ have an increment $\Delta\theta$, $\Delta\theta > 0$, thereby determining the point $Q(x+\Delta x, y+\Delta y)$ lying on the circumference of the circle and in the same quadrant as $P$. In case $P$ lies on one of the axes, $Q$ is to lie in the following quadrant. We then obtain

$$\text{area of } \Delta POQ = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x + \Delta x & y + \Delta y & 1 \end{vmatrix} = \frac{1}{2} (x\Delta y - y\Delta x) > 0.$$

The area of the sector is $\frac{1}{2}a^2\Delta\theta$ and since it is known from plane geometry that these two areas differ by an infinitesimal of higher order we can write as a limit

$$(3) \qquad x \frac{dy}{d\theta} - y \frac{dx}{d\theta} = a^2$$

assuming the limit $dx/d\theta$ or $dy/d\theta$ exists. This is the basic relation mentioned above. Its derivation is independent of the quadrant in which $\theta$ lies.

Recourse to the above limit theorems of plane geometry, as well as the assumption that $dx/d\theta$ or $dy/d\theta$ exists, can be avoided by considering in addition to the two areas above, the area of the quadrilateral $OPRQ$ which is greater than the area of the sector $POQ$. Here the point $R$ is obtained by drawing lines parallel to the axes through $P$ and $Q$ and choosing the point of intersection lying outside the circle (see figure). Then on removing, say, $\Delta x/\Delta\theta$ from the expression $\frac{1}{2}(x\Delta y/\Delta\theta - y\Delta x/\Delta\theta)$ by use of equation (1), the limit $dy/d\theta$, then $dx/d\theta$, and finally (3) are readily established.

To apply this to the determination of the differentiation formulas we also need the relation

$$(4) \qquad x \frac{dx}{d\theta} + y \frac{dy}{d\theta} = 0,$$

which is obtained by differentiating the equation (1) 'implicitly.'

We then find on eliminating $dx/d\theta$ from (3) and (4)

$$(x^2 + y^2)dy/d\theta = a^2x$$

or

$$(5) \qquad dy/d\theta = x.$$

Similarly

$$dx/d\theta = -y.$$

These last two expressions yield the following derivatives on substituting (2)

in them:

$$\frac{d}{d\theta}(a \sin\theta) = x = a\cos\theta$$

or

$$\frac{d}{d\theta}\sin\theta = \cos\theta$$

and similarly

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

If we now consider $\theta$ as a function of $y$, namely,

$$\theta = \arcsin y/a,$$

we can rewrite (5) as

$$\begin{aligned}\frac{d\theta}{dy} = 1/x &= \frac{1}{\sqrt{a^2-y^2}} && -\frac{\pi}{2} < \theta < \frac{\pi}{2}\\ &= \frac{-1}{\sqrt{a^2-y^2}} && \frac{\pi}{2} < \theta < \frac{3\pi}{2}.\end{aligned}$$

We thus find

$$\begin{aligned}\frac{d}{dy}\arcsin y/a &= \frac{1}{\sqrt{a^2-y^2}} && -\frac{\pi}{2} < \theta < \frac{\pi}{2}\\ &= \frac{-1}{\sqrt{a^2-y^2}} && \frac{\pi}{2} < \theta < \frac{3\pi}{2}.\end{aligned}$$

The formulas for the differentiation of the remaining functions can be found in a similar manner.

*Editorial Note.* Another "unorthodox" derivation of these formulae is the following (essentially due to F. D. Murnaghan, this MONTHLY, vol. 53 (1946) p. 424). Consider the unit circle, and let arc length be measured positively in a counterclockwise direction from the point (1, 0). Then the angle $\theta$ (see above figure) is equal to the arc length $s$. But from the arc length formula:

$$ds = -e\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{1}$$

where $e=+1$ when $\theta$ is in the first or second quadrant and $e=-1$ when $\theta$ is in

the third or fourth quadrant. Now since $x=\cos\theta$ and $y=\sin\theta$, and $x^2+y^2=1$;

$$\frac{dy}{dx} = -\frac{x}{y} = -\cot\theta.$$

So from (1)

$$\begin{aligned} d\theta &= -e\sqrt{1+\cot^2\theta}\,d(\cos\theta) \\ &= -e\sqrt{\csc^2\theta}\,d(\cos\theta). \end{aligned} \tag{2}$$

But $\sqrt{csc^2\theta}=e\ csc\ \theta$; so

$$d\theta = -e^2\csc\theta d(\cos\theta) = -\csc\theta d(\cos\theta); \quad \text{or}$$

$$\frac{d(\cos\theta)}{d\theta} = -\sin\theta.$$

A similar argument gives the formula for the derivative of $\sin\theta$.

The chief point to notice is that in every derivation some "work" must be done by way of a limiting process. This "work" may be introduced in any of a number of places, but it can not be escaped. The objective, then, is to make this process as simple and as intuitive as possible. It is on this basis that the relative merit of various derivations should be judged. In the proof just given, advantage is taken of the customary derivation of the formula for the differential of arc length, and this enables one to get the desired result without any additional limit process. Whether or not this procedure is too "slick" is a matter which each teacher will have to decide for himself.

## A TREATMENT OF BONDS BETWEEN INTEREST DATES

P. M. HUMMEL and C. L. SEEBECK, JR., University of Alabama

Since the flat price of a bond fluctuates according to the nearness of the forthcoming interest date, it has long been customary to quote bonds at an *and-interest price.* Consequently a purchaser pays the *quoted price* plus that part of the current bond interest payment which has already been earned. Thus if $Q$ is the quoted price, $R'$ the accrued bond interest, and $P$ the flat price, then

$$P = Q + R'. \tag{1}$$

In practice, the accrued bond interest is usually computed by the simple formula (or its equivalent)

$$R' = fR, \tag{2}$$

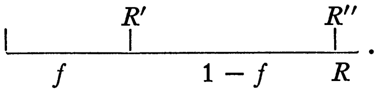
where $R$ is the periodic bond interest payment and $f$ is the fractional part of the interest period that has elapsed since the preceding interest date.

When a bond is bought between interest dates it is customary to consider

the purchase as consisting of two transactions, namely: (1) A loan of $R'$ to the seller which will be repaid, without interest, out of the first bond interest payment, and (2), the purchase of a bond with a present value of $Q$ and a first bond interest payment of $R-R'$. If $C$ is the redemption value, an investment schedule is then set up to amortize (or accumulate) the excess, $Q-C$ (or the deficiency, $C-Q$).

The practical treatment outlined above leaves two important questions unanswered, namely: (1) If the compound interest law is used, what is the accrued bond interest? (2) Should not the investment schedule be made out for the flat price rather than the quoted price?

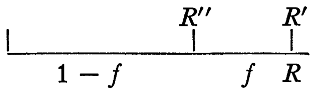
To clarify these issues we first develop a formula for $R'$ based on the compound interest law. Clearly when the quoted price of a bond is given, the yield rate, $i$, which the bond will furnish is established. Consequently the forthcoming bond interest payment should be divided between the buyer and the seller in an equitable manner consistent with the compound interest law at the established rate $i$. To this end let $R'$ be the seller's share, due on the date of sale, and let $R''$ be the buyer's share, due on the forthcoming interest date. These two shares must be equivalent to the bond interest payment due at the end of the interest period as shown on the time diagram below.




An equation of equivalence taken at the end of the interest period gives

$$R'(1+i)^{1-f} + R'' = R.$$

Since both $R'$ and $R''$ are unknown, we need another relation between them in order to determine their values. A second relation is obtained by symmetry. Since the determination of $R'$ and $R''$ is to be equitable to both the buyer and the seller, interchanging the two times, $f$ and $1-f$, would likewise interchange $R'$ and $R''$ as shown on the following time diagram.




An equation of equivalence now gives

$$R' + R''(1+i)^f = R.$$

We now solve these last two equations for $R'$ and get

$$R' = R\frac{(1+i)^f - 1}{i} = Rs_{\overline{f}|i}. \tag{3}$$

In a similar manner, $R'' = Rs_{\overline{1-f}|i}$.

It is easily shown that $s_{\overline{f}|i}$ is always slightly less than $f$, so that the practical

determination of $R'$ is slightly in favor of the seller. Also, (3) disproves the statement made in some texts, that the accrued bond interest is $fR$ but that it is not due until the end of the period, since this statement implies that $R'=fR(1+i)^{1-f}$ which is not identical with (3).

Since (3) is an exact formula for $R'$, it is now possible to develop an exact formula for $Q$. We start with the well known formula

$$P = P_0(1 + i)^f, \tag{4}$$

where $P_0$ is the value of the bond on the preceding interest date and may be computed from the standard bond formula

$$P_0 = C(1 + i)^{-n} + Ra_{\overline{n|}i}. \tag{5}$$

If (1), (3), (4), and (5) are combined, one readily obtains

$$Q = C(1 + i)^{-(n-f)} + Ra_{\overline{n-f|}i}. \tag{6}$$

Thus we see that the exact formula for $Q$ is the standard bond formula (5) with $n$ replaced by $n-f$. This means that $Q$ is the present value, on the date of purchase, of the redemption value plus the present value of *future interest earnings* and does not include that part of the current bond interest payment which has already accrued. Consequently the total purchase price should be $Q+R'$. Since (5) and (6) have the same form, it can be said that (5) is a formula for the quoted price of a bond which is valid whether $n$, the number of interest periods until maturity, is an integer or not.

In actual practice, $Q$ is usually given whereas only an approximate value of $i$ is known. Consequently the exact formulas developed here are of little practical value. However, these exact formulas justify the common accounting practice of treating $Q$ and $R'$ as separate transactions. Therefore when the buyer of a bond considers $R'$ as a temporary loan which will be repaid out of the first bond interest payment, and treats $Q$ as the book value of the bond that is being purchased and consequently sets up an investment schedule with $Q$ as the original value of the bond, the procedure is theoretically correct.

Finally it should be noted that the approximate formulas used in the practical treatment are obtainable from the exact formulas developed here by replacing $(1+i)^f$ by $1+if$, and $a_{\overline{f|}i}$ by $f$. Since $f$ is a fraction it can be shown that these replacements usually differ very little from the exact values.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 786. *Proposed by Michael Goldberg, Washington, D.C.*

Suppose that an equilateral triangle is circumscribed about a regular $n$-gon, where $n=3k\pm 1$, so that one side of the $n$-gon lies on one of the sides of the triangle. Show that the angle subtended by this side of the $n$-gon at the opposite vertex of the triangle is $2\pi/3n$.

E 787. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle $ABC$, show that

$$s^4 + (s-a)^4 + (s-b)^4 + (s-c)^4 - a^4 - b^4 - c^4 = 12S^2,$$

where $a$, $b$, $c$ are the sides, $s$ the semi-perimeter, and $S$ the area.

E 788. *Proposed by Leo Moser, University of Manitoba*

Consider a map on a spherical surface where the countries are determined by $n$ great circles of which no three are concurrent. Show that if $n$ is a multiple of four it is impossible to make a trip visiting each country once and only once, if travelling along a boundary or crossing at a boundary point of more than two countries is forbidden.

E 789. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If $y=\tan x$, show that

$$(d^n y/dx^n)\cos^{n+1} x$$
$$= \begin{vmatrix} \cos x & 0 & \cdots \sin x \\ \cos(x+\pi/2) & \cos x & \cdots \sin(x+\pi/2) \\ \cos(x+2\pi/2) & 2\cos(x+\pi/2) & \cdots \sin(x+2\pi/2) \\ \cdot & \cdot & \cdots \quad \cdot \\ \cos(x+n\pi/2) & n\cos(x+\overline{n-1}\,\pi/2) & \cdots \sin(x+n\pi/2) \end{vmatrix}.$$

E 790. *Proposed by H. S. Wall, University of Texas*

Let

$$f(x, y) = \frac{a + xy(x^{n-2} + x^{n-3}y + x^{n-4}y^2 + \cdots + y^{n-2})}{x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}},$$

where $n$ is an integer greater than unity, and $a>0$. If $x_0>0$, $x_1>0$, and $x_2$, $x_3$, $x_4$, $\cdots$ are computed recurrently by means of the formula

$$x_{p+2} = f(x_p, x_{p+1}), \qquad p = 0, 1, 2, \cdots$$

then

$$\lim_{p=\infty} x_p = a^{1/n}.$$

## SOLUTIONS

### A Conjecture by Srinivasan

E 755 [1947, 39]. *Proposed by Alfred Brauer, University of North Carolina*

Let $a_1$, $a_2$, $a_3$, $a_4$ be relatively prime integers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 = 0. \tag{1}$$

Let $\alpha_\nu(\nu=1, 2, 3, 4)$ by the smallest non-negative residue of $a_\nu$ (mod 6). Then

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \pmod 6. \tag{2}$$

In this paper, Residual Types of Partitions of 0 into Four Cubes (*The Mathematics Student*, vol. 13, 1945, pp. 47–48), A. K. Srinivasan tries to find solutions of (1) for each set of numbers $\alpha_\nu$ satisfying (2). For instance, for $\alpha_1=\alpha_2=0$, $\alpha_3=1$, $\alpha_4=5$, he gives the solution $a_1=12$, $a_2=-54$, $a_3=19$, $a_4=53$. In the following cases he did not succeed in finding examples:

$$\begin{cases} \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 4; \\ \alpha_1 = \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 3; \\ \alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 3, \quad \alpha_4 = 5; \end{cases} \tag{3}$$

and in the cases obtained from (3) if each $\alpha_\nu$ is replaced by $6-\alpha_\nu$. He conjectures that these cases are impossible. Prove that this conjecture is true.

*Solution by the Proposer.* Let

$$a_\nu = 6q_\nu + \alpha_\nu \qquad (\nu = 1, 2, 3, 4).$$

Then

$$a_\nu^3 = 216q_\nu^3 + 108q_\nu^2\alpha_\nu + 18q_\nu\alpha_\nu^2 + \alpha_\nu^3 \equiv \alpha_\nu^3 \pmod 9.$$

Now

$$\begin{aligned} \alpha_\nu^3 &\equiv 0 \pmod 9 \quad \text{if} \quad \alpha_\nu \equiv 0 \pmod 3, \\ \alpha_\nu^3 &\equiv 1 \pmod 9 \quad \text{if} \quad \alpha_\nu \equiv 1 \pmod 3, \\ \alpha_\nu^3 &\equiv 8 \pmod 9 \quad \text{if} \quad \alpha_\nu \equiv 2 \pmod 3. \end{aligned}$$

Since

$$\sum_{\nu=1}^{4} \alpha_\nu^3 \equiv \sum_{\nu=1}^{4} a_\nu^3 \equiv 0 \pmod 9,$$

the cases given in the problem are impossible, because in these cases

$$\sum_{\nu=1}^{4} \alpha_\nu^3 \equiv 3 \text{ or } 6 \pmod 9.$$

Also solved by R. A. Bradley and D. H. Browne. Browne employed the general three-parameter solution of (1) as given by Euler and later simplified by Binet. The Euler-Binet formulas may be found in Carmichael, *Diophantine Analysis* (1915), pp. 62–66, or in Hardy and Wright, *An Introduction to the Theory of Numbers* (1938), pp. 198–203.

**Expansion of a Determinant**

E 756 [1947, 107]. *Proposed by G. Pólya, Stanford University*

Show that

$$\begin{vmatrix} a-x & 1 & 0 & 0 & \cdots & 0 \\ \binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\ \binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \end{vmatrix}$$

$$= \binom{a+n-1}{n} - \binom{2a+n-2}{n-1}x + \binom{3a+n-3}{n-2}x^2 - \cdots + (-1)^n x^n.$$

*Solution by the Proposer.* 1. We consider first the general problem: Being given $a_0, a_1, a_2, \cdots$, to find $u_0, u_1, u_2, \cdots$ so that for indeterminate $t$

$$(a_0 + a_1t + a_2t^2 + \cdots)(u_0 + u_1t + u_2t^2 + \cdots) = 1.$$

We have

$$\begin{aligned} &a_0u_0 &&= 0, \\ &a_1u_0 + a_0u_1 &&= 0, \\ &a_2u_0 + a_1u_1 + a_0u_2 &&= 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_nu_0 + a_{n-1}u_1 + a_{n-2}u_2 + \cdots + a_0u_n &&= 0. \end{aligned}$$

Hence, by Cramer's rule,

$$(-1)^n a_0^{n+1} u_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots 0 \\ a_2 & a_1 & a_0 & 0 & \cdots 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots \cdot \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots a_1 \end{vmatrix}.$$

2. Our problem is the special case in which

$$a_0 = 1, \qquad a_1 = a - x, \qquad a_n = \binom{a}{n} \qquad \text{for } n \geqq 2.$$

In order to evaluate the determinant we have to pick out $u_n$, the coefficient of $t^n$, in the expansion of

$$\frac{1}{(1+t)^a - xt} = \frac{(1+t)^{-a}}{1 - xt(1+t)^{-a}}$$
$$= (1+t)^{-a} + xt(1+t)^{-2a} + x^2t^2(1+t)^{-3a} + \cdots .$$

Thus

$$u_n = \binom{-a}{n} + x\binom{-2a}{n-1} + x^2\binom{-3a}{n-2} + \cdots .$$

Observing that

$$\binom{-a}{n} = \frac{-a(-a-1)\cdots(-a-n+1)}{n!} = (-1)^n\binom{a+n-1}{n},$$

we obtain the proposed equation.

*Editorial Note.* Many remarkable identities may be obtained from this expansion by substituting particular values for $x$ and $a$, *e.g.*, $x=0$, 1, $-1$, $a$ and $a=n$.

### Volume of Cylindrical Wedge by Cavalieri's Theorem

E 757 [1947, 107]. *Proposed by Kirkland Stewart, College of Puget Sound*

A wedge is cut from a right circular cylinder by an oblique plane passing through a diameter of the base of the cylinder. Find the volume of the wedge using Cavalieri's theorem.

*Solution by H. E. Fettis, Dayton, Ohio.* Let the wedge be divided into two equal parts by a plane $M$ through the axis of the cylinder, and let $A$ be the area of the resulting triangular cross-section of the wedge. Construct a prism having as its base a square of side $a$, such that $a^2 = A$, the base lying in the plane $M$, and having an altitude equal to the radius $r$ of the cylinder. Cut from this prism a pyramid whose base is the base of the prism not lying in $M$, and whose vertex

is a point within the square $A$. If $B$ and $B'$ are the areas of cross-section of these two solids made by any plane parallel to $M$ and at a distance $x$ from $M$, then $B$ and $A$ are areas of similar triangles, and

$$B/A = (r^2 - x^2)/r^2.$$

Also,

$$B' = a^2 - (ax/r)^2 = a^2(1 - x^2/r^2) = A(r^2 - x^2)/r^2.$$

Therefore $B=B'$ for all values of $x$, so that the half wedge is, by Cavalieri's theorem, equal to the volume of the prism less the volume of the removed pyramid, which in turn equals

$$a^2r - a^2r/3 = 2a^2r/3.$$

Thus the volume of the wedge is equal to

$$4a^2r/3 = 2r^2h/3,$$

where $h$ is the length of the longest element of the wedge.

Also solved by Elmer Latshaw and the proposer.

*Editorial Note.* Often, in elementary courses in solid geometry, simplicity of treatment and wider scope in application are gained by assuming Cavalieri's volume theorem. For courses where this is done the above problem makes a fine exercise.

It is well known that the volume of a sphere of radius $r$ can very readily be obtained by Cavalieri's theorem by using for the comparison solid a right circular cylinder of radius $r$ and altitude $2r$, with two cones removed having their bases coinciding with those of the cylinder and having common vertex at the center of the cylinder. A nice exercise is to produce, in this case, a polyhedron which may serve as the comparison solid.

**The Kite**

E 758 [1947, 107]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A *kite* consists of the area bounded by a major arc of a circle of radius $r$ and the two tangents drawn at the end points of the arc. Show that (1) the area of the kite is equal to half the product of its perimeter by the radius $r$, (2) $Og/OG=3/2$, where $g$ and $G$ are the centroids of the perimeter and area, and $O$ is the center of the kite's arc, (3) $Og'/OG'=4/3$, where $g'$ and $G'$ are the centroids of the surface and volume of the solid of revolution obtained by revolving the kite about its axis, (4) the plane through $G'$ perpendicular to the axis bisects the lateral area of the solid, (5) the volume of the solid is equal to one third the product of its surface by the radius $r$.

*Solution by the Proposer.* Let $A$, $B$, $M$ be the endpoints and the midpoint, respectively, of the arc of the kite, $S$ the foot (where the tangents meet) of the kite, and $2(\pi-\theta)$ the central angle of the arc of the kite.

*Note.* The relations of parts (2) and (3) conform with results given by Brassine, *Journal de Liouville*, 1843, in a study of polygons circumscribed to a circle and polyhedra circumscribed to a sphere.

### The Difference of Two Squares

E 759 [1947, 223]. *Proposed by Theodore Running, Ann Arbor, Michigan*

Show that $x^n-(x-a)^n$ can be expressed as the difference of two squares in at least one way for all positive integral values of $n$, $x$, and $a$, $a$ less than $x$, not counting the obvious way when $n$ is even.

*Solution by six men from Athens, Georgia, and Columbia, South Carolina.* Exercise 11, page 83, in *Elementary Number Theory*, by Uspensky and Heaslet, states that

(1) *An integer N can be represented as a difference of two squares if it is either odd or divisible by* 4, *otherwise not. The representation is unique if and only if N is an odd prime number.*

Now if $N=x^n-(x-a)^n$, the values $n$, $x$, $a=1$, 3, 2 yield $N=2$, which is not a difference of squares. Further, $n$, $x$, $a=2$, 2, 1 yield $N=3$, a prime which has only the "obvious" representation as a difference of squares. We conclude that the problem as stated is incorrect, and needs to be modified as follows:

(2) *If $x$ and $n$ are odd and $a=4k+2$, then $N=x^n-(x-a)^n$ is not a difference of squares, but if one of these fails then N is indeed a difference of squares. If $n$ is even and greater than* 2, *there is at least one such expression not counting the obvious one.*

The first statement is established by canvassing the cases and observing that that $N=4k+2$ if and only if $x$, $n$ are odd and $a=4k+2$. If $n$ is even and greater than 2, $N$ contains $x^2-(x-a)^2=a(2x-a)>1$ as a factor and is not a prime; by (1) it can be written as a difference of two squares in at least two ways and at least one must be different from the "obvious" one. The restriction $n>2$ is indeed essential since every odd prime may be expressed as a difference of two squares in one and only one way.

*Remark.* The probability that an example constructed at random for $n>2$ would verify the original statement is 15/16.

Also solved by Murray Barbour, Louis Berkofsky, D. H. Browne, Paul Brock, R. E. Crane, Monte Dernham, W. P. DeWitt, N. J. Fine, L. M. Kelly, C. F. Pinzka, Joseph Rosenbaum, and the proposer. A number of these solutions were not complete.

Fine proved the theorem: *Let N be an odd positive integer, R(N) the number of representations $N=u^2-v^2$ $(u>0,\ v>0)$, and d(N) the number of (positive) divisors of N. Then*

$$\begin{aligned} R(N) &= \tfrac{1}{2}d(N), \text{ if } N \text{ is not a square,} \\ &= \tfrac{1}{2}(d(N)-1), \text{ if } N \text{ is a square.} \end{aligned}$$

**A Faltung**

E 760 [1947, 108]. *Proposed by C. D. Olds, San Jose State College*

If

$$u_k = \frac{2\cdot 6\cdot 10 \cdots (4k-2)}{2\cdot 3\cdot 4\cdot 5 \cdots (k+1)},$$

find the value of

$$u_n + u_1u_{n-1} + u_2u_{n-2} + \cdots + u_{n-1}u_1 + u_n.$$

*Solution by D. H. Lehmer, Berkeley, California.* If we set $u_0=1$, then the problem is equivalent to that of finding the coefficient of $x^n$ in the square of the function

$$y = \sum_{n=0}^{\infty} u_n x^n.$$

It is easily verified that

$$y = \{1 - (1-4x)^{1/2}\}/2x,$$

so that

$$y^2 = (y-1)/x = \sum_{n=0}^{\infty} u_{n+1}x^n.$$

Hence the answer to the problem is simply $u_{n+1}$.

Also solved by Joshua Barlaz, Paul Brock, N. J. Fine, William Gustin, H. D. Larsen, F. C. Smith, F. Underwood, and the proposer.

Gustin pointed out that the integer $u_n$ is the number of ways that a sequence of $n+1$ elements may be bracketed under a binary composition. For example, we have the following $u_3=5$ bracketing combinations of a sequence of $3+1=4$ elements: $(((\cdot\cdot)\cdot)\cdot)$, $((\cdot(\cdot\cdot))\cdot)$, $((\cdot\cdot)(\cdot\cdot))$, $(\cdot((\cdot\cdot)\cdot))$, $(\cdot(\cdot(\cdot\cdot)))$.

The problem suggested to Barlaz the associated one of finding the most general function $y=\sum_0^\infty a_nx^n$ such that $y^2=\sum_0^\infty a_{n+1}x^n$. He easily showed that

$$y = \{1 \pm (1-4a_0x)^{1/2}\}/2x.$$

The function $y$ of the above problem is the case where we have the negative sign and $a_0=1$.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4264. *Proposed by G. Pólya, Stanford University*

Given $a>0$, $b>0$, and given that $f(x)$ is a non-linear function such that $f(0)=0$, $f(a)=b$ and that

$$f(x) \geqq 0 \qquad f''(x) \geqq 0 \qquad\qquad 0 \leqq x \leqq a,$$

give an analytic proof that

$$2\pi\int_0^a f(x)[1+(f'(x))^2]^{1/2}dx < \pi b(a^2+b^2)^{1/2}.$$

(The inequality becomes intuitive when both sides are interpreted as areas of curved surfaces.)

4265. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If two tetrahedra are homothetic with respect to their common centroid, the twelve point sphere of one of these tetrahedra is tangent to the twelve point spheres of the four tetrahedra which the planes of its faces cut off from the trihedral angles of the other tetrahedron.

4266. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a sphere $(S)$. If the polar planes of the vertices, $A$, $B$, $C$, $D$ with respect to $(S)$ and the corresponding planes tangent to the circumsphere at $A$, $B$, $C$, $D$ cut each other, respectively, on the faces $BCD$, $CDA$, $DAB$, $ABC$, the tetrahedron is orthocentric. Establish a converse theorem.

4267. *Proposed by C. F. Pinzka, Student, Rutgers University*

Let $p$ be a prime greater than 3, and let $r/ps$ be the sum of the harmonic series, $1+1/2+1/3+\cdots$, to $p$ terms. Prove that $p^3$ divides $r-s$.

4268. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers of upper density greater than $1/k$. (Denote by $f(n)$ the number of $a$'s up to $n$, then the upper density is

defined as $\overline{\lim} f(n)/n$ as $n \to \infty$.) Then for suitable $t$ the equation

$$a_t = a_{i_1} + a_{i_2} + \cdots + a_{i_r} \qquad 1 < r < k$$

is solvable. In fact, there are infinitely many $t$ with this property.

4269. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all numbers $N = abcd$ of four distinct digits, zero excluded, such that the sum

$$ab + ac + ad + bc + bd + cd$$

of products of the digits two at a time shall equal the sum

$$a^2 + b^2 + c^2 + d^2$$

of the squares of the digits. For which among these is it true that the sum of the two-digit numbers $ab$ and $cd$ equals

$$a^2 + c^2 + d^2?$$

## SOLUTIONS

### A Special Sphere of the Tetrahedron

3987 [1941, 152]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The spheres $S_a$, $S_b$, $S_c$, $S_d$ have their respective centers at the vertices of the tetrahedron $ABCD$, and radii whose squares are one-half the sum of the squares of the sides of the face opposite to the vertex considered; and the spheres $S_a'$, $S_b'$, $S_c'$, $S_d'$ have centers at $A_1$, $B_1$, $C_1$, $D_1$, symmetrics of $A$, $B$, $C$, $D$ with respect to $G$ the centroid of the tetrahedron, and radii $A_1A$, $B_1B$, $C_1C$, $D_1D$. Let $C_a$ be the intersection of $S_a$ and $S_a'$, and similarly for $C_b$, $C_c$, $C_d$. Prove that: (1) The four circles $C_a$, $C_b$, $C_c$, $C_d$ lie upon the same sphere $\Sigma$ with its center at $G$ and passing through the intersection of the Longchamps sphere with the anticomplementary sphere of the circumsphere of $ABCD$. (2) Show that $\Sigma$ is the Monge sphere for the Steiner ellipsoid circumscribing $ABCD$.

*Note.* The Longchamps sphere is orthogonal to the spheres $S_a$, $S_b$, $S_c$, $S_d$ See N. A. Court, *L'Enseignement Mathématique*, Geneva, 1930, pp. 31–34; V. Thébault, loc. cit., 1937, pp. 81–89. The anticomplementary sphere of the sphere $(ABCD)$ is the circumsphere of the tetrahedron formed by planes through the vertices of $ABCD$ parallel to the respective opposite faces.

*Solution by the Proposer.* If the lengths of the edges $BC$ and $DA$, $CA$ and $DB$, $AB$ and $DC$ are designated by $a$ and $a'$, $b$ and $b'$, $c$ and $c'$, then the squares of the radii of the spheres $S_a$, $S_b$, $S_c$, $S_d$, are

$$\frac{a^2 + b'^2 + c'^2}{2}, \quad \frac{b^2 + c'^2 + a'^2}{2}, \quad \frac{c^2 + a'^2 + b'^2}{2}, \quad \frac{a^2 + b^2 + c^2}{2}.$$

The radii $A_1A$, $B_1B$, $C_1C$, $D_1D$ of the spheres $S'_a$, $S'_b$, $S'_c$, $S'_d$ are $3M_a/2$, $3M_b/2$, $3M_c/2$, $3M_d/2$, where $M_a$, $M_b$, $M_c$, $M_d$ are the medians of the tetrahedron $ABCD$.

Let $P$ be an arbitrary point on the intersection $C_d$ of the spheres $S_d$ and $S'_d$. The theorem concerning the square of the median, applied to the triangle $DD_1P$, gives

$$\overline{PD}^2 + \overline{PD_1}^2 = 2\overline{GP}^2 + \tfrac{1}{2}\overline{DD_1}^2,$$

or, by virtue of a known expression for $M_d^2$,*

$$\overline{GP}^2 = 3(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/16 = \sigma^2.$$

This relation shows that the circumferences $C_a$, $C_b$, $C_c$, $C_d$ are on a sphere $\Sigma$ with center $G$ and with radius $\sigma$. That this is the Monge sphere for the Steiner ellipsoid circumscribing $ABCD$ has been shown.†

Since further, the sphere $\Sigma$, the Longchamps sphere for the tetrahedron $ABCD$ and the anticomplementary sphere of the sphere $(ABCD)$ form a pencil‡ the demonstration of the theorem is complete.

### A Feuerbach Point Theorem

4059 [1942, 617]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $D$, $E$, $F$ be the points of contact of the inscribed circle $(I)$ with the sides $BC$, $CA$, $AB$ of triangle $ABC$, and $A'$, $B'$, $C'$ the feet of its altitudes. Show that the distances of the points of intersection of the pairs of straight lines such as $B'C'$, $EF$ from the radical axis of $(I)$ and the nine-point circle of triangle $ABC$ are inversely proportional to the distances of the Feuerbach point from the feet of the altitudes.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.*§ Let $ABC$ be taken as the reference triangle for a system of normal trilinear coördinates $(x, y, z)$. Also let $a$, $b$, $c$ denote the lengths of the sides $BC$, $CA$, $AB$; $2p=a+b+c$ the perimeter, $S$ the area, $R$ the radius of the circumcircle, and $r$ the radius of the incircle.

The equations of $B'C'$ and $EF$ are respectively

$$-x\cos A + y\cos B + z\cos C = 0, \quad -a(p-a)x + b(p-b)y + c(p-c)z = 0,$$

and the coördinates (absolute) of their intersection $M$ are

$$x = \frac{2S(c-b)(p-a)}{bc(c-b)}, \qquad y = \frac{2S(c-a)(p-b)}{bc(c-b)}, \qquad z = \frac{2S(a-b)(p-c)}{bc(c-b)}\cdot$$

* V. Thébault, this MONTHLY, 1935, p. 429.

† V. Thébault, L'Enseignement Mathématique, 1937, p. 98.

‡ V. Thébault, ibid.

§ Translated and checked by W. E. Byrne, Virginia Military Institute, Lexington, Virginia.

The required radical axis is tangent to the incircle $(I)$ at the Feuerbach point $\phi$, and its equation is given by

$$\frac{ax}{b-c}+\frac{by}{c-a}+\frac{cz}{a-b}\equiv ux+vy+wz=0.^*$$

The distance from $M$ to this line is

$$\overline{MM'}=\frac{4S(p-b)(p-c)}{bc\,|\,b-c\,|\sqrt{\sigma}}$$

with

$$\sigma=F(u,v,w)\equiv u^2+v^2+w^2-2vw\cos A-2wu\cos B-2uv\cos C.$$

The distance from the incenter $I$ to this radical axis is

$$\frac{r\left|\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}\right|}{\sqrt{\sigma}}=r,$$

so that

$$\sqrt{\sigma}=\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\,.$$

But†

$$\overline{\phi A'}=\frac{|b-c|(p-a)\sqrt{R}}{a\sqrt{R-2r}},$$

and therefore

$$\overline{MM'}\cdot\overline{\phi A'}=\frac{4S(p-a)(p-b)(p-c)}{abc\sqrt{\sigma(1-2r/R)}}=\frac{Sr}{\sqrt{\sigma(R^2-2rR)}},$$

which establishes the proposition.

It has been assumed throughout that $(a-b)(b-c)(c-a)\neq 0$.

**An Envelope of the Third Class**

4154 [1945, 220]. *Proposed by H. F. Sandham, Trinity College, Dublin*

Find the envelope of the axes of conics inscribed in a quadrilateral.

*Solution by the Proposer.* Let $PP'=0$, $QQ'=0$ be the tangential equations of two pairs of opposite vertices of the quadrilateral, and $IJ=0$ the equation of

* See Salmon, *Conic Sections*, §131.

† Victor Thébault, Ann. Soc. Scient. de Bruxelles, 1932, p. 3.

the circular points at infinity. The equation of any inscribed conic is then $\lambda PP'+\mu QQ'=0$. For a suitable $\nu$, the foci of this conic are given by $\Sigma\equiv\lambda PP'+\mu QQ'+\nu IJ=0$. If $r$, $s$, $t$ are the coördinates of the line joining the foci, then $\partial\Sigma/\partial r=\partial\Sigma/\partial s=\partial\Sigma/\partial t=0$. Hence, performing these differentiations and eliminating $\lambda$, $\mu$, $\nu$ from the resulting three equations, we obtain

$$\begin{vmatrix} P(\partial P'/\partial r)+P'(\partial P/\partial r) & Q(\partial Q'/\partial r)+Q'(\partial Q/\partial r) & I(\partial J/\partial r)+J(\partial I/\partial r) \\ & \textit{etc.} & \\ & \textit{etc.} & \end{vmatrix}=0$$

as the required envelope.

The curve is of the third class and is the Jacobian of $PP'=QQ'=IJ=0$. It is at once verified that it touches once each the lines joining the opposite vertices, and the line at infinity.

There are three tangents to the curve from any point. One of the tangents from $I$ is the line at infinity, so that there are just two others. These must be the axes which pass through $I$ of conics inscribed in the quadrilateral. Since the axes of a conic which passes through a circular point coincides with the tangent there, these axes must be the tangents at $I$ to the two inscribed conics which pass through $I$. Now tangents from $I$ to these conics are pairs in an involution of which the tangents to the two conics passing through $I$ are double lines. Since tangents from a circular point to a curve have on each of them one real point which is a real focus of the curve, the above argument shows that the two real foci of the envelope are the double points of the complex involution to which belong the foci of inscribed conics.

Hence, to sum up: The axes of conics inscribed in a quadrilateral envelop a curve of the third class which is the Jacobian of two pairs of opposite vertices and the circular points, and which touches just once the diagonals of the quadrilateral and the line at infinity. The two real finite foci are the double points of the complex involution to which belong the foci of inscribed conics.

*Note.* The corollary that the foci of conics inscribed in a quadrilateral are pairs in a complex involution seems to be new, though the case of the degenerate point pairs is given in Morley's *Inversive Geometry.*

**Four Spheres in a Tetrahedron**

4160 [1945, 281]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show how to construct four spheres passing through a given point and tangent respectively to the planes of three faces of a given tetrahedron so that the points of contact are twelve points of the same sphere.

*Solution by the Proposer.** In a tetrahedron $T\equiv ABCD$, the inscribed sphere $(I, r)$, of center $I$ and radius $r$, touches the faces $BCD$, $CDA$, $DAB$, $ABC$, at $A'$, $B'$, $C'$, $D'$. A sphere $(I, \rho)$, of arbitrary radius $\rho>r$, concentric to $(I, r)$, in-

* Translation by W. E. Byrne, Virginia Military Institute.

tersects these same faces in equal circles $(A')$, $(B')$, $(C')$, $(D')$, which the rays $(A'B, A'C, A'D)$, $(B'C, B'D, B'A)$, $(C'D, C'A, C'B)$, $(D'A, D'B, D'C)$ meet in the points $(X_b, X_c, X_d)$, $(Y_c, Y_d, Y_a)$, $(Z_d, Z_a, Z_b)$, $(V_a, V_b, V_c)$. The triangles $X_bX_cX_d$, $Y_cY_dY_a$, $Z_dZ_aZ_b$, $V_aV_bV_c$ are equal since their circumscribed circles are equal and the angles $BA'C$, $CA'D$, $DA'B$ formed by the lines which join the vertices of $T$ to the points of contact of the sphere $(I, r)$ on the planes of the faces are the same for the four faces. [1], [2].

The planes $Y_aZ_aV_a, X_bZ_bV_b, X_cY_cV_c, X_dY_dZ_d$ form a tetrahedron $T_1 \equiv A_1B_1C_1D_1$ homothetic to the tetrahedron $T' \equiv A'B'C'D'$. The planes of the faces of $T$, which are perpendicular to the radii $IA'$, $IB'$, $IC'$, $ID'$ of the sphere $(I, r)$ circumscribed about tetrahedron $T'$, are anti-parallel to the opposite faces of $T_1$ in the trihedral angles $(A_1)$, $(B_1)$, $(C_1)$, $(D_1)$. The planes of the faces of $T$ meet the edges of $T_1$ at the vertices of the tangential triangles $X_b'X_c'X_d'$, $Y_c'Y_d'Y_a'$, $Z_d'Z_a'Z_b'$, $V_a'V_b'V_c'$ of the equal triangles $X_bX_cX_d$, $Y_cY_dY_a$, $Z_dZ_aZ_b$, $V_aV_bV_c$. These tangential triangles are also equal. The homothetic center of the tetrahedron $T$ and the tangential tetrahedron $T_1'$ of $T_1$ is the point $L$ whose distances from the planes of the faces of $T_1'$ are proportional to the radii of the circles circumscribed about these faces [3], [4]. This point $L$, the second Lemoine point of $T_1'$, is also the homothetic center of tetrahedrons $T'$ and $T_1$ [5]. Hence with respect to tetrahedron $T_1$ the sphere $(I, \rho)$ belongs to a system of Tucker spheres of axis $O_1L$, $O_1$ being the circumcenter of tetrahedron $T_1$.

If we construct spheres $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$, inscribed in the trihedral angles $(A)$, $(B)$, $(C)$, $(D)$ of $T$ and tangent respectively to the three adjacent faces at $(Y_a, Z_a, V_a)$, $(X_b, Z_b, V_b)$, $(X_c, Y_c, V_c)$, $(X_d, Y_d, Z_d)$, the points $A'$ and $A_1$, $B'$ and $B_1$, and $C'$ and $C_1$, $D'$ and $D_1$ belong to the radical axes of the spheres $[(\omega_b), (\omega_c), (\omega_d)]$, $[(\omega_c), (\omega_d), (\omega_a)]$, $[(\omega_d), (\omega_a), (\omega_b)]$, $[(\omega_a), (\omega_b), (\omega_c)]$. The radical center of the spheres $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ is therefore the point $L$ which remains fixed when the radius $\rho$ of the sphere $(I, \rho)$ varies. The spheres $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ are orthogonal to a sphere $(L)$, of center $L$, which reduces to a point when $(\omega_a) \cdots$, pass through $L$.

Conversely, if a sphere $(L)$ is given, the spheres $(O_a)$, $(O_b)$, $(O_c)$, $(O_d)$, orthogonal to $(L)$ and inscribed respectively in the trihedral angles $(A)$, $(B)$, $(C)$, $(D)$, are tangent to the adjacent faces in twelve points situated on a sphere concentric to the inscribed sphere $(I, r)$ of tetrahedron $T$. As a special case, when the sphere $(L)$ is a point sphere, the spheres $(\omega_a'')$, $(\omega_b'')$, $(\omega_c'')$, $(\omega_d'')$ passing through $L$ and having their centers on $AI$, $BI$, $CI$, $DI$, between $A$ and $I$, $B$ and $I$, $C$ and $I$, $D$ and $I$, are tangent to the four faces of $T$ at twelve points situated on the same sphere concentric with the inscribed sphere $(I, r)$. (Sphere of Adams.)

*Note* 1. We have fixed the position of the points $(X_b, X_c, X_d), \cdots$, but as the lines $BA'$, $CA'$, $DA'$ meet the circle $(A')$ in six points, there are four sets of spheres like $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ to consider. Furthermore, to each of the pedal tetrahedrons such as $T'$ of the eight centers of spheres tangent to the four faces of $T$, there corresponds a point $L$. There are in all eight positions of $L$ to which there may be associated spheres such as $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$, each touching three

faces of $T$.

*Note* 2. The following remarks concerning the construction of the spheres which satisfy the problem may be added [6].

Given a tetrahedron $ABCD$ and the second point of Lemoine $L$, the planes drawn through $L$ perpendicular to the lines $AI$, $BI$, $CI$, $DI$ joining the vertices to the center of the inscribed sphere, cut the cones of vertices $A$, $B$, $C$, $D$ circumscribed about the inscribed sphere in four circles which are all on the same sphere of center $I$. (Sphere of Adams.)

An isogonic tetrahedron has the lines joining the vertices of the tetrahedron to the points of contact of the opposite faces with the inscribed sphere of the tetrahedron concurrent [7]. This point of intersection coincides with the second Lemoine point $L$ defined above. In this case we have a complete analogy between the plane figure [8] and the three-dimensional figure.

**References**

1. A. S. Bang, Tidsskrift for Mathematik, 1897, p. 48.
2. N. A. Court, Modern Pure Solid Geometry, Macmillan, New York, 1935. §§244, 245.
3. V. Thébault, Ann. de la Soc. Scient. de Bruxelles, 1922, p. 173.
4. R. Bouvaist, Mathesis, t. 55, pp. 352–356.
5. P. Delens, Mathesis, 1937, p. 447.
6. V. Thébault, Mathesis, t. 54, Supplément, p. 20, reference to R. Bouvaist.
7. N. A. Court, ibid, §879.
8. V. Thébault, Mathesis, t. 54, Supplément, p. 9.

**A Property of the Adams Sphere of a Tetrahedron**

3998 [1941, 341]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A sphere $(S)$ is tangent to the faces of a tetrahedron $ABCD$ at the points $A'$, $B'$, $C'$, $D'$ and the straight lines $AA'$, $BB'$, $CC'$, $DD'$ are concurrent in the point $P$. The cones $(\Gamma_A)$, $(\Gamma_B)$, $(\Gamma_C)$, $(\Gamma_D)$ with vertices at $A$, $B$, $C$, $D$ circumscribe $(S)$. The planes through $P$ parallel to the planes $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ cut the respective cones in four circles which lie on a sphere concentric with $(S)$.

*Solution by the Proposer.* In case $(S)$ is the inscribed sphere, the point $P$ is the second Lemoine point mentioned in the above solution of 4160 and the proof of the present proposition is included in the final remarks there. If $(S)$ is one of the externally tangent spheres, $P$ is one of the associates of the second Lemoine point, and the proof follows without essential change. See also V. Thébault, *Comptes-Rendus*, 1940, p. 377; *Mathesis*, 1941, *Supplément*, p. 20; this MONTHLY, 1942, p. 170.

**Upper Bound for Terms of the Binomial Expansion**

4186 [1946, 45]. *Proposed by Fritz Herzog, Michigan State College*

It is known from the theory of probability that for $x$ fixed, $0<x<1$, the largest of the $n+1$ terms ${}_nC_r x^r(1-x)^{n-r}$, $r=0, 1, \cdots, n$, is asymptotically equal to $[2\pi nx(1-x)]^{-1/2}$, as $n\to\infty$. Show that for all integral values of $n$ and $r$ with

$n \geqq 1$, $0 \leqq r \leqq n$ and for all values of $x$ with $0 < x < 1$

$$ {}_nC_r x^r(1-x)^{n-r} < 1/[2enx(1-x)]^{1/2} $$

and that this represents the best inequality in the sense that the numerator on its right cannot be replaced by any number less than unity.

*Solution by the Proposer.* We put

$$ R(n, r; x) = ({}_nC_r)^2 n x^{2r+1}(1-x)^{2n-2r+1} \tag{1} $$

and proceed to show that l.u.b. $R(n, r; x) = 1/(2e)$, the l.u.b. being taken over the ranges of $n, r$, and $x$, indicated in the problem. For fixed $n$ and $r$ we denote the maximum of $R(n, r; x)$ in the interval $0 < x < 1$ by $S(n, r)$ and have then to show that l.u.b. $S(n, r) = 1/2e$, this l.u.b. being taken over all integral $n$ and $r$ with $n \geqq 1$ and $0 \leqq r \leqq n$. By differentiation of (1) it is easily seen that the maximum of $R(n, r; x)$ as a function of $x$ in the interval $0 < x < 1$ is obtained when $x = (2r+1)/(2n+2)$, so that

$$ S(n, r) = ({}_nC_r)^2 n(2r+1)^{2r+1}(2n-2r+1)^{2n-2r+1}/(2n+2)^{2n+2}. \tag{2} $$

We now put, for $r = 0, 1, \cdots, n-1$; $q(n, r) = S(n, r+1)/S(n, r)$ and obtain from (2)

$$ q(n, r) = \frac{(n-r)^2(2r+3)^{2r+3}(2n-2r-1)^{2n-2r-1}}{(r+1)^2(2r+1)^{2r+1}(2n-2r+1)^{2n-2r+1}}. \tag{3} $$

Considering $r$ for a moment as a continuous variable ($0 \leqq r \leqq n-1$), we obtain from (3)

$$ \begin{aligned} d \log q(n, r)/dr &= -2/(r+1) + 2 \log [1 + 2/(2r+1)] \\ &\quad - 2/(n-r) + 2 \log [1 + 2/(2n-2r-1)] \\ &= f(2r+1) + f(2n-2r-1), \end{aligned} \tag{4} $$

where $f(t) = -4/(t+1) + 2 \log (1 + 2/t)$, $t > 0$. (Since $0 \leqq r \leqq n-1$, $2r+1 \geqq 1$ and $2n-2r-1 \geqq 1$.) From $f(t) \to 0$, as $t \to +\infty$, and $f'(t) = 4/(t+1)^2 - 4/t(t+2) < 0$, it follows that $f(t) > 0$ for $t > 0$. We thus conclude from (4) that $d \log q(n, r)/dr > 0$ and hence that $q(n, r)$ increases with $r$ ($r = 0, 1, \cdots, n-1$). In particular, we have for $0 \leqq r < (n-1)/2$

$$ q(n, r) < q(n, n-1-r). \tag{5} $$

On the other hand, from (1) we have $R(n, n-r; x) = R(n, r; 1-x)$ so that $S(n, n-r) = S(n, r)$ and $q(n, n-1-r) = 1/q(n, r)$. The last relation together with (5) yields $q(n, r) < 1 < q(n, n-1-r)$ for $0 \leqq r < (n-1)/2$. (In the case $n$ odd, $q(n, r) = 1$ for $r = (n-1)/2$.) Thus, for fixed $n$, $S(n, r)$ decreases with $r$ for $0 \leqq r < (n+1)/2$ and increases with $r$ for $(n-1)/2 < r \leqq n$. Therefore, the largest of the $S(n, r)$ ($r = 0, 1, \cdots, n$) is

$$ S(n, 0) = S(n, n) = n(2n+1)^{2n+1}/(2n+2)^{2n+2}, \tag{6} $$

and it remains to show that l. u. b. $S(n, 0)=1/2e$, the l. u. b. being taken over all positive integral values of $n$. Considering $n$ in (6) for a moment as a continuous variable, we have $d \log S(n, 0)/dn=1/n-2 \log [1+1/(2n+1)]>1/n-2/(2n+1)>0$ (since $\log (1+u)<u$ for $u>0$). Hence $S(n, 0)$ increases with $n$ for $n\geqq 1$. On the other hand, from (6) we have $S(n, 0)=[n/(2n+2)] [1+1/(2n+1)]^{-(2n+1)}$, whence $\lim S(n, 0)=1/2e$, as $n\rightarrow\infty$. This proves l. u. b. $S(n, 0)=1/2e$. Since $S(n, 0)$ is actually less than $1/2e$ for $n=1, 2, \cdots$, the $<$ sign in the proposed inequality (rather than $\leqq$) is justified.

The following references to the asymptotic relation from the theory of probability, mentioned in the beginning of the problem, may be of interest: E. Czuber, *Wahrscheinlichkeitsrechnung*, 3rd ed., 1914, v. I, p. 134, (7), Arne Fisher, *Mathematical Theory of Probabilities*, 1922, v. I, pp. 101–102, and H. C. Plummer, *Probability and Frequency*, 1940, p. 34, (44.1).

### Greatest Integer Function

4199 [1946, 225]. *Proposed by N. J. Fine, Indianapolis, Ind.*

Let $r$ be any integer greater than unity, $n$ a non-negative integer, and $\alpha$ a non-negative integer less than $r$. Let $\nu$ be the number of digits in the expression of $n$ in the scale $r$ which are not less than $r-\alpha$, and let $\sigma$ be the sum of the digits. Show that

$$\sum_{k=1}^{\infty}\left[\frac{n+\alpha r^{k-1}}{r^k}\right]=\nu+\frac{n-\sigma}{r-1},$$

where the brackets denote the greatest integer function.

*Solution by R. C. Buck, Harvard University.* Let $n(k)$ be the individual digits so that

$$n=\sum_{k=0}^{\infty} n(k)r^k, \qquad \sigma=\sum_{k=0}^{\infty} n(k).$$

Then,

$$\left[\frac{n+\alpha r^{k-1}}{r^k}\right]=\sum_{\lambda=k}^{\infty} n(\lambda)r^{\lambda-k}+\theta_k,$$

where

$$\theta_k=\left[r^{-k}\sum_{\lambda=1}^{k-2} n(\lambda)r^{\lambda}+\{n(k-1)+\alpha\}r^{-1}\right].$$

If $n(k-1)+\alpha<r$, then $\theta_k=0$; otherwise $\theta_k=1$. Summing from 1 to infinity on $k$, we have

$$\sum_{k=1}^{\infty}\left[\frac{n+\alpha r^{k-1}}{r^k}\right]=\sum_{k=1}^{\infty}\sum_{\lambda=k}^{\infty} n(\lambda)r^{\lambda-k}+\sum_{k=1}^{\infty}\theta_k$$

$$= \sum_{\lambda=1}^{\infty} n(\lambda) \sum_{k=1}^{\lambda} r^{\lambda-k} + \nu$$

$$= \sum_{\lambda=1}^{\infty} n(\lambda) \frac{r^{\lambda}-1}{r-1} + \nu$$

$$= \frac{n-\sigma}{r-1} + \nu.$$

If $\alpha=0$, this reduces to $\sum(n/r^k)=(n-\sigma)/(r-1)$. If $r$ is a prime, this gives the highest power of $p$ dividing $n$!

Solved also by Paul Bateman, Paul Brock, C. D. Olds, and the Proposer.

**Partial Derangements**

4202 [1946, 278]. *Proposed by Vladimir Karapetoff, New York, N. Y.*

In a certain game of chance, consecutive numbers from 1 to $n$ are written on a table. The same number of discs are provided with consecutive numbers written on them. The discs are turned with the numbers down so that the players cannot see the numbers written on them. A player covers all the numbers on the table with the discs at random, because he does not see the numbers on them. The discs are then turned over and the score is made on the basis of the number of discs whose numbers agree with the numbers on the table which they are covering. It is required to deduce an expression for the chance that $k$ of the $n$ discs covered the correct numbers.

I. *Solution by E. S. Keeping, University of Alberta.* The number of possible arrangements of discs is $n$! The number of arrangements with $k$ matches and no more is $A_k$, where $A_k$ is the coefficient of $x_1, x_2 \cdots x_n t^k$ in *

$$\begin{aligned}
\phi &= (x_1t + x_2 + \cdots + x_n)(x_1 + x_2t + \cdots + x_n) \cdots (x_1 + x_2 + \cdots + x_nt) \\
&= [\textstyle\sum x_i + (t-1)x_1][\sum x_i + (t-1)x_2] \cdots [\sum x_i + (t-1)x_n] \\
&= \sum_{g=0}^{n} (\textstyle\sum x_i)^g (t-1)^{n-g}[x_1x_2 \cdots x_{n-g} + \cdots ],
\end{aligned}$$

the last bracket including all the sets of $n-g$ different variables out of $x_1, x_2, \cdots, x_n$.

Now the coefficient of $t^k$ in $(t-1)^{n-g}$ is $\binom{n-g}{k}(-1)^{n-g-k}$ and the coefficient of $x_1x_2 \cdots x_n$ in $(\sum x_i)^g$ is $g$! The number of ways of picking out $g$ different variables from $x_1, x_2, \cdots, x_n$ is $\binom{n}{g}$, and for each such choice there is one and only one term in the last bracket of $\phi$ which will give a term $x_1x_2 \cdots x_n$ in $\phi$. Therefore the required coefficient is

$$A_k = \sum_{g=0}^{n-k} \binom{n}{g} g! \binom{n-g}{k} (-1)^{n-g-k}.$$

* Each $t$ in such a term means that some $x_j$ came from the $j$th factor $(x_1+x_2+\cdots+x_jt+\cdots+x_n)$.

The probability of $k$ matches is therefore

$$\frac{A_k}{n!} = \sum_{g=0}^{n-k} \frac{(-1)^{n-g-k}}{k!(n-k-g)!}$$

$$= \frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-k}\frac{1}{(n-k)!}\right).$$

II. *Solution by C. F. Pinzka, Student, Rutgers University.* The $k$ correct pairs may be chosen in $C(n, k)$ ways, the number of combinations of $n$ objects taken $k$ at a time. For each of these $C(n, k)$ correct pairings, there are $D(n-k)$ incorrect pairings of the remaining discs and numbers, where $D(n-k)$ is the number of derangements of $n-k$ objects. This is known (See W. W. Ball, *Mathematical Recreations and Essays*, 11th ed., pp. 46–47) to be

$$D(n-k) = (n-k)!\sum_{i=0}^{n-k}(-1)^i(1/i!).$$

The total number of ways of arranging all the discs on all the numbers is $n!$, and the required probability is therefore

$$\frac{C(n, k)D(n-k)}{n!} = \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!}.$$

Solved also by D. W. Alling, Paul Bateman, R. C. Buck, W. B. Campbell, Clara M. Feller, and Free Jamison.

*Editorial Note.* See also 4146 [1946, 107–110], where many references are given. Another reference is Wilks, *Mathematical Statistics*, p. 208. See also the related problems, E 719 [1947, 45–46] and E 589 [1944, 287].

**The Generalized Coin Problem**

4203 [1946, 278]. *Proposed by N. J. Fine, Washington, D. C.*

This is a generalization of E 651 [1945, 42]. If one is allowed $n$ weighings on a beam balance, what is the maximum number $A_n$ of coins, exactly one of which is bad, from which one can isolate the bad coin and determine whether it is heavy or light? (Cf. E 712 [1946, 156].)

*Solution by the Proposer.* It will be shown that $A_n = (3^n - 3)/2$.

I. Given $k$ coins, exactly one of which is bad, separated into two sets $X$ and $Y$ (not necessarily non-empty) such that the odd coin is known to be heavy or light according as it is in $X$ or $Y$; if $k \leqq 3^n$, one can isolate the bad coin and identify it as light or heavy in not more than $n$ weighings. Put $x$ coins of $X$ and $y$ coins of $Y$ in each scale pan, where $x$ and $y$ are so chosen that $x+y \leqq 3^{n-1}$, $k-2x-2y \leqq 3^{n-1}$. If the scales do not balance the bad coin is among the $x$ coins on the heavy side or the $y$ coins on the light side. If the scales balance, the bad coin is among the unweighed coins. In either case a complete induction is possible, the statement being evidently true for $n=1$.

To show by induction that $3^n$ is maximum (even if one has extra good coins available) suppose that $x$ coins of $X$ and $y$ coins of $Y$ are weighed against $x'$ coins of $X$, $y'$ coins of $Y$ and $z'$ good coins, leaving $w$ unweighed. The possibility of balancing (with induction) implies that $w \leqq 3^{n-1}$. The possibility of unbalance (with induction) implies that the larger of $x+y'$ and $x'+y$ does not exceed $3^{n-1}$. Hence the total number cannot exceed $3^n$.

II. Given $(3^n-1)/2$ coins and a single additional good coin, one can isolate and identify the bad coin in $n$ weighings. Weigh the good coin and $(3^{n-1}-1)/2$ coins against $(3^{n-1}+1)/2$, leaving unweighed $(3^{n-1}-1)/2$. If the scales balance we have $(3^{n-1}-1)/2$ unidentified coins plus a good one, so that the induction can go through. If the scales do not balance, we have $(3^{n-1}-1)/2+(3^{n-1}+1)/2 = 3^{n-1}$ coins to which I applies.

To show that the stated number is maximum (even if more good coins are available) suppose $x$ unidentified and $y$ good coins are weighed against $x+y$ unidentified coins, leaving $z$ coins unweighed. The possibility of balancing (with induction) implies that $z \leqq (3^{n-1}-1)/2$. The possibility of not balancing implies, by I, that $y+2x \leqq 3^{n-1}$. Hence the total cannot exceed $(3^{n-1}-1)/2+3^{n-1}=(3^n-1)/2$.

III. Given $(3^n-3)/2$ unidentified coins (with no extra good coins) one can isolate and identify the bad coin in $n$ weighings. Weigh $(3^{n-1}-1)/2$ against the same number, leaving as many unweighed. If the scales balance, we have $(3^{n-1}-1)/2$ unidentified coins to work with, and in addition more than one good coin. By II only $n-1$ more weighings are needed. If the scales do not balance, we have $3^{n-1}-1$ coins to which I applies.

To prove that this number is maximum, suppose that $x$ coins are weighed against $x$, leaving $y$ unweighed. The possibility of balancing implies, by II, that $y \leqq (3^{n-1}-1)/2$. The possibility of not balancing implies, by I, that $2x \leqq 3^{n-1}$. But $3^{n-1}$ is odd, so that $2x \leqq 3^{n-1}-1$. Hence the total number does not exceed $(3^{n-1}-1)/2+3^{n-1}-1=(3^n-3)/2$.

Solved also by Murray Barbour, R. L. Brooks and C. A. B. Smith, H. Dowker and A. Seidenberg, Clara M. Feller, Ralph Keffer, Victor Perlo, J. Rosenbaum, and G. Szekeres.

*Editorial Note.* C. F. Pinzka referred to *The Mathematical Gazette*, 1945, pp. 227–229, and 1946, pp. 231–234, where two solutions are given. Rosenbaum's solution associates the individual coins with integers in the ternary scale and gives a scheme whereby the number may be obtained from the results of the weighings, similar to the second solution in the *Gazette*. Brooks and Smith give an analogous solution using vector terminology.

Every solver assumed either that the coins are marked so as to be individually identifiable at any stage, or that each scale pan is divided in some way so that it is possible to put on it coins of different sets without mixing. From the clause, "all of which appear exactly alike," in the statement of E 712, a more natural interpretation seems to be that after coins have been placed together in a scale pan they are to be considered as a single set. With this restriction the maxi-

mum seems to be given by

$$A_n = (7\cdot 3^{n-2} - 1)/2 = 3\cdot 3^{n-2} + 3^{n-3} + 3^{n-4} + \cdots + 3 + 1.$$

In two weighings we can determine whether the three sets of $3^{n-2}$ coins are of equal weight, and hence all good, or whether one set contains a light or a heavy coin. If they are all good, a third weighing (against $3^{n-3}$ good coins) will determine whether the set of $3^{n-3}$ coins is all good or contains a heavy or a light coin. Continuing thus, since one coin is known to be bad, for some $s$ ($s=2, 3, 4, \cdots, n$) we reach in $s$ weighings a set of $3^{n-s}$ coins containing a heavy or a light coin. By I in Fine's solution above, $n-s$ further weighings will isolate the bad coin. (When either $X$ or $Y$ is empty, the method of I comes under our present restriction.)

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*Science Since 1500.* By H. T. Pledge. New York, Philosophical Library, 1947. (Originally published by H. M. Stationery Office, London, 1939). 357 pages. $5.00.

Interest in the history of science has grown so rapidly in recent years that the publication of this book in America should be welcome. The reader will find it both valuable and entertaining, provided he is forewarned of its numerous shortcomings.

The style is uneven, being usually good but often so bad as to obscure the sense. Surely "the mutual ignorance of the chief scientists" (p. 55) is not idiomatic English to express their lack of personal acquaintance with one another. Spelling, punctuation and printing show similar carelessness, though much more rarely. There are about thirty inserts, in the form of illustrations, graphs, charts and maps, which are interesting in themselves but have little connection with the text.

More serious defects are connected with the author's laudable but difficult attempt to provide a unifying thread through the history of modern science. Such a thread he finds in the relation between industry and science, or as he often says, between the crafts and theory. Thus Chapter V—Mechanics, Astronomy and Optics in the 17th Century—begins

the story of this, the most important single episode in the history of science, is a complex one. Improvements in the science of mechanics are severely tested by difficulties in astronomy, brought to the front by advances in the manipulation of lenses. Other experiments made possible by skill

in the handling of glass give definiteness to the concept of a gas. This, and advances in hydraulics, enrich the idea of the physical; and the concept of mass crystallises out. With its aid, and with that of advances in mathematics, the astronomical difficulty is overcome. Doubly the advance depends upon the properties of glass.

And the whole idea is summed up in a thought-provoking sentence (p. 322) "abstract science is not knowledge of nature but of the artefacts of industry from nature."

On this chosen thread the author has attempted to string a truly amazing number of facts, the indices of his book containing nearly four thousand entries. He is thereby confronted by two complementary dangers, which he does not altogether avoid. The unifying thread may not be sufficiently strong, so that facts are set down without much connection. This is certainly the case in the closing chapters, on "Real Materials." On the other hand, a desire for unity may lead an author to discern similarities where none exist. The present book is full of comparisons between mathematics and religion, psychology and bacteriology, and so forth, which have excited ridicule, *e.g.*, in the review in *Nature*, April 13, 1940. As an example, consider the following comparison (p. 324) between mathematics and geography:

we suggested in the first chapter that elbow-room rather than order came, with the Explorations, to be the dominant idea as to space. We suggested that, with its tacit implication of relativity, this led to Copericanism in the 16th century. . . . But meanwhile (in the nineteenth century) white man was being forced to realise that, on this planet, elbow-room is not unlimited: and, as if reflecting this, the idea of order rather than of room began to come to the front again, with the added point that there might be types of order more general than any sort of space. For even the abstract spaces of Frechet recognise the limitation that there must be a meaning for the "vicinity" of points.

Such an argument, apart from its lack of clarity, goes much too far in an attempt to explain the history of mathematics in terms of navigational instruments and discoverable territory. It will prove almost anything. Fréchet's spaces are based on Cantor's theory of aggregates, arising from a study of Fourier's series, which contain the trigonometrical functions used in navigation. So we have reached the opposite conclusion: it is the Portuguese explorations which are responsible for modern abstract spaces!

The author himself is uncomfortably aware that many of his linkages are fanciful and sometimes apologizes for them in the wrong places. Thus, in a paragraph on theories of heat (p. 111):

in the early 18th century, gin engulfed Britain in an overwhelming wave of drunkenness. We may, fancifully, see in these new sciences of chemistry and of heat one reaction to this! For not only is distillation one of the best methods of preparing pure substances, but distillers, Black found, were aware, as users of fuel, of a quantitative aspect of heat other than mere temperature. Two branches of technology were thus exchanging influences, at the very start, with this new theoretical science of heat. Black, in fact, ended another case of the isolation of craft knowledge and theorist's knowledge; and the union of traditions gave him the doctrine of the latent heat of freezing and of vaporisation (1757–62). This was of special interest to Watt.

But what is fanciful in a statement that the process of manufacturing gin has

contributed to our knowledge of heat and chemistry? It is true, important, and a first-class example of the author's general method.

From what has already been said, it is clear that in spite of serious flaws the book has great positive merit. Most of our quotations have been chosen to display some fault; yet they are both informative and interesting. In fact, a book which gives a perspicuous and reasonably inclusive account, in less than five lines, of the history of Cepheid variables in determining distances to the stars (p. 293), of the early enthusiasm and opposition encountered by the practice of asepsis (p. 164) and vaccination (p. 166), of the relation of logical positivism to science (p. 190), or in less than twenty lines, of the history of the insolubility of the quintic (p. 175), of the theoretical and practical discovery of Dirac's positron (p. 282), and so on almost endlessly, in such a way as to produce in the reader, through at least two-thirds of its course, a sense of mounting excitement and of pride in human achievement, such a book is worthy of respect, let its defects be what they may.

The five chapters devoted to mathematics are: IV: Mathematics before the Calculus; VI: Mathematics 1600–1800; X: Mathematical Physics (to the end of the nineteenth century); and XII and XIII: 19th-Century Mathematics. As in the rest of the book, their extreme conciseness makes it impossible to give an account of what they include. To put it roughly: everybody is mentioned at least once.

The above remarks about the book as a whole apply equally well to the mathematical chapters. It is easy to quote passages of distinct merit which contain a serious flaw. The following paragraph (p. 187) is perhaps the best example:

> another worker in algebraic numbers, Dedekind, joined hands with Weierstrass in developing a method for dealing with the position (the need for "arithmetisation" of analysis). It lay in working out an analytically usable definition of irrational numbers, by extending that of Eudoxus. Dedekind's "cut" (1858, published 1872) divided the *rational* numbers into a L(eft) and R(ight) class, each with at least one member, such that every $L$ is less than every $R$. A rational number makes a cut in which it is either the greatest of the $L$'s or the least of the $R$'s: which both exist. An irrational is then defined as one for which neither exists, but which (Weierstrass) can be defined, and reached in practice, by a convergent infinite sequence among the $L$'s.

Taken as a whole, the paragraph is excellent, in line with the best passages in other parts of the book. But the words "which both exist" (instead of "one *or* the other of which exists," or perhaps "both of which possibilities always exist") constitute an elementary mistake which could easily prove baffling.

It is probable that the book will be useful; but for what type of reader? Certainly not for the undergraduate, who cannot defend himself against its errors and has never heard of Fréchet. But if, as signs indicate, the history of science will soon find its place in the curriculum of the American college, then the intending lecturer will do well to turn to this book. It will provide him with a framework, and with information which he can supplement and correct from more voluminous and more exact sources.

S. H. Gould

*Advanced Mathematics for Engineers.* Second Edition. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc., 1947. 12+508 pages. \$5.00.

The first edition (1938) of this book was reviewed in this MONTHLY (v. 48, 204, 1941) by Professor E. B. Allen. This second edition differs but slightly in content from the first, the character of which is familiar enough to justify omission here of detailed comment. Suffice it to say that the qualified undergraduate engineering student who has finished the calculus and wishes to take a next course in mathematics pertinent to his engineering studies will find this book instructive and to his taste. The critical remarks which follow are not to be understood as detracting from the many merits of this book.

To begin with, superficially, the format will not be found as pleasing nor the quality of the paper as good as in the first edition. However, it should be emphasized that the publishers have taken pains with the book; it has been reset, legibly printed, and is singularly free from typographic errors. This edition does no disservice to the reputation of the publishers for the much admired standard of their technical publications.

All of the problems of the first edition have been retained but with considerable rearrangement within each list. A great many new exercises have been added throughout the book, and several are of interest. Others are "routine," and it is not always clear just what these and certain of the rearrangements are intended to accomplish. The answers to all the problems are given at the end of the book. The engineering student will no doubt continue to be attracted by the practical aura of the exercises; some first rate students although finding many of the exercises worthwhile for drill purposes may miss both those significantly hard and also those delicate problems by which mettle is proved.

Beside extensive revisions of the problem lists throughout the book there are certain revisions in the text proper. In Chapter I ("Ordinary Differential Equations") the same presentation occurs except for certain revisions to several sections of Article 7: (1) a numerical change in the efflux of the problem in Section $c$ makes the discussion more instructive; (2) there is a literal revision of Section $d$; (3) a more careful statement of Hooke's law and subsequent discussion are given in Section $f$; (4) Figures 3 and 4 have been redrawn. In Chapter II ("Hyperbolic Functions") an article on the "geometric representation of hyperbolic functions" and accompanying figure have been deleted. In Chapter III ("Elliptic Integrals") Article 31, dealing with a mechanical brake problem, is new. Chapter IV ("Infinite Series") and Chapter VI ("Gamma and Bessel Functions") are substantially unchanged. In Chapter V ("Fourier Series") a new Section 49 called "Combination of Series" has been added; this turns out to be merely a quotation of the sine and cosine half-range series for a particular linear polynomial. In Chapter VII ("Partial Derivatives and Differential Equations") the latter parts of Articles 70 and 71 dealing respectively with the solutions of the vibrating string and one-dimensional heat-flow equations are entirely rewritten

and improved. Now (pp. 282–284) a numerical illustration of the "telephone equations" replaces the former illustration (pp. 265–267) of the "telegraph equations." Finally, the chapter is extended by the inclusion of a discussion of the partial differential equation of a vibrating membrane (Art. 77). The contents of Chapter VIII ("Vector Analysis") and Chapter IX ("Probability") remain essentially unchanged.

Chapters X and XI remain the same as in the first edition. These are intended to serve as first introductions to the study of functions of a complex variable and of the Heaviside operational calculus. A distinguishing feature in each of these chapters will be recalled by readers of the first edition. A third of the former chapter is given over to a readable introductory account of the idea of the Schwarz-Christoffel transformation, with some of the familiar applications. The second half of the last chapter may serve as an introduction, by way of purely formal procedures, to the classical Bromwich "justification" analysis.

Finally, an appendix of five pages' length has been added on "Units and Dimensional Analysis." It is perhaps convenient to have these tables of dimensions of various physical quantities collected here. The remarks on "dimensional analysis" are cursory; a few illustrations but no explicit exercises for the student are given. The table of contents and the index have been carefully revised.

In the first edition of this book there were a few rather well-chosen references to the engineering literature which were incorporated in the text. These of course remain; one can but wish that circumstances had permitted several additions to them in the years since 1938. The reader of the appendix is referred to Eshbach's handbook and Bridgman's monograph for further information on dimensional analysis; other than these only two new references to the literature have been added. The first is to a paper by I. Opatowski which appeared in this MONTHLY (v. 48, 443, 1941), the essence of which is now included as Article 31; the other reference, to some work of one of the authors, occurs in connection with an exercise in Chapter IX. It may also be remarked that it would be helpful if authors would consistently cite references to journals and books completely and in some one of the accepted ways; defects of this kind need remedying in this book.

The highways of modern engineering science are so many-laned and the traffic is so heavy and fast that the mathematical vehicles upon which we sometimes sanguinely mount our students seem to be "Model T." Let us hope that this book, and others written with the same intent, will eventually lead (with the engineer's cooperation) to courses in what really is advanced mathematics for undergraduates in engineering which will go far toward modernizing their mathematical transportation as they start off on their careers.

The appearance of the second edition of this book, implying the growing number of such courses and the wide adoption of the book itself, would seem to call for congratulations all around.

S. G. HACKER

*Cours Complet de Mathêmatiques Élêmentaires, Tome 1, Arithmêtique.* By J. Haag. Paris, Gauthier-Villars, 1945. 6+103 pages. 80 fr.

This is a nice little book in the theory of arithmetic from a not too advanced standpoint. The chapter headings mention the four rational operations, decimal notation, prime numbers, fractions, decimal fractions, squares and square roots, ratio and proportion, mensuration, theory of errors. There are no problems, these doubtless being provided in a supplement.

C. C. MacDuffee

## NEW BOOKS RECEIVED

*Applied Bessel Functions.* By F. E. Relton. London and Glasgow, Blackie and Son, Ltd., 1946. 7+191 pages. 17s. 6d.

*Advanced Calculus.* By D. V. Widder. New York, Prentice-Hall, Inc., 1947. 16+432 pages. $5.00.

*Brief Analytic Geometry.* Second Edition. By T. E. Mason and C. T. Hazzard. Boston, Ginn and Co., 1947. 9+205 pages. $2.50.

*Calculating Machines.* By D. R. Hartree. Cambridge University Press, 1947. 40 pages. $0.75.

*Curso de Matemática en Forma de Problemas.* By J. Gallego-Díaz. Madrid, Dossat, 1944. 12+333 pages.

*Differential and Integral Calculus.* Functions of One Variable. By F. D. Murnaghan. Brooklyn, Remsen Press, 1947. 10+502 pages. $5.00.

*Elements of Symbolic Logic.* By H. Reichenbach. New York, The Macmillan Company, 1947. 13+444 pages. $5.00.

*Great Engines and Great Planes.* By W. W. Stout. Detroit, Chrysler Corporation, 1947. 8+133 pages.

*An Introduction to Mechanics.* By J. W. Campbell. New York and London, Pitman Publishing Corporation, 1947. 18+372 pages. $4.50.

*Introduction to the Theory of Equations.* Second Edition. By L. W. Griffiths. New York, John Wiley and Sons, Inc., 1947. 9+278 pages. $3.50.

*Intermediate Algebra.* Revised Edition. By H. L. Rietz, A. R. Crathorne, and L. J. Adams. New York, Henry Holt and Co., 1947. 10+294 pages. $2.40.

*Mathematics as a Culture Clue and Other Essays.* (Collected Works, Vol. 1). By C. J. Keyser. New York, Scripta Mathematica, 1947. 7+277 pages. $3.75.

*The Physical Principles of Wave Guide Transmission and Antenna Systems.* By W. H. Watson. Oxford University Press, 1947. 10+208 pages. $7.00.

*Proceedings of the First Canadian Mathematical Congress, Montreal,* 1945. Toronto, University of Toronto Press, 1946. 44+367 pages. $3.25.

*Romping Through Mathematics.* By R. W. Anderson. New York, Alfred Knopf, 1947. 150 pages. $2.50.

*The Strange Story of the Quantum.* By B. Hoffman. New York, Harper and Brothers, 1947. 11+239 pages. $3.00.

# CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

## CLUB REPORTS, 1946–47

### Pi Mu Epsilon, University of Nebraska

During 1946–47, seven meetings were held by the *Nebraska Alpha* Chapter of *Pi Mu Epsilon*, including a business meeting, the initiation banquet and the annual spring picnic. The following talks were given:

*The Laplace transformations*, by Dale Rippe

*Continuous functions which are nowhere differentiable*, by Mr. W. T. Lanser

*The simplest problem in the calculus of variations*, by Mr. Lloyd Jackson

*Pascal's triangle*, by Mr. Dale M. Mesner.

Ten men were initiated at the annual banquet held December 8, while twenty-three members were initiated on May 13 at the spring picnic.

The annual Freshman and Sophomore competitions were held May 10. The Freshman prize was won by Mr. William Mundell, the Sophomore prize by Mr. William Bade.

Officers for the year 1946–47 were: President, Dick Bresee; Vice-President, Marilyn Stahl; Secretary, Louise Gardels; Treasurer, Simon Delisi.

Officers for 1947–48, who took office in March, are: President, Martha Clark; Vice-President, R. N. Scheidt; Secretary, Maurice Lamoree; Treasurer, Harold B. Frost. The faculty advisor is Miss Lulu L. Runge.

### Pi Mu Epsilon, University of Alabama

The activities of the *Alabama Alpha* Chapter of *Pi Mu Epsilon* centered around applications of mathematics. Several papers showed how mathematics was used by various members of the chapter during the war. The topics were:

*The Army University at Biarritz*, by Dr. H. S. Thurston

*Geometric proofs of some trigonometric theorems*, by Dr. C. L. Seebeck, Jr.

*Mathematics in meteorology*, by Ria J. Clinkscales, a former U.S.N. meteorologist

*Mathematics in navigation*, by Herschel E. Morrison, former U. S. A. A. F. navigation instructor

*Differentiating and integrating circuits*, by Dr. Ferdinand H. Mitchell

*Internal stresses in beams*, by T. W. Wilder.

The chapter started the year with 31 members and initiated 22 new members during the year. Other activities included a party and a picnic.

The officers for 1946–47 were: Director, Emily Jones; Vice-Director, Hazel Reynolds; Treasurer, Dr. H. S. Thurston; Publicity Chairman, Linda Simpson; Secretary, Dr. C. L. Seebeck, Jr.; Social Chairman, Kathleen Cannon.

The officers elected for 1947–48 are: Director, Kathleen Cannon; Vice-Director, Linda Simpson; Treasurer, Dr. H. S. Thurston; Secretary, Dr. C. L. Seebeck, Jr.; Publicity Chairman, Dr. F. A. Lewis; Social Chairman, Susie Lee Ward.

**Kappa Mu Epsilon, Central Missouri State College**

The theme of the program of *Missouri Beta* Chapter of *Kappa Mu Epsilon* for the year 1946–47 was *the uses of mathematics.* The following papers were presented:

*Mathematics in the social sciences,* by Dr. Royal J. Briggs, Professor of Economics

*Mathematics in the natural sciences,* by Dr. Laura J. Nahm, Professor of Zoology

*Some interesting variations of the Pythagorean theorem,* by Edgar Curtis

*Actuarial mathematics,* by Gordon Cross

*Mathematical recreations,* by Mrs. Virginia Moore

*Magic squares,* by Ronald Evans

*Radar,* by James Remley.

The annual spring banquet was held on April 30. On May 7, 1947, members of the *Missouri Beta* Chapter went to the campus of William Jewell College, Liberty, Missouri, and installed the *Missouri Gamma* Chapter of *Kappa Mu Epsilon.*

The officers for the year 1946–47 were: President, Ernest Hoover; Vice-President Ronald Evans; Secretary, Robert Ellis; Treasurer, Norman Hoover.

The officers for 1947–48 will be: President, Mrs. Virginia Moore; Vice-President, Samuel Herndon; Secretary, Patricia Stewart; Treasurer, Berna Deane Rist.

**Pi Mu Epsilon, University of Oklahoma**

The *Oklahoma Alpha* Chapter of *Pi Mu Epsilon,* having been inactive since the summer of 1943, held a reorganization meeting in October 1946.

The following papers were presented as part of the programs of the regular meetings during the year:

*Ballistic meteorology,* by Mr. Otis S. Spears

*Graeffe's method for the solution of algebraic equations,* by Mr. Clarence R. Gates

*Some aspects of topology,* by Mr. Roy B. Deal

*The life of Newton and the development of the calculus,* by Mr. H. Milton Peek

*Non-Euclidean geometry,* by Mr. John D. Lennes.

The annual spring banquet was held on May 9, 1947 at which the new members were initiated. Dr. N. A. Court, Professor of Mathematics, gave the principal address, *Perplexities of a Potato-pusher.*

The chapter also sponsored a renewal of its annual all-school mathematics

contest. This contest is open to any undergraduate student and consists of problems in applied calculus and analytic geometry. A prize of ten dollars in books and a second prize of five dollars in books were given to the first and second place winners. Mr. Lester D. Fisher, junior engineer, and Mr. Kay N. Burns, sophomore engineer, were guests of honor at the spring banquet.

The officers serving for 1946–47 were: Director, Mr. Charles J. Pipes; Assistant Director, Mr. Gene Levy; Secretary-Treasurer, Mr. Garth A. Abbott.

**Pi Mu Epsilon, Louisiana State University**

*Louisiana Alpha* of *Pi Mu Epsilon* at Louisiana State University met early in October, 1946, for an organization meeting and to map out a program for the year.

The following papers were given at various meetings throughout the year:

*Checking hunches*, by Dr. Frank A. Rickey
*A fish tale*, by Dr. W. V. Parker
*Intuitive discussion of the point-calculus of Grassman*, by Dr. H. L. Smith
*There is another way*, by Dr. Paul K. Rees
*The generalized water-fetching problem*, by Mr. John C. Currie
*The degenerate forms of conic sections*, by Mr. Ernest Ikenberry.

The winner of the annual Freshman Honors Examination was Mr. Earl P. Babin, and the winner of the annual Senior Award was Mr. Charles W. McArthur.

One hundred and two new members were initiated at the May meeting.

Eight new volumes have been added to the Pi Mu Epsilon collection of books, which is housed in the Mathematics Library.

The following officers served for the year: Director, Mr. Ben Mitchell; Vice-Director, Mr. Charles McCleskey; Secretary, Miss Beverly Jean Russell; Treasurer, Miss Kathryn Jumonville; Corresponding Secretary, Dr. Houston T. Karnes.

The new officers will not be elected until the fall semester.

**Mathematics Club, University of Buffalo**

The activities of the club centered around lectures on different phases of mathematics given by the students at monthly meetings, the topics were:

*Celestial navigation*, by Charles Kurland
*Modular arithmetic*, by Harold Schwartz
*The discovery of logarithms*, by Jean Ackerman
*"Math" magic*, by Kathleen Butz
*Contributions of the Greeks to mathematics*, by Edward Fadell
*Approximations of definite integrals*, by Mildred Scaffidi
*New numbers*, by Sara Zubkoff
*Calendars*, by Katherine Konst
*Continuous deformation of solids*, by William Braun, Jr.
*Probability*, by Harold Schwartz.

A Christmas party was held at the December meeting at which time games were played and refreshments were served.

At the April meeting the members of the club were hosts for the annual meeting of the students from the various local high schools. The purpose of this meeting was to show prospective university students what the Mathematics Department offers.

The officers for 1946–47 were: President, Jane Noller; Vice-President, Ruth Cohen; Secretary, Shirley Schwartz; Treasurer, Robert Locke; Faculty Advisor, Dr. H. F. Montague.

The officers elected for 1947–48 are: President, Jean Ackerman; Vice-President, Edward Fadell; Secretary, Mildred Scaffidi; Treasurer, William Braun, Jr.; Faculty Advisor, Dr. Harriet F. Montague.

### Kappa Mu Epsilon, Central Michigan College

The fall and spring initiations brought twenty-seven new members to the *Michigan Beta* Chapter of *Kappa Mu Epsilon*, which increased the total number of active members to fifty.

The lectures and readings presented at our monthly meetings were:

Reading of Stephen Leacock's, *The Human Element of Mathematics* and *Boarding House Geometry*, by Miss Gertrude Pratt

*The fundamental importance of the theory of numbers*, by Mr. Dana Sudborough

*Computation of firing data for field artillery*, by Thomas Selby, presented at the National *Kappa Mu Epsilon* convention, and later reprinted in the spring edition, 1947, of the *Pentagon*.

The social activities of the chapter included a sleigh ride and den party, and the annual spring picnic held at School Section Lake.

Officers for the fall semester 1946 were: President, Dorothy Michener; Vice-President, Louis Stasaski; Secretary, Raymond Williams; Treasurer, Coleen Edison; Corresponding Secretary, Dr. C. C. Richtmeyer.

Officers for the spring semester 1947 were: President, Thomas Selby; Vice-President, Leon Kimball; Secretary, Genevieve Waszkiewicz; Treasurer, Mary Welsh; Corresponding Secretary, Mr. Lester Serier.

### Pi Mu Epsilon, University of Washington

*Pi Mu Epsilon* has begun its reorganization from inactive status. The following men have been elected to associate memberships: F. Andrews, F. Ballantine, W. J. Firey, J. L. Hildebrand, E. Schlesinger, and R. Kraft. A speech on some phase of mathematics must be delivered by an associate member before an assembled body preliminary to promotion to full membership. In conformity with this rule F. Ballantine discussed a generalization of a problem appearing in this MONTHLY entitled *The stability of balanced objects*. Mr. Kraft will give a discussion of stellar spectroscopy.

Officers pro-tem are: Director, M. Stippes; Vice-Director, R. Bradford.

## NEWS AND NOTICES

EDITED BY HARRY POLLARD, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to Harry Pollard, White Hall, Cornell University, Ithaca, New York.*

### COMPUTING MACHINERY SECTION ANNOUNCED BY QUARTERLY

Beginning with the October issue, the quarterly journal, *Mathematical Tables and Other Aids to Computation*, will publish a new feature section, "Automatic Computing Machinery," designed to disseminate information and news on research and development in the field of high-speed automatic calculating machinery. Material should fall under the general headings of Bibliography, Technical Developments, Discussion (including correspondence), and News. Contributions to this section are invited, and should be addressed to Dr. E. W. Cannon, Head of the Mathematics Group, Machine Development Laboratory, National Bureau of Standards, Washington, D. C.

The journal, edited on behalf of the Committee on Mathematical Tables and Other Aids to Computation, is published quarterly by the National Research Council, and can be obtained from the National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C. A calendar year subscription is $4.00; single numbers, $1.25. Payments should be made to the National Academy of Sciences.

### AWARDS IN PRELIMINARY ACTUARIAL EXAMINATIONS

The winners of the prize awards offered by the Actuarial Society of America and the American Institute of Actuaries to the nine undergraduates ranking highest in the combined score on Part 1 and Part 2 of the 1947 Preliminary Actuarial Examinations are as follows:

*First Prize of $200*

James H. Chung, University of Toronto

*Additional Prizes of $100*

James F. A. Briggs, Yale University
George Y. Cherlin, Rutgers University
Frank H. David, Harvard University
Thomas M. Galt, University of Manitoba
Charles F. Pinzka, Rutgers University
Philip C. Rapp, University of Buffalo
Morton K. Schwartz, Brown University
James G. C. Templeton, University of Toronto

The two actuarial organizations have authorized a similar set of nine prize awards for the 1948 Examinations.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination: Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.

Part 2. General Mathematics Examination: Algebra, trigonometry, coordinate geometry, differential and integral calculus.

Part 3. Special Mathematics Examination: Finite differences, probability and statistics.

The 1948 Examinations will be administered by the College Entrance Examination Board at centers throughout the United States and Canada on May 14–15, 1948. Further information on these examinations, and an application for taking them will be found in a booklet entitled, *Preliminary Actuarial Examinations*, which may be secured from either of the following organizations:

The Actuarial Society of America
393 Seventh Avenue
New York 1, New York.

American Institute of Actuaries
135 South LaSalle Street
Chicago 3, Illinois.

## COLLECTED WORKS OF G. D. BIRKHOFF

Plans for the publication of a three-volume edition of the collected mathematical papers of the late Professor George D. Birkhoff have been laid by the American Mathematical Society. It is proposed to publish the papers by the photo-offset process so as to reduce the cost and to make the papers more widely available. A tentative prepublication price of $18.00 has been set for the three volumes. Advance subscriptions payable now or later may be sent to Professor J. R. Kline, American Mathematical Society, University of Pennsylvania, Philadelphia 4, Pennsylvania.

## TULANE UNIVERSITY GRADUATE PROGRAM

The graduate work in mathematics at Tulane is being expanded to incorporate a program leading to the doctorate.

## PERSONAL ITEMS

The following have received Guggenheim fellowship appointments: Professor Warren Ambrose of the Massachusetts Institute of Technology; Professor Garrett Birkhoff of Harvard University; Professor P. R. Halmos of the University of Chicago; Professor Saunders MacLane of the University of Chicago; Professor A. H. Taub of the University of Washington.

The following have received National Research Council fellowship appointments in mathematics: Mr. Hing Tong of Columbia University; Mr. Herman Rubin of the University of Chicago; Dr. Daniel Zelinsky of the University of Chicago.

Professor E. J. Cartan of the University of Paris has been elected to membership in the Royal Society.

Professor Emeritus G. H. Hardy of Cambridge University has been elected to membership in the French Academy of Science.

Dr. Samuel Karlen of Princeton University has been awarded the first Harry Bateman Research Fellowship by California Institute of Technology.

Professor H. A. Kramers of the University of Leyden has been elected correspondent of the French Academy of Science.

Professor E. J. McShane of the University of Virginia has been awarded an honorary doctorate of science by Tulane University.

Dr. Irving Reiner has been appointed a member at the Institute for Advanced Study for 1947–48.

Professor J. H. Van Vleck of Harvard University has received an honorary doctorate of science from the University of Wisconsin.

Professor Hermann Weyl of the Institute for Advanced Study and Professor George Pólya of Stanford University have been elected correspondents of the French Academy of Science.

Assistant Professor Florence E. Allen of the University of Wisconsin has retired.

Dr. Harriet W. Allen of Air Reduction Company, Stamford, Connecticut, has been appointed to an associate professorship in physics at Connecticut College.

Assistant Professor Warren Ambrose of the University of Michigan has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Assistant Professor B. H. Arnold of Montana State College has been appointed to an assistant professorship at Oregon State College.

Assistant Professor D. H. Ballou of Middlebury College, Middlebury, Vermont, has been promoted to an associate professorship.

Associate Professor T. A. Bancroft of the University of Georgia has accepted a position as director of the Statistical Laboratory, Alabama Polytechnic Institute.

Dr. J. D. Bankier of McMaster University, Hamilton, Ontario, has been promoted to an assistant professorship.

Assistant Professor Joseph Barnett of the Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Professor C. F. Barr has been made Head of the Mathematics Department at the University of Wyoming.

Dr. Grace E. Bates of Mount Holyoke College has been promoted to an assistant professorship.

Assistant Professor E. E. Betz of the United States Naval Academy has been promoted to an associate professorship.

Assistant Professor F. C. Biesale of the University of Utah has been promoted to an associate professorship.

Assistant Professor M. T. Bird of Allegheny College has been appointed to an assistant professorship at San Jose State College, San Jose, California.

Associate Professor Z. W. Birnbaum of the University of Washington has been appointed Director of the Institute of Mathematical Statistics which has been established at the University of Washington.

Associate Professor David Blackwell of Howard University has been promoted to a professorship and has been appointed Chairman of the Department of Mathematics. He is visiting Cornell University from June 10 to September 20 to work on probability.

Associate Professor A. W. Boldyreff of Wittenberg College has been appointed to an associate professorship at the University of New Mexico.

Associate Professor J. C. Brixey of the University of Oklahoma has been promoted to a professorship.

Associate Professor G. W. Brown of the Iowa State College of Agriculture and Mechanical Arts has been promoted to a professorship.

Assistant Professor R. H. Bruck of the University of Wisconsin has been promoted to an associate professorship.

Professor H. E. Buchanan of Tulane University is retiring as Chairman of the Department of Mathematics, but is returning in 1947–48 to teach.

Dr. E. L. Buell of Northwestern University has been appointed mathematician at the Aerial Measurements Laboratory, Northwestern Technical Institute, Evanston, Illinois.

Associate Professor G. P. Burns of Marshal College, Huntington, West Virginia, has accepted a position as research physicist at the Naval Research Laboratory, Washington, D. C.

Assistant Professor G. F. Carrier of Brown University has been promoted to an associate professorship.

Edmund Churchill of Rutgers University has been appointed to an assistant professorship at Antioch College.

Assistant Professor J. A. Clarkson of the University of Pennsylvania has been promoted to an associate professorship.

Professor A. B. Coble of the University of Illinois has been appointed to a professorship at Haverford College.

Dr. I. S. Cohen of the University of Pennsylvania has been promoted to an assistant professorship.

G. R. Costello has accepted a position as mathematician with the National Advisory Committee on Aeronautics, Cleveland, Ohio.

Dr. S. H. Crandall of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. D. A. Darling of the California Institute of Technology has been appointed research associate at Cornell University.

Dr. Norman Davids of Johns Hopkins University has been appointed to an associate professorship at Pennsylvania State College.

Georgia K. Del Franco of the University of Miami has been promoted to an assistant professorship.

Dr. W. W. Denton of Minden City, Michigan, has been apppointed to an

assistant professorship at the University of Arizona.

Assistant Professor J. B. Diaz of Carnegie Institute of Technology has been appointed to an assistant professorship at Brown University.

Dr. Bernard Dimsdale has accepted a position as mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Professor P. A. M. Dirac of Cambridge University has been appointed visiting professor at the Institute for Advanced Study for the current academic year.

W. W. Dolan of the University of Oklahoma has been promoted to an assistant professorship.

Associate Professor H. L. Dorwart of Washington and Jefferson College, Washington, Pennsylvania, has been promoted to a professorship.

Associate Professor T. L. Downs of the United States Naval Academy has been appointed to an associate professorship at Washington University.

Professor W. L. Duren, Jr., has been appointed Chairman of the Department of Mathematics in the College of Arts and Sciences at Tulane University.

Dr. William H. Durfee of Dartmouth College has been promoted to an assistant professorship.

Dr. Jacques Dutka of Princeton University has been appointed to an assistant professorship at Rutgers University.

M. W. Eudey of the University of California has accepted a position as operations analyst with the Army Air Force, Washington, D. C.

Assistant Professor C. J. Everett of the University of Wisconsin is on leave of absence and will be at the Los Alamos Scientific Laboratories.

Albert Furman of the University of Chicago has been appointed to an assistant professorship at Kansas State College of Agriculture and Mechanical Arts.

Professor Abel Gauthier of the University of Montreal has been appointed head of the Institute of Mathematics.

Assistant Professor Gladys Gibbens of the University of Minnesota has been promoted to an associate professorship.

Associate Professor M. E. Gillis of the University of Florida has been appointed to a professorship at Blue Mountain College, Blue Mountain, Mississippi.

Dr. M. A. Girschick of the Bureau of the Census has accepted a position as research mathematician with Douglas Aircraft Corporation, Santa Monica, California.

Associate Professor Wallace Givens of the Illinois Institute of Technology has been appointed to a professorship at the University of Tennessee.

Dr. C. H. Graves, formerly of the Office of Price Administration, has accepted a position as operations analyst with the Army Air Forces, Air Defense Command, Mitchell Field, New York.

H. J. Greenberg of Brown University has been promoted to an assistant professorship.

Bernard Greenspan of Drew University, Madison, New Jersey, has been promoted to an assistant professorship.

Associate Professor P. C. Hammer of Oregon State College has accepted a position as mathematician at the Los Alamos Scientific Laboratory.

I. H. Harris of the University of Illinois has been appointed to a professorship at Oklahoma Baptist University.

Assistant Professor Philip Hartman of Johns Hopkins University has been promoted to an associate professorship.

Assistant Professor G. E. Hay of the University of Michigan has been promoted to an associate professorship.

J. J. Hayes of the University of Utah has been promoted to an assistant professorship.

Dr. Katherine E. Hazard of the New Jersey College for Women, Rutgers University, has been promoted to an assistant professorship.

Assistant Professor Anna S. Henriques of the University of Utah has been promoted to an associate professorship.

W. C. Hoffman of the University of California at Los Angeles has accepted a position as mathematician at the Naval Electronics Laboratory, San Diego, California.

Professor Ralph Hull of the University of Nebraska has accepted a position as mathematician at Boeing Aircraft Company, Seattle, Washington.

Associate Professor Nathan Jacobson of Johns Hopkins University has been appointed to an associate professorship at Yale University.

Associate Professor E. D. Jenkins of Eastern Kentucky State Teachers College has been appointed to an associate professorship at Kent State University.

Assistant Professor Walter Jennings of Virginia Polytechnic Institute has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Assistant Professor R. E. Johnson of Mount Holyoke College has been appointed to an associate professorship at Smith College.

Professor B. W. Jones of Cornell University will be on leave of absence for the current academic year and will be at California Institute of Technology.

Dr. William Karush of the University of Chicago has been promoted to an assistant professorship.

E. H. Larguier of the University of Michigan has been appointed to a professorship at Spring Hill College, Spring Hill, Alabama.

M. M. Lemme of Purdue University has been appointed to an associate professorship at Illinois Wesleyan University. He will also be assistant dean of admissions.

Assistant Professor R. J. Levit of the University of Georgia has been promoted to an associate professorship.

Associate Professor C. C. Lin of Brown University has been appointed to an associate professorship at Massachusetts Institute of Technology.

Associate Professor H. W. Linscheid of the College of Emporia, Emporia, Kansas, has been appointed to an associate professorship at the Southwestern Institute of Technology, Weatherford, Oklahoma.

A. J. Lorenz of St. Louis University has been promoted to an assistant pro-

fessorship.

R. E. Luce of Rutgers University has been promoted to an assistant professorship.

R. T. Luginbuhl of the City Bank Farmers Trust Company has been promoted to the position of actuarial assistant.

Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has accepted a position with Douglas Aircraft Company, Santa Monica, California.

F. A. McMahon of Manhattan College has been promoted to an assistant professorship.

Dr. N. S. Mendelsohn of Queens University has been appointed to an assistant professorship at the University of Manitoba.

Dr. W. A. Mersman of Taylor Instrument Company, Rochester, New York, has accepted a position as mathematician with the National Advisory Committee on Aeronautics, Moffett Field, California.

Dr. D. S. Miller of Yale University has been appointed to an assistant professorship at the University of Rochester.

A. M. Mood of Iowa State College of Agriculture and Mechanical Arts has been promoted to a professorship.

Leo Moser of the University of Toronto has been appointed lecturer at the University of Manitoba.

Associate Professor Ivan Niven of Purdue University has been appointed to an associate professorship at the University of Oregon.

Dr. Anne F. O'Neill of Smith College has been promoted to an assistant professorship.

Dr. B. J. Pettis of Yale University has been appointed to an associate professorship at Tulane University of Louisiana.

Assistant Professor Everett Pitcher of Lehigh University has been promoted to an associate professorship.

Associate Professor E. J. Purcell of the University of Arizona has been promoted to a professorship.

Assistant Professor Haim Reingold of the Illinois Institute of Technology has been promoted to an associate professorship.

Assistant Professor Helene Reschovsky of Russell Sage College, Troy, New York, has been promoted to an associate professorship.

Professor C. E. Rhodes of Washington College, Chestertown, Maryland, has been appointed to a professorship at Alfred University, Alfred, New York.

Associate Professor L. F. S. Ritcey of United College, Winnipeg, Manitoba, has been appointed to a professorship in actuarial science at the University of Manitoba.

Professor H. P. Robertson of Princeton University has been appointed to a professorship at California Institute of Technology.

Dr. H. M. Schaerf of Montana State College has been appointed to an assistant professorship at Washington University.

A. E. Schild of Carnegie Institute of Technology has been promoted to an

assistant professorship.

Dr. H. M. Schwartz of the Franklin Institute has accepted a position as scientist with the Brookhaven National Laboratory, Patchogue, New York.

Assistant Professor Domina E. Spencer of Tufts College has been appointed to an assistant professorship of physics at Brown University.

Associate Professor N. E. Steenrod of the University of Michigan has been appointed to an associate professorship at Princeton University.

Professor M. H. Stone of the University of Chicago has been lecturing at the University of Brazil.

W. M. Stone of Iowa State College of Agriculture and Mechanical Arts has been appointed to an assistant professorship at Oregon State College.

Dr. Walter Strodt of Columbia University has been promoted to an assistant professorship.

F. W. Thalgott of the University of Illinois has accepted a position as mechanical engineer at the Clinton Laboratories, Oak Ridge, Tennessee.

Assistant Professor C. J. Thorne of the University of Utah has been promoted to an associate professorship.

Dr. W. J. Thron of Washington University has been promoted to an assistant professorship.

Dr. E. B. Tolsted of Brown University has been appointed to an assistant professorship at Pomona College, Claremont, California.

Assistant Professor H. C. Trimble of Iowa State Teachers College has been appointed to an associate professorship at Florida State College.

Professor G. R. Trott of Blue Mountain College, Blue Mountain, Mississippi, has been appointed to an associate professorship at the University of Mississippi.

Professor Bird M. Turner of West Virginia University has retired with the title emeritus.

Dr. G. B. Van Schaack of Union College has been appointed to an assistant professorship at Washington University, St. Louis, Missouri.

Assistant Professor Kenichi Watanabe of the University of Hawaii has been appointed assistant professor in physics at Wabash College.

Associate Professor F. P. Welch of Mississippi State College has been appointed to a professorship at Washington and Lee University.

R. L. Wine of the University of Oklahoma has been appointed to an assistant professorship at Washington and Lee University.

Dr. Y. C. Wong of the University of Pennsylvania has been appointed to a professorship at the National Sun Yat-Sen University, Canton, China.

Assistant Professor G. S. Young of Purdue University has been appointed to an assistant professorship at the University of Michigan.

Professor Oscar Zariski of the University of Illinois has been appointed to a professorship at Harvard University.

The following appointments to instructorships are announced:

Allegheny College: E. A. Sturley

Brown University: Dr. F. M. Stewart
Grinnell College: R. E. Cross
Harvard University: Dr. W. J. Leveque, Dr. G. R. MacLane
Haverford College: D. L. Thomsen
Rutgers University: A. G. Anderson, A. F. Bartholomay, Dr. Erwin Biser, Dr. R. M. Cohn, A. W. Goodman
Syracuse University: Dr. Erik Hemmingsen
Texas Technological College: Mrs. Virginia B. Roberts
Washington University: Dr. C. W. Mathews
Western Illinois State Technological College, Macomb, Ill.: J. J. Stipanowich
University of California at Los Angeles: L. J. Paige, J. D. Swift
University of Minnesota: Irwin Stoner
University of Pennsylvania: D. M. Adelman
University of Tennessee: R. L. Wilson
University of Wisconsin: Dr. A. M. Mark
University of Wisconsin at Racine: M. R. Moore

Dean Arthur Léveillé, who was Head of the Institute of Mathematics at the University of Montreal, died March 15, 1947.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-seventh regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, on Saturday, March 8, 1947. Professor H. J. Hamilton, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was one hundred and thirty, including the following forty-nine members of the Association: L. J. Adams, O. W. Albert, E. F. Beckenbach, Clifford Bell, Garrett Birkhoff, L. T. Black, H. F. Bohnenblust, Frances L. Campbell, L. M. Coffin, F. E. Cothran, D. R. Curtis, P. H. Daus, R. P. Dilworth, H. P. Edmundson, Iva B. Ernsberger, C. M. Fulton, J. W. Green, W. S. Gustin, H. J. Hamilton, R. B. Herrera, R. E. Horton, D. H. Hyers, C. G. Jaeger, Glenn James, G. R. Kaelin, Margaret B. Lehman, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, R. P. Peterson, W. T. Puckett, H. R. Pyle, L. T. Ratner, E. C. Rex, G. E. F. Sherwood, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, H. L. Steinberg, A. E. Taylor, F. B. Thompson, W. I. Thompson, S. E. Urner, F. A. Valentine, Morgan Ward, R. L. White, Mabel G. Whiting, Euphemia R. Worthington.

At the business meeting the following officers were elected for the next academic year: Chairman, D. V. Steed, University of Southern California;

Vice-Chairman, E. F. Beckenbach, University of California at Los Angeles; Program Committee, J. W. Green (Chairman), O. W. Albert, C. W. Trigg, and the Secretary, Paul H. Daus, ex-officio. The regular meeting was scheduled for March 13, 1948, at the University of Redlands.

The following papers were presented:

1. *Symposium on opportunities for mathematically trained college graduates*, conducted by Professor I. S. Sokolnikoff, University of California at Los Angeles.

Participants (introduced by Professor Sokolnikoff) were Dr. William Bollay, North American Aviation, Inc.; Dr. J. B. Smyth, U. S. Navy Electronic Laboratory, San Diego; Dr. J. W. Odle, Naval Ordnance Test Station, Inyokern.

Professor Sokolnikoff set the stage by relating something of the history of the growing demand for trained mathematicians in industrial and civil-service positions. The other speakers described the opportunities in their respective fields, and the necessary training the candidate should have. They all stressed the importance of a broad background in analysis and the urgent need for training in applied mathematics in such fields as statistics, hydrodynamics, and aerodynamics, potential theory, elasticity, and electric circuit theory.

2. *The recent development of lattice theory*, by Professor Garrett Birkhoff, Harvard University.

Professor Birkhoff gave a historical description of the recent development of lattice theory. The features of a number of interesting results, many of which are due to young American mathematicians, were described. The only new result was the following relation between lattice theory and the wave question in two-dimensional space-time. A real-valued function on a lattice may be called a *valuation* if it satisfies the identity $v[x]+v[y]=v[x\cup y]+v[x\cap y]$, satisfied by dimension in projective geometry and by probability. Let space-time be partially ordered as in special relativity. Then the solutions of the wave equation are the different valuations of space-time.

3. *Averaging operators for functions*, by Professor F. H. Bohnenblust, California Institute of Technology.

Professor Bohnenblust, considered a general notion of averaging whereby the average of a function is a function (not necessarily constant). If the average of $f$ be denoted by $\bar{f}$, it is assumed that: (1) $\overline{f_1+f_2}=\bar{f}_1+\bar{f}_2$; (2) $\overline{cf}=c\bar{f}$, if $c$ is a constant; (3) $\bar{f}$ depends continuously (in some sense) on $f$; (4) $(\overline{\bar{f}g})=\bar{f}\bar{g}$; (5) $\bar{\bar{f}}=\bar{f}$. After considering an example in which $\bar{f}$ was determined by omitting certain of the harmonics in the Fourier representation of $f$, Professor Bohnenblust stressed the importance of condition (4) by showing how it can point the way to a standard representation of the averaging operator. Condition (5) appears then merely as an additional normalizing requirement.

4. *Collegiate mathematics in Mexico*, by Dr. Alfredo Banos, Associate Professor of Physics, University of California at Los Angeles, introduced by Professor A. E. Taylor.

An account of the development of collegiate mathematics in Mexico necessarily includes an account of the history of the National University of Mexico since the founding, in 1910, of the present day institution. Because the National University was, in its early formative years, almost exclusively a conglomeration of professional schools, such as the School of Law, the School of Medicine, the School of Engineering, and so forth, and because even today the Mexican universities do not have, properly speaking, the counterpart of the American College of Letters and Science, it turns out that an account of so-called collegiate mathematics must necessarily describe the development of the teaching of mathematics in the technical and engineering schools of the country as well as the teaching of mathematics in the Preparatory School of the National University and all other similar schools. In recent years, however, particularly after the founding of the Faculty of Sciences at the National University (1939), there has been a tendency to establish scientific curricula closely paralleling the trend in this country, and thus it is that, today, the Mexican University offers courses of instruction leading to degrees in mathematics entirely equivalent to the graduate and undergraduate degrees offered in the United States.

5. *Computing machines*, by Myron Tribus, Lecturer in Engineering, University of California at Los Angeles, introduced by Professor Clifford Bell.

The recent developments in computing machines were described and some details of the unique features of the new machines were given. It was suggested that the machines of the future may be capable of performing 20,000 to 40,000 computations per second. The machines are considered as being capable of eight basic operations: (a) programming, (b) addition, (c) subtraction, (d) multiplication, (e) division, (f) looking up in tables, (g) storage or memory, and (h) printing results. Existing machines perform these operations at slower speeds, have limited memory capacity, and require considerable time to program. The use of a binary base simplifies equipment and permits faster operation. It is expected that more attention will have to be given to the mathematics of finite difference equations and methods of solving equations, in order to utilize the full possibilities of the machines.

6. *On the interior of the convex hull of a euclidean set*, by W. S. Gustin, University of California at Los Angeles.

Let $E$ be an arbitrary set in an $n$-dimensional euclidean space. It is known that any point in the convex hull of $E$ lies in the convex hull of some subset of $E$ containing at most $n+1$ points. It is shown that any point in the interior of the convex hull of $E$ lies in the interior of the convex hull of some subset of $E$ containing at most $2n$ points.

P. H. Daus, *Secretary*

## APRIL MEETING OF THE OHIO SECTION

The thirty-first annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 3, 1947. Professor S. A. Rowland, Chairman of the Section, presided.

Eighty persons registered attendance, including the following fifty-four members of the Association: W. E. Anderson, F. R. Bamforth, Grace M. Bareis, H. M. Beatty, Foster Brooks, V. B. Caris, F. E. Carr, A. B. Carson, E. H. Clarke, Florentina M. Clinton, Rufus Crane, Wayne Dancer, R. H. Downing, H. P. Fawcett, H. E. Fettis, Frances Freese, B. E. Gatewood, B. C. Glover, E. L. Godfrey, L. J. Green, C. H. Heinke, R. G. Helsel, R. C. Hildner, L. A. Jehn, Margaret E. Jones, L. C. Knight, A. C. Ladner, S. W. McCuskey, Margaret Mauch, H. E. Menke, E. J. Mickle, C. V. Newsom, H. C. Parrish, H. S. Pollard, J. F. Randolph, S. E. Rasor, Hortense Rickard, R. F. Rinehart, L. D. Rodabaugh, Louis Ross, S. A. Rowland, K. C. Schraut, Samuel Selby, G. W. Starcher, H. E. Stelson, Irving Sussman, H. S. Toney, E. P. Vance, R. W. Wagner, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, C. R. Wylie, Jr., P. M. Young.

The following officers were elected for the coming year: Chairman, H. S. Pollard, Miami University; Secretary-Treasurer, Foster Brooks, Kent State University; Member of Executive Committee, E. J. Mickle, Ohio State University; Member of Program Committee, R. B. Wildermuth, Capital University. It is expected that the next meeting will be held on Saturday, April 3, 1948.

The following program was presented:

1. *The Association's interest in pre-college training*, by Professor S. A. Rowland, Ohio Wesleyan University.

This paper is concerned especially with the situation in Ohio, but is doubtless applicable elsewhere. For a good many years the standard of achievement of pre-college students in mathematics has been deteriorating. The state-supported colleges and universities do not have complete freedom of action to maintain high standards of admission since by law they must admit graduates of approved high schools, and the approval of high schools is not controlled by them. Few private colleges have enough endowment to be independent as to standards of admission.

This Association has fostered high achievement in college mathematics, but faulty preparation for college is now undermining some of our best efforts. Possibly our most important task now is to embark upon an educational campaign, and to establish cooperation with the Ohio College Association and with other educational organizations in an endeavor to get the legislature to establish a State College Entrance Board independent of the State Department of Education. Such a board would not attempt to dictate the requirements for a high school diploma, but would set up entrance examinations which would ensure reasonable uniformity of preparation for college. The high schools could provide an intensive fifth year for such students as needed it and wanted it in order to pass the college entrance examinations.

2. *A proof of five-color sufficiency*, by Professor L. D. Rodabaugh, University of Akron.

Use is made in this paper of the well-known duality between regions and

common segments of their boundaries on the one hand, and points and their joins (no two points being permitted to have more than one join) on the other. There is described a *completion procedure*, which is to be applied to any admissible network of points and joins, the result being defined as a *complete network*. There follow the statements and proofs of nine properties possessed by every complete network, after which it is easy to prove the five-color sufficiency.

The proof does not depend on the formula of Euler which states that $V-E+F=2$, and has immediate application to any set of regions simply and completely covering the plane or sphere, regardless of the order of connectivity of any of the regions. The techniques used in this paper can, however, be used to prove this formula for any set of simply connected regions simply and completely covering the sphere. If the four-color problem be considered, the techniques employed can be used to prove many of the properties of the so-called "first non-colorable map." One illustration is given.

3. *An experimental evaluation of the base e*, by Professor H. S. Pollard, Miami University.

Buffon's needle problem deals with the probability that a needle, thrown at random on a table ruled with parallel lines, will cross one of these lines. This probability is a function of the length of the needle, the distance between the rulings, and $\pi$. By determining the relative frequency of crossings, many experimental calculations of the value of $\pi$ have been made.

This paper discusses a method of evaluating experimentally the number $e$. When a needle, which rotates about its mid-point, comes to rest, its direction will locate a definite point $x$ on an axis which does not pass through the mid-point of the needle. The probability density of the variable $x$ is given by the expression $1/[\pi(1+x^2)]$. The mathematical expectation of $\phi(x)=\cos x$ is then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx$$

whose value may be shown to be $e^{-1}$. Hence $e^{-1}$ may be evaluated by making successive spins of the needle and calculating the mean of the values of $\cos x$ at the points $x$ so located.

4. *A proof of Wilson's theorem*, by Robert E. Seall, Denison University, introduced by Professor F. B. Wiley.

The theorem $(p-1)^k+1\equiv 0 \pmod{p}$, $p$ any prime, is established as a corollary to two lemmas.

LEMMA 1. *If $n$ is any positive integer prime to each of the integers 2 to $k$, then*

$$\sum_{i=1}^{n-1} i^{k-1} \equiv 0 \pmod{n}.$$

The proof of this comes from writing $2^k, 3^k, \cdots, n^k$ as $(1+1)^k, (1+2)^k, \cdots, (1+n-1)^k$, expanding each by the binomial theorem, and then summing all to

obtain

$$\frac{k}{1}\sum i + \frac{k(k-1)}{2!}\sum i^2 + \cdots + \frac{k}{1}\sum i^{k-1} = n^k - n.$$

Now give $k$ the successive values 2 to $k$ in this last expression, and then solve the resulting system of equations, linear in the $\sum$'s for $\sum i^{k-1}$.

LEMMA 2. *If $S_{n,k}$ represents the sum of all possible products of $k$ unlike factors chosen from the positive integers* 1 *to $n$, then*

$$kS_{n,k} = S_{n,k-1}\sum i - S_{n,k-2}\sum i^2 + \cdots + (-1)^{k+1}\sum i^k.$$

The truth of this may be established by beginning with the last term and adding each preceding term in turn.

5. *Intuition in mathematics*, by Professor Tibor Radó, Ohio State University

In teaching, as well as in research, the initial reaction of a person toward a specific mathematical issue is largely determined by what may be termed intuition. From this point of view, intuition is an important component of motivation for the student, the teacher, and the research man. These general remarks were illustrated with a number of specific examples.

6. *The teacher training program for teachers in elementary schools*, by Professor H. P. Fawcett, Ohio State University.

This was a report from a committee that has been working on this problem for some time. It is hoped that some changes may be effected in the program which will strengthen the preparation of the teacher in mathematics.

RUFUS CRANE, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Muncie, Oct. 17, 1947
IOWA, Fairfield, April 1, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Beloit, May 8, 1948

# BOOK NEWS

PRINTED BY THE GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 9


CONTENTS




NOVEMBER 1947

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)



EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions and advertising in the Monthly, should be sent to the Secretary-Treasurer, W. B. Carver, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 20th of the month preceding that in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.
(Devoted to the Interests of Collegiate Mathematics)



---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

**1. The concept of cardinal number.** Cantor's continuum problem is simply the question: How many points are there on a straight line in Euclidean space? In other terms, the question is: How many different sets of integers do there exist?

This question, of course, could arise only after the concept of "number" had been extended to infinite sets; hence it might be doubted if this extension can be effected in a uniquely determined manner and if, therefore, the statement of the problem in the simple terms used above is justified. Closer examination, however, shows that Cantor's definition of infinite numbers really has this character of uniqueness, and that in a very striking manner. For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (*e.g.*, their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets $A$ and $B$ it is possible (at least theoretically) to change the properties and relations of each element of $A$ into those of the corresponding element of $B$, whereby $A$ is transformed into a set completely indistinguishable from $B$, hence of the same cardinal number. For example, assuming a square and a line segment both completely filled with mass points (so that at each point of them exactly one mass point is situated), it follows owing to the demonstrable fact that there exists a one-to-one correspondence between the points of a square and of a line segment, and, therefore, also between the corresponding mass points, that the mass points of the square can be so rearranged as exactly to fill out the line segment, and vice versa. Such considerations, it is true, apply directly only to physical objects, but a definition of the concept of "number" which would depend on the kind of objects that are numbered could hardly be considered as satisfactory.

So there is hardly any choice left but to accept Cantor's definition of equality between numbers, which can easily be extended to a definition of "greater" and "less" for infinite numbers by stipulating that the cardinal number $M$ of a set $A$ is to be called less than the cardinal number $N$ of a set $B$ if $M$ is different from $N$ but equal to the cardinal number of some subset of $B$. On the basis of these definitions it becomes possible to prove that there exist infinitely many different infinite cardinal numbers or "powers," and that, in particular, the number of subsets of a set is always greater than the number of its elements; furthermore it becomes possible to extend (again without any arbitrariness) the arithmetical operations to infinite numbers (including sums and products with any infinite

number of terms or factors) and to prove practically all ordinary rules of computation.

But, even after that, the problem to determine the cardinal number of an individual set, such as the linear continuum, would not be well defined if there did not exist some "natural" representation of the infinite cardinal numbers, comparable to the decimal or some other systematic denotation of the integers. This systematic representation, however, does exist owing to the theorem that for each cardinal number and each set of cardinal numbers[1] there exists exactly one cardinal number immediately succeeding in magnitude and that the cardinal number of every set occurs in the series thus obtained.[2] This theorem makes it possible to denote the cardinal number immediately succeeding the set of finite numbers by $\aleph_0$ (which is the power of the "denumerably infinite" sets), the next one by $\aleph_1$, *etc.*; the one immediately succeeding all $\aleph_i$ where $i$ is an integer, by $\aleph_\omega$, the next one by $\aleph_{\omega+1}$, *etc.*, and the theory of ordinal numbers furnishes the means to extend this series farther and farther.

**2. The continuum problem, the continuum hypothesis and the partila results concerning its truth obtained so far.** So the analysis of the phrase "how many" leads unambiguously to quite a definite meaning for the question stated in the second line of this paper, namely, to find out which one of the $\aleph$'s is the number of points on a straight line or (which is the same) on any other continuum in Euclidean space. Cantor, after having proved that this number is certainly greater than $\aleph_0$, conjectured that it is $\aleph_1$, or (which is an equivalent proposition) that every infinite subset of the continuum has either the power of the set of integers or of the whole continuum. This is Cantor's continuum hypothesis.

But, although Cantor's set theory has now had a development of more than sixty years and the problem is evidently of great importance for it, nothing has been proved so far relative to the question what the power of the continuum is or whether its subsets satisfy the condition just stated, except (1) that the power of the continuum is not a cardinal number of a certain very special kind, namely, not a limit of denumerably many smaller cardinal numbers,[3] and (2) that the proposition just mentioned about the subsets of the continuum is

---

[1] As to the question why there does not exist a set of all cardinal numbers, see footnote 12.

[2] In order to prove this theorem the axiom of choice (see: A. Fraenkel, Einleitung in die Mengenlehre, 3rd ed. Berlin, 1928, p. 288 ff.) is necessary, but it may be said that this axiom is, in the present state of knowledge, exactly as well founded as the system of the other axioms. It has been proved consistent, provided the other axioms are so. (See my paper quoted in footnote 13.) It is exactly as evident as the other axioms for sets in the sense of arbitrary multitudes and, as for sets in the sense of extensions of definable properties, it also is demonstrable for those concepts of definability for which, in the present state of knowledge, it is possible to prove the other axioms, namely, those explained in footnotes 17 and 21.

[3] See F. Hausdorff, Mengenlehre, 1st ed. (1914), p. 68. The discoverer of this theorem. J. König, asserted more than he had actually proved (see Math. Ann. 60 (1904), p. 177).

true for a certain infinitesimal fraction of these subsets, the analytical[4] sets.[5] Not even an upper bound, however high, can be assigned for the power of the continuum. Nor is there any more known about the quality than about the quantity of the cardinal number of the continuum. It is undecided whether this number is regular or singular, accessible or inaccessible, and (except for König's negative result) what its character of confinality[4] is. The only thing one knows, in addition to the results just mentioned, is a great number of consequences of, and some propositions equivalent to, Cantor's conjecture.[6]

This pronounced failure becomes still more striking if the problem is considered in its connection with general questions of cardinal arithmetic. It is easily proved that the power of the continuum is equal to $2^{\aleph_0}$. So the continuum problem turns out to be a question from the "multiplication table" of cardinal numbers, namely, the problem to evaluate a certain infinite product (in fact the simplest non-trivial one that can be formed). There is, however, not one infinite product (of factors $>1$) for which only as much as an upper bound for its value can be assigned. All one knows about the evaluation of infinite products are two lower bounds due to Cantor and König (the latter of which implies a generalization of the aforementioned negative theorem on the power of the continuum), and some theorems concerning the reduction of products with different factors to exponentiations and of exponentiations to exponentiations with smaller bases or exponents. These theorems reduce[7] the whole problem of computing infinite products to the evaluation of $\aleph_\alpha^{cf(\aleph_\alpha)}$ and the performance of certain fundamental operations on ordinal numbers, such as determining the limit of a series of them. $\aleph_\alpha^{cf(\aleph_\alpha)}$, and therewith all products and powers, can easily be computed[8] if the "generalized continuum hypothesis" is assumed, *i.e.*, if it is assumed that $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for every $\alpha$, or, in other terms, that the number of subsets of a set of power $\aleph_\alpha$ is $\aleph_{\alpha+1}$. But, without making any undemonstrated assumption, it is not even known whether or not $m<n$ implies $2^m<2^n$ (although it is trivial that it implies $2^m \leqq 2^n$), nor even whether $2^{\aleph_0}<2^{\aleph_1}$.

**3. Restatement of the problem on the basis of an analysis of the foundations of set theory and results obtained along these lines.** This scarcity of results, even as to the most fundamental questions in this field, may be due to some extent to purely mathematical difficulties; it seems, however (see Section 4 below), that there are also deeper reasons behind it and that a complete solution of

---

[4] See the list of definitions at the end of this paper.

[5] See F. Hausdorff, Mengenlehre, 3rd ed. (1935), p. 32. Even for complements of analytical sets the question is undecided at present, and it can be proved only that they have (if they are infinite) either the power $\aleph_0$ or $\aleph_1$ or continuum (see; C. Kuratowski, Topologie I, Warszawa-Lwow, 1933, p. 246.)

[6] See W. Sierpinski, Hypothese du Continu, Warsaw, 1934.

[7] This reduction can be effected owing to the results and methods of a paper by A. Tarski published in Fund. Math. 7 (1925), p. 1.

[8] For regular numbers $\aleph_\alpha$ one obtains immediately:

$$\aleph_\alpha^{cf(\aleph_\alpha)} = \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

these problems can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the meanings of the terms occurring in them (such as "set," "one-to-one correspondence," *etc.*) and of the axioms underlying their use. Several such analyses have been proposed already. Let us see then what they give for our problem.

First of all there is Brouwer's intuitionism, which is utterly destructive in its results. The whole theory of the $\aleph$'s greater than $\aleph_1$ is rejected as meaningless.[9] Cantor's conjecture itself receives several different meanings, all of which, though very interesting in themselves, are quite different from the original problem, and which lead partly to affirmative, partly to negative answers;[10] not everything in this field, however, has been clarified sufficiently. The "half-intuitionistic" standpoint along the lines of H. Poincare and H. Weyl[11] would hardly preserve substantially more of set theory.

This negative attitude towards Cantor's set theory, however, is by no means a necessary outcome of a closer examination of its foundations, but only the result of certain philosophical conceptions of the nature of mathematics, which admit mathematical objects only to the extent in which they are (or are believed to be) interpretable as acts and constructions of our own mind, or at least completely penetrable by our intuition. For someone who does not share these views there exists a satisfactory foundation of Cantor's set theory in its whole original extent, namely, axiomatics of set theory, under which the logical system of Principia Mathematica (in a suitable interpretation) may be subsumed.

It might at first seem that the set theoretical paradoxes would stand in the way of such an undertaking, but closer examination shows that they cause no trouble at all. They are a very serious problem, but not for Cantor's set theory. As far as sets occur and are necessary in mathematics (at least in the mathematics of today, including all of Cantor's set theory), they are sets of integers, or of rational numbers (*i.e.*, of pairs of integers), or of real numbers (*i.e.*, of sets of rational numbers), or of functions of real numbers (*i.e.*, of sets of pairs of real numbers), *etc.*; when theorems about all sets (or the existence of sets) in general are asserted, they can always be interpreted without any difficulty to mean that they hold for sets of integers as well as for sets of real numbers, *etc.* (respectively, that there exist either sets of integers, or sets of real numbers, or . . . *etc.*, which have the asserted property). This concept of set, however, according to which a set is anything obtainable from the integers (or some other well defined

---

[9] See L. E. J. Brouwer, Atti del IV Congresso Internazionale dei Matematici (Roma 1908), p. 569.

[10] See L. E. J. Brouwer, Over de gondslagen der wiskunde (Amsterdam and Leipzig, 1907) I, 9; III, 2.

[11] See H. Weyl, Das Kontinuum, 2nd ed. 1932. If the procedure of construction of sets described there (p. 20) is iterated a sufficiently large (transfinite) number of times, one gets exactly the real numbers of the model for set theory spoken of below in Section 4, in which the continuum-hypothesis is true. But this iteration would hardly be possible within the limits of the half intuitionistic standpoint.

objects) by iterated application[12] of the operation "set of,"[13] and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent.[14]

But, furthermore, the axioms underlying the unrestricted use of this concept of set, or, at least, a portion of them which suffice for all mathematical proofs ever produced up to now, have been so precisely formulated in axiomatic set theory[15] that the question whether some given proposition follows from them can be transformed, by means of logistic symbolism, into a purely combinatorial problem concerning the manipulation of symbols which even the most radical intuitionist must acknowledge as meaningful. So Cantor's continuum problem, no matter what philosophical standpoint one takes, undeniably retains at least this meaning: to ascertain whether an answer, and if so what answer, can be derived from the axioms of set theory as formulated in the systems quoted.

Of course, if it is interpreted in this way, there are (assuming the consistency of the axioms) *a priori* three possibilities for Cantor's conjecture: It may be either demonstrable or disprovable or undecidable.[16] The third alternative (which is only a precise formulation of the conjecture stated above that the difficulties of the problem are perhaps not purely mathematical) is the most likely, and to seek a proof for it is at present one of the most promising ways of attacking the problem. One result along these lines has been obtained already, namely, that Cantor's conjecture is not disprovable from the axioms of set theory, provided that these axioms are consistent (see Section 4).

It is to be noted, however, that even if one should succeed in proving its undemonstrability as well, this would (in contradistinction, for example, to the proof for the transcendency of $\pi$) by no means settle the question definitively.

---

[12] This phrase is to be understood so as to include also transfinite iteration, the totality of sets obtained by finite iteration forming again a set and a basis for a further application of the operation "set of."

[13] The operation "set of $x$'s" cannot be defined satisfactorily (at least in the present state of knowledge), but only be paraphrased by other expressions involving again the concept of set, such as: "multitude of $x$'s," "combination of any number of $x$'s," "part of the totality of $x$'s"; but as opposed to the concept of set in general (if considered as primitive) we have a clear notion of this operation.

[14] It follows at once from this explanation of the term "set" that a set of all sets or other sets of a similar extension cannot exist, since every set obtained in this way immediately gives rise to further application of the operation "set of" and, therefore, to the existence of larger sets.

[15] See, *e.g.*, P. Bernays, A system of axiomatic set theory, J. Symb. Log. 2 (1937), p. 65; 6 (1941), p. 1; 7 (1942), p. 65; p. 133; 8 (1943), p. 89. J. von Neumann, Eine Axiomatisierung der Mengenlehre, J. reine u. angew. Math. 154 (1925), p. 219; cf also: *ibid.*, 160 (1929), p. 227; Math. Zs 27 (1928), p. 669. K. Gödel, The Consistency of the Continuum Hypothesis (Ann. Math. Studies No. 3), 1940.

[16] In case of the inconsistency of the axioms the last one of the four *a priori* possible alternatives for Cantor's conjecture would occur, namely, it would then be both demonstrable and disprovable by the axioms of set theory.

Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to point out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained.

For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[17] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of." These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers. The simplest of these strong "axioms of infinity" assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $>\aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes).[18] Other axioms of infinity have been formulated by P. Mahlo.[19] Very little is known about this section of set theory, but at any rate these axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set up so far.

That these axioms have consequences also far outside the domain of very great transfinite numbers, which are their immediate object, can be proved; each of them (as far as they are known) can, under the assumption of consistency, be shown to increase the number of decidable propositions even in the field of Diophantine equations. As for the continuum problem, there is little hope of solving it by means of those axioms of infinity which can be set up on the basis of principles known today (the above-mentioned proof for the undisprovability of the continuum hypothesis, *e.g.*, goes through for all of them without any change). But probably there exist others based on hitherto unknown principles; also there may exist, besides the ordinary axioms, the axioms of infinity and

---

[17] Similarly also the concept "property of set" (the second of the primitive terms of set theory) can constantly be enlarged, and furthermore concepts of "property of property of set" *etc.* be introduced whereby new axioms are obtained, which, however, as to their consequences for propositions referring to limited domains of sets (such as the continuum hypothesis) are contained in the axioms depending on the concept of set.

[18] See E. Zermelo, Fund. Math. 16 (1930), p. 29.

[19] See: Ber. Verh. Sächs. Ges. Wiss. 63 (1911), pp. 190–200; 65 (1913), pp. 269–276. From Mahlo's presentation of the subject, however, it does not appear that the numbers he defines actually exist.

the axioms mentioned in footnote 14, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts.

Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success," that is, its fruitfulness in consequences and in particular in "verifiable" consequences, *i.e.*, consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytical number theory frequently allows us to prove number theoretical theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

**4. Some observations about the question: In what sense and in which direction may a solution of the continuum problem be expected?** But are such considerations appropriate for the continuum problem? Are there really any strong indications for its unsolubility by the known axioms? I think there are at least two.

The first one is furnished by the fact that there are two quite differently defined classes of objects which both satisfy all axioms of set theory written down so far. One class consists of the sets definable in a certain manner by properties of their elements,[20] the other of the sets in the sense of arbitrary multitudes, irrespective of, if, or how they can be defined. Now, before it is settled what objects are to be numbered, and on the basis of what one-to-one correspondences, one could hardly expect to be able to determine their number (except perhaps in case of some fortunate coincidence). If, however, someone believes that it is meaningless to speak of sets except in the sense of extensions of definable properties, or, at least, that no other sets exist, then, too, he can hardly expect more than a small fraction of the problems of set theory to be solvable without making use of this, in his opinion essential, characteristic of sets, namely, that they are

[20] Namely, definable "in terms of ordinal numbers" (*i.e.*, roughly speaking, under the assumption that for each ordinal number a symbol denoting it is given) by means of transfinite recursions, the primitive terms of logic, and the $\epsilon$-relation, admitting, however, as elements of sets and of ranges of quantifiers only previously defined sets. See my papers quoted in footnotes 13 and 19, where an exactly equivalent, although in its definition slightly different, concept of definability (under the name of "constructibility") is used. The paradox of Richard, of course, does not apply to this kind of definability, since the totality of ordinals is certainly not denumerable.

all derived from (or in a sense even identical with) definable properties. This characteristic of sets, however, is neither formulated explicitly nor contained implicitly in the accepted axioms of set theory. So from either point of view, if in addition one has regard to what was said above in Section 2, it is plausible that the continuum problem will not be solvable by the axioms set up so far, but, on the other hand, may be solvable by means of a new axiom which would state or at least imply something about the definability of sets.[21]

The latter half of this conjecture has already been verified; namely, the concept of definability just mentioned (which is itself definable in terms of the primitive notions of set theory) makes it possible to derive the generalized continuum hypothesis from the axiom that every set is definable in this sense.[22] Since this axiom (let us call it "A") turns out to be demonstrably consistent with the other axioms, under the assumption of the consistency of these axioms, this result (irrespective of any philosophical opinion) shows the consistency of the continuum hypothesis with the axioms of set theory, provided that these axioms themselves are consistent.[23] This proof in its structure is analogous to the consistency-proof for non-Euclidean geometry by means of a model within Euclidean geometry, insofar as it follows from the axioms of set theory that the sets definable in the above sense form a model for set theory in which furthermore the proposition A and, therefore, the generalized continuum hypothesis is true. But the definition of "definability" can also be so formulated that it becomes a definition of a concept of "set" and a relation of "element of" (satisfying the axioms of set theory) in terms of entirely different concepts, namely, the concept of "ordinal numbers," in the sense of elements ordered by some relation of "greater" and "less," this ordering relation itself, and the notion of "recursively defined function of ordinals," which can be taken as primitive and be described axiomatically by way of an extension of Peano's axioms.[24] [Note that this does not apply to my original formulation presented in the papers quoted above, because there the general concept of "set" with its element relation occurs in the definition of "definable set," although the definable sets remain the same if, afterwards, in the definition of "definability" the term "set" is replaced by "definable set."]

---

[21] D. Hilbert's attempt at a solution of the continuum problem (see Math. Ann. 95 (1926), p. 161), which, however, has never been carried through, also was based on a consideration of all possible definitions of real numbers.

[22] On the other hand, from an axiom in some sense directly opposite to this one the negation of Cantor's conjecture could perhaps be derived.

[23] See my paper quoted in footnote 13 and note Proc. Nat. Ac. Sci. 25 (1939), p. 220. I take this opportunity to correct a mistake in the notation and a misprint which occurred in the latter paper: in the lines 25 to 29 of page 221, 4 to 6 and 10 of page 222, 11 to 19 of page 223, the letter $\alpha$ should be replaced (in all places where it occurs) by $\mu$. Also, in Theorem 6 on page 222 the symbol "$\equiv$" should be inserted between $\phi_\alpha(x)$ and $\phi_{\bar{\alpha}}(x')$. For a carrying through of the proof in all details the paper quoted in footnote 13 is to be consulted.

[24] For such an extension see A. Tarski, Ann. Soc. Pol. Math. 3 (1924), p. 148, where, however, the general concept of "set of ordinal numbers" is used in the axioms; this could be avoided, without any loss in demonstrable theorems, by confining oneself from the beginning to recursively definable sets of ordinals.

A second argument in favor of the unsolubility of the continuum problem on the basis of the ordinary axioms can be based on certain facts (not known or not existing at Cantor's time) which seem to indicate that Cantor's conjecture will turn out to be wrong;[25] for a negative decision the question is (as just explained) demonstrably impossible on the basis of the axioms as known today.

There exists a considerable number of facts of this kind which, of course, at the same time make it likely that not all sets are definable in the above sense.[26] One such fact, for example, is the existence of certain properties of point sets (asserting an extreme rareness of the sets concerned) for which one has succeeded in proving the existence of undenumerable sets having these properties, but no way is apparent by means of which one could expect to prove the existence of examples of the power of the continuum. Properties of this type (of subsets of a straight line) are: (1) being of the first category on every perfect set,[27] (2) being carried into a zero set by every continuous one-to-one mapping of the line on itself.[28] Another property of a similar nature is that of being coverable by infinitely many intervals of any given lengths. But in this latter case one has so far not even succeeded in proving the existence of undenumerable examples. From the continuum hypothesis, however, it follows that there exist in all three cases not only examples of the power of the continuum,[29] but even such as are carried into themselves (up to denumerably many points) by *every* translation of the straight line.[30]

And this is not the only paradoxical consequence of the continuum hypothesis. Others, for example, are that there exist: (1) subsets of a straight line of the power of the continuum which are covered (up to denumerably many points) by *every* dense set of intervals, or (in other terms) which contain no undenumerable subset nowhere dense on the straight line,[31] (2) subsets of a straight line of the power of the continuum which contain no undenumerable zero set,[32] (3) subsets of Hilbert space of the power of the continuum which contain no undenumerable subset of finite dimension,[33] (4) an infinite sequence $A^i$ of decompositions of any set $M$ of the power of the continuum into continuum

[25] Views tending in this direction have been expressed also by N. Lusin in Fund. Math. 25 (1935), p. 129 ff. See also: W. Sierpinski, *ibid.*, p. 132.

[26] That all sets are "definable in terms of ordinals" if *all* procedures of definition, *i.e.* also quantification and the operation $\mathfrak{F}$ with respect to *all* sets, irrespective of whether they have or can be defined, are admitted could be expected with more reason, but still it would not at all be justified to assume this as an axiom. It is worth noting that the proof that the continuum hypothesis holds for the definable sets or follows from the assumption that all sets are definable, does not go through for this kind of definability, although the assumption that these two concepts of definability are equivalent is, of course, demonstrably consistent with the axioms.

[27] See W. Sierpinski, Fund. Math. 22 (1934), p. 270 and C. Kuratowski, Topologie I, p. 269 ff.

[28] See N. Lusin and W. Sierpinski, Bull. Internat. Ac. Sci. Cracovie 1918, p. 35, and W. Sierpinski, Fund Math. 22 (1934), p. 270.

[29] For the 3rd case see: *l.c.* 6, p. 39, Th. 1.

[30] See W. Sierpinski, Ann. Scuol. Norm. Sup. Pisa 4 (1935), p. 43.

[31] See N. Lusin, C. R. Paris 158 (1914), p. 1259.

[32] See W. Sierpinski, Fund. Math. 5 (1924), p. 184.

[33] See W. Hurewicz, Fund. Math. 19 (1932), p. 8.

many mutually exclusive sets $A_x^i$ such that, in whichever way a set $A_{x_i}^i$ is chosen for each $i$, $\Pi_i(M-A_{x_i}^i)$ is always denumerable.[34] Even if in (1)–(4) "power of the continuum" is replaced by "$\aleph_1$" these propositions are very implausible; the proposition obtained from (3) in this way is even equivalent with (3).

One may say that many of the results of point-set theory obtained without using the continuum hypothesis also are highly unexpected and implausible.[35] But, true as that may be, still the situation is different there, insofar as in those instances (such as, *e.g.*, Peano's curves) the appearance to the contrary can in general be explained by a lack of agreement between our intuitive geometrical concepts and the set-theoretical ones occurring in the theorems. Also, it is very suspicious that, as against the numerous plausible propositions which imply the negation of the continuum hypothesis, not one plausible proposition is known which would imply the continuum hypothesis. Therefore one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

**Definitions of some of the technical terms**

Definitions 4–12 refer to subsets of a straight line, but can be literally transferred to subsets of Euclidean spaces of any number of dimensions; definitions 13–14 refer to subsets of Euclidean spaces.

1. I call "character of confinality" of a cardinal number $m$ (abbreviated by "$cf(m)$") the smallest number $n$ such that $m$ is the sum of $n$ numbers $<m$.
2. A cardinal number $m$ is regular if $cf(m)=m$, otherwise singular.
3. An infinite cardinal number $m$ is inaccessible if it is regular and has no immediate predecessor (*i.e.*, if, although it is a limit of numbers $<m$, it is not a limit of fewer than $m$ such numbers); it is inaccessible in the stronger sense if each product (and, therefore, also each sum) of fewer than $m$ numbers $<m$ is $<m$. [See: W. Sierpinski and A. Tarski, Fund. Math. 15 (1930), p. 292; A. Tarski, Fund. Math. 30 (1938), p. 68. From the generalized continuum hypothesis follows the equivalence of these two notions. This equivalence, however, is a much weaker and much more plausible proposition. $\aleph_0$ evidently is inaccessible in both senses. As for finite numbers, 0 and 2 and no others are inaccessible in the stronger sense (by the above definition), which suggests that the same will hold also for the correct extension of the concept of inaccessibility to finite numbers.]
4. A set of intervals is dense if every interval has points in common with some interval of the set. [The end-points of an interval are not considered as points of the interval.]
5. A zero-set is a set which can be covered by infinite sets of intervals with arbitrarily small lengths-sum.
6. A neighborhood of a point $P$ is an interval containing $P$.
7. A subset $A$ of $B$ is dense in $B$ if every neighborhood of any point of $B$ contains points of $A$.
8. A point $P$ is in the exterior of $A$ if it has a neighborhood containing no point of $A$.
9. A subset $A$ of $B$ is nowhere dense on $B$ if those points of $B$ which are in the exterior of $A$ are dense in $B$. [Such sets $A$ are exactly the subsets of the borders of the open sets in $B$, but the term "border-set" is unfortunately used in a different sense.]
10. A subset $A$ of $B$ is of the first category in $B$ if it is the sum of denumerably many sets nowhere dense in $B$.

[34] See S. Braun and W. Sierpinski, Fund. Math. 19, (1932), p. 1, proposition ($Q$). This proposition and the one stated under (3) in the text are equivalent with the continuum hypothesis.

[35] See, *e.g.*, L. Blumenthal, Am. Math. Monthly 47 (1940), p. 346.

11. Set $A$ is of the first category on $B$ if the intersection $A \cdot B$ is of the first category in $B$.
12. A set is perfect if it is closed and has no isolated point (*i.e.*, no point with a neighborhood containing no other point of the set).
13. Borel-sets are defined as the smallest system of sets satisfying the postulates:
    (1) The closed sets at Borel-sets.
    (2) The complement of a Borel-set is a Borel-set.
    (3) The sum of denumerably many Borel-sets is a Borel-set.
14. A set is analytic if it is the orthogonal projection of some Borel-set of a space of next higher dimension. [Every Borel-set therefore is, of course, analytic.]
15. Quantifiers are the logistic symbols standing for the phrases: "for all objects $x$" and "there exist objects $x$." The totality of objects $x$ to which they refer is called their range.
16. The symbol "$\hat{x}$" means "the set of those objects $x$ for which $\cdots$ "

---

# PERSONNEL AND TRAINING PROBLEMS IN STATISTICS

S. S. WILKS, Princeton University

**1. Introduction.** A report* has been prepared by the Committee** on Applied Mathematical Statistics of the National Research Council to: (1) analyze some factors contributing to the recent extraordinary growth of interest in the use of statistical methods, (2) present some information on the current and future needs of statistically trained personnel, (3) examine the impact of these needs on present training facilities, and (4) indicate some steps which might establish a training program adequate to meet these growing needs. This is a summary of the report.

**2. Statistical organizations.** As a simple indication of growth of interest in statistical methods, the Committee describes the formation and recent growth of statistical organizations. The American Statistical Association, founded more than 100 years ago, had a membership of 1700 in 1935. By the end of 1946 it had nearly 4000 members. The Institute of Mathematical Statistics, formed in

* "Personnel and Training Problems Created by the Recent Growth of Applied Statistics in the United States," a report by the Committee on Applied Mathematical Statistics, National Research Council, Washington, D. C., NRC Reprint and Circular Series No. 128, May, 1947, 17 pp.

** Luther P. Eisenhart, Chairman, Former Chairman Division of Physical Sciences, National Research Council

Samuel S. Wilks, Secretary, Professor of Mathematical Statistics, Princeton University

Chester I. Bliss, Biometrician, Connecticut Agricultural Experiment Station

Edward U. Condon, Director, National Bureau of Standards

Harold O. Gulliksen, Professor of Psychology, Princeton University

Lowell J. Reed, Vice-President of the University, and Professor of Biostatistics, School of Hygiene and Public Health, The Johns Hopkins University

Charles F. Roos, President, The Econometric Institute, Inc.

Walter A. Shewhart, Research Engineer, Bell Telephone Laboratories

Hugh M. Smallwood, Director, Department of Physical Research, General Laboratories, U. S. Rubber Company

Frederick F. Stephan, Professor of Sociology and Statistics, Cornell University

1935 to promote the development of statistical theory, had 900 members by the end of 1946. The Econometric Society with a membership of more than 750 was organized in 1930 to promote the application of mathematics and statistical methods in economics. The Psychometric Society is a similar organization for psychology. It was organized in 1935 and now has more than 200 members. The Biometrics Section of the American Statistical Association was formed in 1938 for sponsoring similar work in the biological sciences. It now has more than 1100 members. The most recent statistical organization is the American Society for Quality Control which is concerned with applications of statistical methods in industry. It was organized early in 1946 and now has approximately 2000 members, mostly engineers of various kinds. There are other organizations with considerable interest in statistical methods such as the American Marketing Association, American Public Health Association, American Sociological Society, and Population Association of America.

**3. Demand for statisticians.** According to the report, there is a heavy demand for both academic and non-academic statistical personnel. The non-academic fields which account for most of the recent growth of interest in statistical methods are: (1) industrial statistical control (in quality control, research and development), (2) research in the biological sciences, (3) collection and analyses of government statistics, (4) market research and commercial surveys, and (5) psychological testing. The Committee, in its report, discusses each of these fields in some detail. Demands are continuing and increasing for statistical personnel in some of the older fields such as finances and taxes, labor and employment, prices and production. Demands for more statistical training for social scientists are increasing.

The Committee made an inquiry among 30 leading authorities at 27 universities in mathematical and applied statistics as to requests received for statistical personnel for a period of approximately six months after the end of the war. These authorities reported a total of 135 requests for personnel for academic positions in mathematical and applied statistics ranging from instructorships to full professorships. No attempt was made to have each respondent identify each request so as to eliminate duplication. But one person reported that he had requests from 21 college and university mathematics departments for Ph.D.'s in mathematical statistics. Another reported 12 requests for Ph.D.'s in agronomy with minors in statistics. There were reported 90 requests from government agencies and 140 requests from industry. The training requirements for these requests ranged from B.A.'s to Ph.D.'s in mathematical and applied statistics.

At least a rough comparison may be made between demands for personnel in mathematics, physics, and statistics. As of December 31, 1945, the National Roster of Scientific and Specialized Personnel had registrations of 9972, 9608 and 2018 in mathematics, physics, and statistics, respectively. For the period September 1, 1945, to May 31, 1946, the number of requests for personnel in these three fields per 1000 persons registered were 4.4, 23.9, and 30.7, respectively.

**4. Training in statistics.** The Committee devoted more than a third of its report to problems of education and training in statistics. These problems were discussed at two levels: (1) the undergraduate level and (2) the graduate level. It was stated that substantial progress had been made in the teaching of statistics at the graduate level in a number of universities; but that it was still inadequate to meet the growing demands for statistical personnel. The Committee charged that the teaching of statistics at the undergraduate level is still in a very chaotic condition. Graduate teaching in mathematical statistics is more standardized than that in applied statistics. The Committee listed basic requirements in mathematics for graduate training in mathematical statistics as follows: real and complex variables, linear and quadratic forms, matrix algebra, $n$-dimensional euclidean geometry, measure and integration theory. The courses are essential for the theory of probability which is the foundation for courses in advanced mathematical statistics covering distribution theory, estimation theory, testing of statistical hypotheses and multivariate statistical theory.

Of the 27 universities included in the Committee's inquiry, only ten claimed a graduate program leading to a Ph.D. degree in mathematical statistics, and fourteen claimed an adequate training program at the advanced level for some field of applied statistics. It was found that only four of the universities have special stipends for graduate work in mathematical statistics. The situation in applied statistics is hardly more adequate.

In its discussion of the teaching of elementary statistics the Committee emphasized the duplication of material in elementary statistics courses as they are now taught in various departments of a given college or university, as well as the heterogeneity of the quality of teaching. The Committee expressed its opinion that the standardization and improvement of the teaching of statistics at the undergraduate level is a basic requirement for the solution to the problem of training statistical personnel. Specifically, it proposes that there should be developed a basic course in statistics at the freshman level for students who will go into the natural and social sciences, standardized to the same extent as introductory courses in mathematics, physics, and chemistry.

According to the Committee, one of the most puzzling problems regarding statistics is how it should be organized within a university. Two plans which are being tried out at certain universities were discussed: (1) the statistical laboratory and research center which would serve as an informal campus statistical center, and (2) the department of statistics. Plan (1) is necessarily rather informal and depends for its success on the voluntary cooperation of personnel from various departments who are interested in research and teaching of statistics. Plan (2) would be more formal and desirable but its success would depend on joint membership of some of its personnel with other departments. This is particularly important for the effective teaching of applied statistics which should be carried out in conjunction with departments interested in applications of statistical methods.

**5. Summary.** The Committee summarized its conclusions as follows:

1. There should be developed a basic introductory course in statistics at preferably the freshman level for colleges and universities throughout the country.

2. The laboratory work in the average course in statistics is inadequate, particularly at the elementary level; experimental work should replace much of the computation at this level.

3. The minimum requirement in effective organization is a central statistical laboratory with which all of those teaching or doing research in statistics would be associated, even though informally in some cases.

4. More success is to be expected from a department of statistics associated with a statistical laboratory, and having some members in common with other departments.

5. The number of institutions needed for giving first-class training through the graduate level are: (a) 5 to 10 in mathematical statistics, (b) 25 to 30 in varying fields of applied statistics.

6. An institution giving complete training in either mathematical or applied statistics should give some training in the other.

7. Institutional stipends for graduate students specializing in mathematical and applied statistics are inadequate.

8. In strengthening its statistical work at the advanced and research levels any given university should consider which field it can develop most effectively, so as to avoid duplication and inefficiency from a national point of view.

9. The immediate critical shortage of highly qualified teachers can be eased only by suitable training of high-grade personnel now in fields of application, or mathematics.

10. There should be an adequate number of postdoctoral fellowships in statistics.

11. Arrangements should be established whereby postgraduate students, research workers, and teachers on leave would be able to obtain work experience in certain government agencies, industrial laboratories, and business research organizations.

12. To help offset the present critical shortage of qualified personnel in applied statistics, it would be desirable to promote conferences at advanced levels and short courses at the elementary level in various fields.

# A SIMPLE ANALYTIC PROOF OF A GENERAL $\chi^2$ THEOREM

C. L. WEAVER, Boston, Mass.

Many of the applications of the $\chi^2$ distribution can be justified as particular cases of a general theorem. It is the purpose of this paper to state and give a simple analytic proof of this theorem. Two of the more basic $\chi^2$ principles are stated as corollaries of the general theorem.

The usual procedure in developing the applications of $\chi^2$ is to treat each application more or less independently. When proofs are included they usually involve geometry in $n$-space, the moment generating function, the Laplace transform, or the algebra of quadratic forms. The proof given here makes use of none of these methods. It does make use of the Jacobian in the change of variables in a multivariate differential (or multiple integral).

DEFINITION. We shall first define the $\chi^2$ distribution with $\nu$ degrees of freedom as that distribution having the probability element

$$K_\nu(\chi^2)^{(\nu/2)-1}e^{-\frac{1}{2}\chi^2}d(\chi^2), \qquad 0 \leqq \chi^2 < \infty.$$

Substituting $\chi^2 = 2w$ and equating the integral of the above differential over the given range of $\chi^2$ to unity, it follows by the definition of the gamma integral that

$$K_\nu = \left\{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right\}^{-1}.$$

Let us consider $y_i$, $i=1$ to $n$, to be $n$ independent random variables. Each $y_i$ is normally distributed with mean $a_0+a_1t_{1i}+a_2t_{2i} \cdots +a_lt_{li}$ and variance $\sigma_i^2$. The $t_{1i}$ to $t_{li}$ may be functionally related but are assumed to be linearly independent. Of the set of $l+1$ parameters $a_j$ we know the true values of any $l_1$ (the set $a_r$) and do not know the values of the other $l_2$ (the set $a_s$). It is assumed that $l_2 < n$.

THEOREM. *If the set $\hat{a}_s$ be the maximum likelihood estimates of the set $a_s$, then*

$$P = \sum_i \frac{1}{\sigma_i^2}\left\{y_i - \sum_r a_rt_{ri} - \sum_s \hat{a}_st_{si}\right\}^2,$$

*is distributed as $\chi^2$ with $n-l_2$ degrees of freedom, and independently of the set $\hat{a}_s$ which are jointly distributed in the $l_2$-variate normal distribution*

$$C_1e^{-\frac{1}{2}Q}\prod_s d\hat{a}_s$$

*with*

$$Q = \sum_i \frac{1}{\sigma_i^2}\left\{\sum_s (\hat{a}_s - a_s)t_{si}\right\}^2.$$

*This $Q$ is distributed independently of $P$ as $\chi^2$ with $l_2$ degrees of freedom. If*

$$R = \sum_i \frac{1}{\sigma_i^2}\left\{y_i - \sum_j a_j t_{ji}\right\}^2,$$

*then $R=P+Q$, and $R$ is distributed as $\chi^2$ with $n$ degrees of freedom.*

Proof of the Theorem: To simplify the discussion (and this has already been done in the expressions for $P$ and $R$ in the statement of the theorem) we shall introduce the variable $t_{0i}$ in the expression for the means with the understanding that $t_{0i}=1$.

The joint distribution of the $y_i$ is

$$C_2 e^{-\frac{1}{2}R} dy_1 dy_2 \cdots dy_n$$

and this will be a maximum when $R$ is a minimum. The $\hat{a}_s$ are therefore determined so as to make

$$\sum_i \frac{1}{\sigma^2}\left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\}^2$$

a minimum.

The $l_2$ estimates $\hat{a}_s$ are thus determined by the $l_2$ linear equations

$$\sum_i \frac{t_{si}}{\sigma_i^2}\left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\} = 0. \tag{1}$$

We next note that

$$y_i - \sum_j a_j t_{ji} = \left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\} + \left\{\sum_s (\hat{a}_s - a_s) t_{si}\right\},$$

we square both sides, divide by $\sigma_i^2$, and sum over $i$, to find by using (1) that

$$R = P + Q.$$

Now let

$$y_i = \sum_r a_r t_{ri} + \sum_s \hat{a}_s t_{si} + \lambda_i \sqrt{P}, \qquad i = 1 \text{ to } n.$$

By the definition of $P$ and (1) we have the $n$ variables $\lambda_i$ satisfying the $l_2+1$ equations

$$\sum_i \frac{\lambda_i^2}{\sigma_i^2} = 1, \qquad \sum_i \frac{\lambda_i t_{si}}{\sigma_i^2} = 0. \tag{2}$$

We now can write the joint probability element of $y_1, y_2, \cdots, y_n$ as

(3) $$C_2 e^{-\frac{1}{2}P} e^{-\frac{1}{2}Q} dy_1 dy_2 \cdots dy_n.$$

Next we shall change variables in this differential from $y_1, y_2, \cdots, y_n$ to $P$, the set $\hat{a}_s, \lambda_1, \lambda_2, \cdots, \lambda_{n-l_2-1}$.

The Jacobian of the transformation becomes

$$J_1 = \begin{vmatrix} \dfrac{\partial y_1}{\partial P} \cdots \dfrac{\partial y_1}{\partial \hat{a}_s} \cdots \dfrac{\partial y_1}{\partial \lambda_1} \cdots \dfrac{\partial y_1}{\partial \lambda_{n-l_2-1}} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial y_n}{\partial P} \cdots \dfrac{\partial y_n}{\partial \hat{a}_s} \cdots \dfrac{\partial y_n}{\partial \lambda_1} \cdots \dfrac{\partial y_n}{\partial \lambda_{n-l_2-1}} \end{vmatrix}.$$

We have

$$\frac{\partial y_i}{\partial P} = \frac{\lambda_i}{2\sqrt{P}},$$

$$\frac{\partial y_i}{\partial \hat{a}_s} = t_{si},$$

$$\frac{\partial y_i}{\partial \lambda_k} = 0 \qquad \text{for } k \neq i, \text{ and } = \sqrt{P} \text{ for } k = i.$$

Therefore

$$J_1 = \begin{vmatrix} \dfrac{\lambda_1}{2\sqrt{P}} \cdots t_{s1} \cdots \sqrt{P} & 0 & \cdots \\ \dfrac{\lambda_2}{2\sqrt{P}} \cdots t_{s2} \cdots 0 & \sqrt{P} & \cdots \\ \cdots\cdots\cdots\cdots & & \\ \dfrac{\lambda_n}{2\sqrt{P}} \cdots t_{sn} \cdots 0 & 0 & \cdots \end{vmatrix}$$

$$= P^{(n-l_2)/2-1} g_1(\lambda_1, \lambda_2, \cdots, \lambda_n) = P^{(n-l_2)/2-1} g_2(\lambda_1, \lambda_2, \cdots, \lambda_{n-l_2-1}).$$

To obtain $g_2$ we use (2) to eliminate $\lambda_{n-l_2}$ to $\lambda_n$ from $g_1$. We now can write (3) as

$$C_2 P^{(n-l_2)/2-1} e^{-\frac{1}{2}P} dP e^{-\frac{1}{2}Q} \prod_s d\hat{a}_s g_2(\lambda_1, \lambda_2, \cdots, \lambda_{n-l_2-1}) d\lambda_1 \cdots d\lambda_{n-l_2-1}.$$

This shows that $P$, the set $\hat{a}_s$ ,and the $\sigma$ variables are three independent sets of variables. Therefore $P$ is distributed as

(4) $$C_3 P^{(n-l_2)/2-1} e^{-\frac{1}{2}P} dP,$$

and the set $\hat{a}_s$ is distributed independently as

(5) $$C_1 e^{-\frac{1}{2}Q} \prod_s d\hat{a}_s.$$

These are the required $\chi^2$ and normal $l_2$-variate distributions for $P$ and the set $\hat{a}_s$.

To prove that $R$ is distributed as $\chi^2$ with $n$ degrees of freedom we need only note that $R$ is independent of $l_2$ and the $\hat{a}_s$., and that $P$ becomes $R$ when $l_2=0$.

The definition of $Q$ in the theorem shows that $Q$ is a function of the set $\hat{a}_s$. Since the latter is distributed independently of $P$, so is $Q$.

To find the distribution of $Q$ we start with (5), letting

$$\sum_s (\hat{a}_s - a_s)t_{si} = \delta_i\sqrt{Q}, \qquad i = 1 \text{ to } n \tag{6}$$

and change variables from the $l_2$ variables $\hat{a}_s$ to $\delta_1, \delta_2, \cdots \delta_{l_2-1}, Q$. If $J_2$ is the Jacobian of this transformation, then $1/J_2$ is the Jacobian of the inverse transformation.

$$\frac{1}{J_2} = \begin{vmatrix} \dfrac{\partial\delta_1}{\partial\hat{a}_1} & \cdots & \dfrac{\partial\delta_{l_2-1}}{\partial\hat{a}_1} & \dfrac{\partial Q}{\partial\hat{a}_1} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial\delta_1}{\partial\hat{a}_{l_2}} & \cdots & \dfrac{\partial\delta_{l_2-1}}{\partial\hat{a}_{l_2}} & \dfrac{\partial Q}{\partial\hat{a}_{l_2}} \end{vmatrix}.$$

We have

$$\frac{\partial\delta_i}{\partial\hat{a}_s} = \frac{t_{si}}{\sqrt{Q}},$$

$$\frac{\partial Q}{\partial\hat{a}_s} = -\sum_i \frac{2}{\sigma_i^2}\left\{\sum_s (\hat{a}_s - a_s)t_{si}\right\} t_{si}$$

$$= -\sqrt{Q}\sum_i \frac{2\delta_i t_{si}}{\sigma_i^2}.$$

Hence

$$\frac{1}{J_2} = \left(\frac{1}{\sqrt{Q}}\right)^{l_2-1} \sqrt{Q}\, g_3(\delta_1, \delta_2, \cdots, \delta_n),$$

and

$$J_2 = Q^{(l_2/2)-1} g_4(\delta_1, \delta_2, \cdots, \delta_{l_2-1}).$$

In going from $1/g_3$ to $g_4$ we have eliminated $\delta_{l_2}$ to $\delta_n$ by use of (6) and

$$\sum_i \frac{\delta_i^2}{\sigma_i^2} = 1. \tag{7}$$

The latter follows from using (6) in the expression for $Q$ in the statement of the theorem. It should be noted that (6) consists of $n$ equations in the $l_2$ variables

$\hat{a}_s - a_s$. It is possible to solve (6) for $\delta_{l_2+1}$ to $\delta_n$ in terms of $\delta_1$ to $\delta_{l_2}$ with the factor $\sqrt{Q}$ cancelling out in the process. Equation (7) then gives the additional relationship to eliminate $\delta_{l_2}$.

Expression (5) may now be written as

$$C_1 Q^{(l_2/2)-1} e^{-\frac{1}{2}Q} dQ g_4(\delta_1, \delta_2, \cdots, \delta_{l_2-1}) d\delta_1 \cdots d\delta_{l_2-1}.$$

This shows that $Q$ is distributed independently of the $\delta$ variables as

$$C_4 Q^{(l_2/2)-1} e^{-\frac{1}{2}Q} dQ.$$

The latter is the $\chi^2$ distribution with $l_2$ degrees of freedom.

It may be noted that the part of the analysis that leads to $R = P + Q$ includes an indirect proof of the theorem that if $\chi_1^2$ and $\chi_2^2$ are distributed independently as $\chi^2$ with $\nu_1$ and $\nu_2$ degrees of freedom, respectively, then $\chi^2 = \chi_1^2 + \chi_2^2$ is distributed as $\chi^2$ with $\nu_1 + \nu_2$ degrees of freedom. Also the part of the theorem concerning the distribution of $Q$ is a direct proof of the theorem concerning the $\chi^2$ distribution with $k$ degrees of freedom of minus two times the exponent in a normal $k$-variate distribution. The latter would be the special case $n = l_2 = k$.

COROLLARY 1. *If $u_i$, $i = 1$ to $n$, are $n$ independent normal variables with common mean $a$ and common variance $\sigma^2$, then*

$$\chi^2 = \frac{1}{\sigma^2} \sum_i (u_i - a)^2$$

*is distributed as $\chi^2$ with $n$ degrees of freedom.*

COROLLARY 2. *If $u_i$ are defined as in Corollary 1 and $n\bar{u} = \sum_i u_i$, then*

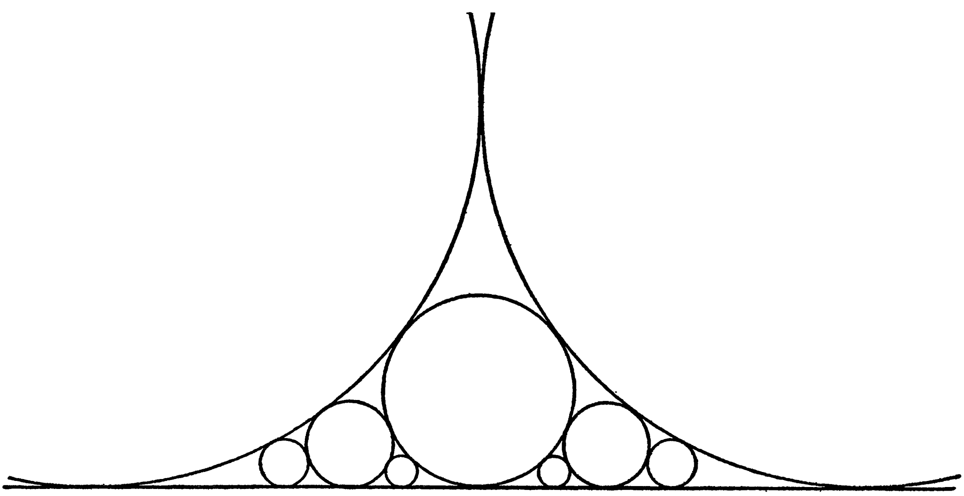
$$\chi^2 = \frac{1}{\sigma^2} \sum_i (u_i - \bar{u})^2$$

*is distributed as $\chi^2$ with $n - 1$ degrees of freedom.*

# A FAMILY OF INTEGERS AND A THEOREM ON CIRCLES

G. T. WILLIAMS, Brookhaven National Laboratory, N. Y.
D. H. BROWNE, Buffalo, N. Y.

**1. Introduction.** Between two tangent unit circles lying on a line $L$, a third circle is inscribed tangent to both circles and to $L$ (see figure). Then two more circles are inscribed in the newly-created spaces, each tangent to two circles and to $L$. Now four circles are inscribed similarly along $L$, and so on *ad inf.*



We propose to determine the area-sum of circles in this configuration.*

**2. The theorems.**

THEOREM 1. *If two circles of radii $1/r^2$ and $1/s^2$ are tangent to each other and to $L$, the radius of the smaller circle tangent to both circles and to $L$ is $1/(r+s)^2$, that of the larger circle being $1/(r-s)^2$.*

This is readily established by comparing the projections of the three lines of centers on $L$. Thus, in the figure, the first inscribed circle has a radius of $\frac{1}{4}$, the next two have radii of $1/9$, and so on.

DEFINITION. Calling the primary position (the two unit circles) the 0th stage, at the $n$th stage we have $2^n+1$ circles whose radii, reading across, we designate as

$$\{1/A_\nu^n\}^2 \qquad (\nu = 0, 1, \cdots, 2^n),$$

* There is a treatment at some length of these circles in a paper on Fractions by L. R. Ford in this MONTHLY, vol. 45, pp. 586–601 (November, 1938). Some of the results of the present paper will be found there. The problem of the sum of the areas, however, was not considered.—EDITOR.

where the $A$ are successively the positive integers

$$\begin{array}{c} 1\quad 1 \\ 1\ 2\ 1 \\ 1\ 3\ 2\ 3\ 1 \\ 1\ 4\ 3\ 5\ 2\ 5\ 3\ 4\ 1 \\ 1\ 5\ 4\ 7\ 3\ 8\ 5\ 7\ 2\ 7\ 5\ 8\ 3\ 7\ 4\ 5\ 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

These obviously satisfy the relations,

$$\begin{aligned} A_\nu^n &= A_{2^n-\nu}^n, \\ A_{2\nu}^{n+1} &= A_\nu^n, \\ A_{2\nu+1}^{n+1} &= A_\nu^n + A_{\nu+1}^n. \end{aligned}$$

These numbers have a host of properties, some, involving $A_\nu^n/A_\nu^{n+1}$, being akin to properties involved in the theory of Farey Series (*cf.* Hardy & Wright, *Introduction to Theory of Numbers*, p. 23), but we mention only a few which do not bear directly on the principal theorem.

THEOREM 2. *If* $S_n$ *denotes* $\sum_{\nu=0}^{2^n} A_\nu^n$, then $S_n=3^n+1$.

For it is evident that $S_n=S_{n-1}+2(S_{n-1}-1)=3S_{n-1}-2$.

Since $A_0^n=1$ for all $n$, we have immediately, $A_1^n=n+1$. Now, since the remaining $A$ are linear combinations of previous ones, we have an inductive proof that

THEOREM 3. $A_\nu^n$ *is a linear function of* $n$ *for constant* $\nu$.

Indeed, from the recurrence, we find $A_2^n=n$, $A_3^n=2n-1$, $A_4^n=n-1$, $A_5^n=3n-4$, and so on. These linear coefficients also behave interestingly, but a discussion of them would carry us too far afield.

THEOREM 4. *For fixed* $n$, *the maximum* $A_\nu^n$ *is* $f_n$, *the nth Fibonacci number* ($\{f_n\}_0^\infty=1, 2, 3, 5, \cdots$), *and is given by* $\nu=(2^n-(-1)^n)/3$, *and symmetrically.*

THEOREM 5.

$$A_\nu^n \mid A_{\nu-1}^n + A_{\nu+1}^n \qquad \text{(all } n, \nu).$$

The proof is inductive. By assuming the proposition for $n-1$ and all $\nu$, we deduce its validity for $n$ and all $\nu$. When the subscript is odd, the theorem holds trivially. When it is even, we observe that

$$A_{2\nu-1}^n + A_{2\nu+1}^n = A_{\nu-1}^{n-1} + 2A_\nu^{n-1} + A_{\nu+1}^{n-1}$$

and by the inductive hypothesis,

$$A_{\nu}^{n-1} \mid A_{\nu-1}^{n-1} + A_{\nu+1}^{n-1}.$$

But

$$A_{\nu}^{n-1} = A_{2\nu}^{n}.$$

Hence it is established generally.

We now prove what is the fundamental property of this table of numbers.

THEOREM 6. *Every coupled pair of relatively prime integers occurs in the table of A's.*

Again the proof is inductive and depends on the two conditions,

(*i*) If an integer $m$ occurs next to each of the $\phi(m)$ integers $1, a_2, a_3, \cdots, m-1$ prime to and less than $m$,

(*ii*) then ultimately it must occur next to each of the integers

$$q = jm + 1 \qquad (i = 1, a_2, \cdots, m-1; j = 1, 2, \cdots)$$

as can be seen by simple inductive reasoning. Plainly these numbers comprise all $q$ such that $q>m$, $(q, m)=1$. Hence *all* relatively prime pairs involving $m$ occur if (*i*) holds for $m$. But if (*i*) is true for all integers less than $m$ it is true for $m$, since the $\phi(m)$ integers prime to $m$ are included among $1, 2, \cdots, m-1$ for which (*i*) and (*ii*) both hold. The induction is completed by noting the truth of (*i*) when $m=2$.

THEOREM 7. *Ultimately, every integer $m>1$, appears precisely $\phi(m)$ times in a row of the table.*

This is true since $m$ is formed by adding an integer to each of the $\phi(m)$ integers $1, a_2, \cdots, m-1$. (*Cf.* this MONTHLY, 1947, 112, Problem 4236.)

Our concluding remarks return to the configuration of circles which initiated this discussion.

THEOREM 8. *The area-sum of the configuration is*

$$\pi + \frac{\pi\zeta(3)}{\zeta(4)} = 6.6307288,$$

*where $\zeta(s) = \sum 1/n^s$ is the Riemann Zeta function.*

The area becomes by Theorem 7

$$\lim_{n\to\infty} \sum \frac{\pi}{(A^n)^4} = \pi + \pi \sum_{m=1}^{\infty} \frac{\phi(m)}{m^4}$$

and from p. 249, *loc. cit.*, the result follows.

As might be expected, the same argument shows that the sum of radii diverges.

## MATHEMATICAL NOTES

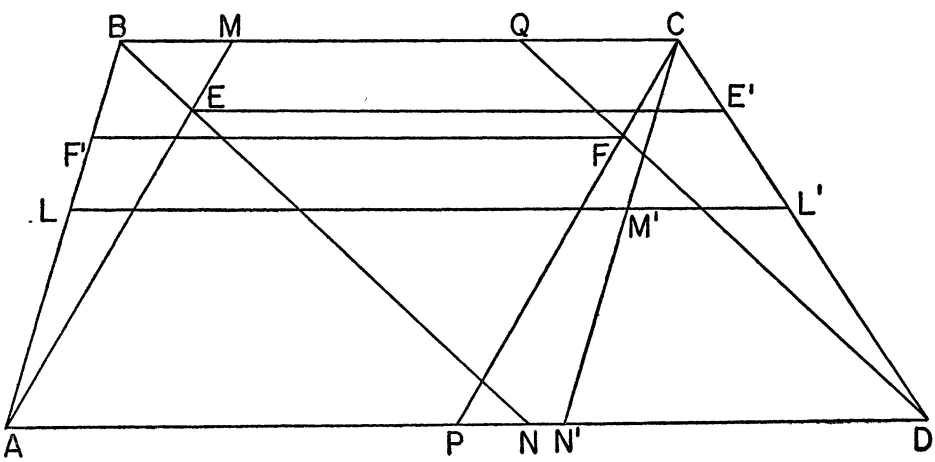
EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

### THEOREM ON THE TRAPEZOID

VICTOR THÉBAULT, Tennie, Sarthe, France.

The theorem which we are going to prove is a generalization, believed to be new, of a well known property in the geometry of the trapezoid.




LEMMA. If *L and L' divide the non-parallel sides AB and CD of a trapezoid in the same ratio*

$$BL/BA = k = CL'/CD,$$

*then*

$$LL' = (1 - k)BC + kAD.$$

*Proof.* If the parallel to $AB$ through the vertex $C$ cuts $LL'$ and $AD$ at $M'$ and $N'$, by similar triangles as shown in the figure, we have

$$k = CL'/CD = M'L'/N'D = M'L'/(AD - AN') = M'L'/(AD - BC);$$

hence

$$M'L' = k(AD - BC)$$

and

$$LL' = LM' + M'L' = BC + M'L' = (1 - k)BC + kAD.$$

THEOREM. *If through the vertices $A$ and $C$, $B$ and $D$, of a trapezoid $ABCD$, pairs of parallels are drawn meeting at $E$ and $F$ ($E$ is the intersection of the lines*

*drawn from $A$ and $B$, $F$ that of the lines drawn from $C$ and $D$), the distances of $E$ and $F$ to $CD$ and $AB$, measured along parallels to the parallel sides of the trapezoid, are equal.*

*Proof.* The parallels drawn from $A$ and $C$ meet $BC$ and $AD$ at $M$ and $P$, those drawn from $B$ and $D$ meet $AD$ and $BC$ at $N$ and $Q$, and these parallels have any directions. See the figure.

Let $EE'$ and $FF'$ be the distances of the intersections $E$ and $F$ of $AM$ and $BN$, $CP$ and $DQ$ to $CD$ and $AB$, measured along parallels to the parallel sides of the trapezoid.

If

$$BC = a, \qquad AD = b, \qquad AP = MC = m, \qquad BQ = ND = n,$$

then

$$k \equiv ME/MA = (a - m)/(a + b - m - n),$$
$$k' \equiv QF/QD = (a - n)/(a + b - m - n),$$

for

$$ME/EA = BM/AN = (a - m)/(b - n),$$
$$QF/FD = QC/PD = (a - n)/(b - m).$$

If we now apply the lemma to the trapezoid $AMCD$, the trapezoid $ABQD$, and the lines $EE'$ and $FF'$ parallel to their parallel sides, we find

$$\begin{aligned} EE' &= (1 - k)MC + kAD = (1 - k)m + kb \\ &= (ab - mn)/(a + b - m - n) \\ &= (1 - k')n + k'b = (1 - k')BQ + k'AD = FF', \end{aligned}$$

and this proves the theorem.

COROLLARY. *In a trapezoid the intersection of the diagonals is the midpoint of the segment between the points where the non-parallel sides meet a line drawn through that intersection and parallel to the parallel sides of the trapezoid.*

This corresponds to the case when $E \equiv F$ is the intersection of the diagonals; then $m = n = 0$ and

$$E'F' = 2ab/(a + b),$$

that is $E'F'$ *is the harmonic mean of the parallel sides of the trapezoid.*

## ON A THEOREM OF J. GRIFFITHS*

LUCIEN DROUSSENT, Clermont-Ferrand, France

The English geometer J. Griffiths has established [1] the following theorem:

*The Euler circle, circumcircle, orthocentroidal circle, and the orthoptic circle of the Steiner ellipse of a triangle have the orthic axis as common radical axis.*

* Translated from the French by Howard Eves.

The following definitions should be recalled.† The *Euler circle* is the nine-point circle; the *orthocentroidal circle* is the circle having the join of the orthocenter and the centroid as diameter; the *Steiner ellipse* is the inscribed ellipse which touches the sides of the triangle at their midpoints; the *orthoptic circle* of an ellipse is the locus of points of intersection of perpendicular tangents to the ellipse, and is frequently referred to as the director circle or the Monge circle of the ellipse; the *orthic axis* is the axis of perspectivity of the given triangle and its orthic triangle.

With the aid of this theorem we shall obtain some new properties of cyclic quadrangles. Consider a quadrangle $ABCD$ inscribed in a circle of center $O$. Now the Euler circles $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ of the four triangles $BCD$, $CDA$, $DAB$, $ABC$ pass through a point $P$, called the *anticenter* of the quadrangle, and $P$ is also the point of concurrence of the Wallace lines of the vertices $A$, $B$, $C$, $D$ with the respect to the same triangles [2]. If $H_a$, $H_b$, $H_c$, $H_d$ are the orthocenters of the four triangles, then the altitudes $DH_a$, $DH_b$, $DH_c$ of the triangles $DBC$, $DCA$, $DAB$ intersect the sides $BC$, $CA$, $AB$ of triangle $ABC$ in three collinear points on the Wallace line of $D$ with respect to this triangle. This line passes through the midpoint of the segment $DH_d$, and there meets the Euler circle $(\omega_d)$ of triangle $ABC$ [3]. Since it is clear that this point must be the anticenter $P$, we have refound the following known property.

*The orthocenters of the four triangles formed by the vertices of a cyclic quadrangle are the symmetrics of the vertices with respect to the anticenter $P$* [4].

The lines $OH_a$, $OH_b$, $OH_c$, $OH_d$ are the Euler lines of the four triangles $BCD$, $CDA$, $DAB$, $ABC$. If $G_a$, $G_b$, $G_c$, $G_d$ are the centroids, and $\omega_a$, $\omega_b$, $\omega_c$, $\omega_d$ the centers of the Euler circles of these four triangles, the quadrangles $G_aG_bG_cG_d$ and $\omega_a\omega_b\omega_c\omega_d$ are homothetic with respect to $O$ to the quadrangle $ABCD$, the homothetic ratios being $1/3$ and $1/2$ respectively. The lines $H_bH_c$, $G_bG_c$, $\omega_b\omega_c$, $BC$ are then all parallel, and if $\alpha$ is the midpoint of $AD$, the line $P\alpha$, being the radical axis of $(\omega_b)$ and $(\omega_c)$, is perpendicular to these four lines.

Now, by Griffiths' Theorem, the orthocentroidal circles $(G_bH_b)$ and $(G_cH_c)$, and the circumcircle $(O)$, have for radical center the point of intersection of the orthic axes of the two triangles $CDA$ and $DAB$, and this point is also the radical center of the two Euler circles $(\omega_b)$ and $(\omega_c)$, and the circle $(O)$. This point, then, lies on the radical axis of $(G_bH_b)$ and $(G_cH_c)$ and also on the radical axis of $(\omega_b)$ and $(\omega_c)$. These two radical axes, each being perpendicular to $BC$, must therefore coincide. The radical axis of $(G_bH_b)$ and $(G_cH_c)$ is thus the line $P\alpha$, whence we have the following result.

*The six radical axes of the orthocentroidal circles of the four triangles formed by the vertices of a cyclic quadrangle are the lines joining the anticenter of the quadrangle to the midpoints of the six sides of the quadrangle. These six circles thus have*

† Inserted by the translator.

*the anticenter of the quadrangle as a common radical center and also as the center of a common orthogonal circle.*

Since the circumcircle and the Euler circle of a triangle are inverse with regard to the polar circle of the triangle [5], it follows that these three circles have a common radical axis, and Griffiths' Theorem could equally well have also included the polar circle of the triangle. By exactly the same argument as above, we may, then, in the last theorem replace "orthocentroidal circles" by either "polar circles" or "orthoptic circles of the Steiner ellipses."

The polar circle of a triangle is real only if the triangle is obtuse, and it is easy to see that either two or four of the polar circles of the four triangles formed by the vertices of a cyclic quadrangle must be real.


**References**

1. J. Griffiths, Nouvelles Annales de Mathématiques, 1864, p. 345, and 1865, p. 322.
2. See, *e.g.*, Emile Lemoine, Nouvelles Annales de Mathématiques, 1869, p. 47, or R. A. Johnson, *Modern Geometry*, p. 251 ff. (H.E.)
3. R. A. Johnson, art. 327. (H.E.)
4. R. A. Johnson, art. 265. (H.E.).
5. R. A. Johnson, art. 278. (H.E.)


## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.*

### PASCAL'S TRIANGLE AND NEGATIVE EXPONENTS

L. V. ROBINSON, University of South Carolina

A most interesting and useful fact apparently overlooked heretofore in the use of Pascal's triangle is that its methods are equally valid for negative, as well as positive, integral exponents. To illustrate this extension in evaluating the binomial coefficients for all integral values of the exponent, it is sufficient to refer to the following scheme:

$$\begin{array}{llllllll}
(a+b)^0 \longrightarrow & & & & & 1 & \longleftarrow (a-b)^{-1} & \\
(a+b)^1 \longrightarrow & & & & 1 & 1 & \longleftarrow (a-b)^{-2} & \\
(a+b)^2 \longrightarrow & & & 1 & 2 & 1 & \longleftarrow (a-b)^{-3} & \\
(a+b)^3 \longrightarrow & & 1 & 3 & 3 & 1 & \longleftarrow (a-b)^{-4} & \\
(a+b)^4 \longrightarrow & 1 & 4 & 6 & 6 & 1 & \longleftarrow (a-b)^{-5} & \\
(a+b)^5 \longrightarrow 1 & .5 & 10 & 10 & 5 & 1 & &
\end{array}$$

Thus the horizontal numbers are the coefficients for $(a+b)^n$ and the diagonal ones are for $(a-b)^{-n}$. For example, the binomial expansion of $(a-b)^{-2}$ is, accordingly,

$$(a-b)^{-2} = a^{-2} + 2a^{-3}b + 3a^{-4}b^2 + 4a^{-5}b^3 + 5a^{-6}b^4 + \cdots, \qquad b < a.$$

It seems never inappropriate to point out also to students the use of Pascal's triangle in evaluating the probability of a given number of heads, say, from an equal, or greater, number of tosses of a coin.

## UNIFORM CONVERGENCE AND CONTINUITY

L. C. GREEN, Haverford College

Undergraduate students often have difficulty with the concept of uniform convergence and the content of the theorems which arise in connection with it. In particular the relation of uniform convergence to continuity is not always clear to the beginner. This relation may be considered as embodied in the following:

THEOREM. *The limit function $L(x)$ of a sequence or series of continuous functions which converge uniformly in an interval is itself a continuous function in that interval.*

The following simple sequences and the series of which they are the partial sums have been found helpful to the student in understanding the content of the theorem above.

| | | Uniform Convergence | Non-uniform Convergence |
|---|---|---|---|
| $L(x)$ Continuous | Terms of sequence continuous | Case treated by the theorem | $f_n(x)=\sin nx \quad 0\leqq x\leqq \frac{\pi}{n}$<br>$f_n(x)=0 \quad \frac{\pi}{n}<x$ |
| | Terms of sequence discontinuous | $f_n(x)=x^n \quad \lvert x\rvert<\frac{1}{2}$<br>$f_n(x)=2x^n \quad \frac{1}{2}\leqq\lvert x\rvert\leqq a<1$ | $f_n(x)=\sin nx \quad 0\leqq x\leqq \frac{\pi}{2n}$<br>$f_n(x)=0 \quad \frac{\pi}{2n}<x$ |
| $L(x)$ Discontinuous | Terms of sequence continuous | An example here would contradict the theorem | $f_n(x)=(\cos^2 x)^n$ |
| | Terms of sequence discontinuous | $f_n(x)=\frac{1-x^n}{1-x} \quad \lvert x\rvert<\frac{1}{2}$<br>$f_n(x)=2\,\frac{1-x^n}{1-x} \quad \frac{1}{2}\leqq\lvert x\rvert\leqq a<1$ | $f_n(x)=e^{1/x}\frac{1-x^n}{1-x} \quad 0\neq\lvert x\rvert\leqq a<1$<br>$f_n(0)=1$ |

## INTEGRATION BY PARTS

K. W. Folley, Wayne University

A schematic method for integrating by parts the product of two functions of $x$ is illustrated in the following examples.

Example 1. The product of a power of $x$ by a function which can be integrated successively.

$$I_1 = \int x^5 \cos x dx.$$

Let $f = x^5$ and $g = \cos x$. Write the derivatives of $f$ with respect to $x$ in one column and the integrals of $g$ with respect to $x$ in a second column, continuing to the row in which $f^{(n)} = 0$.

$$\begin{array}{lcl}
x^5 & & \cos x \\
 & + & \\
5x^4 & & \sin x \\
 & - & \\
20x^3 & & -\cos x \\
 & + & \\
60x^2 & & -\sin x \\
 & - & \\
120x & & \cos x \\
 & + & \\
120 & & \sin x \\
 & - & \\
0 & & -\cos x
\end{array}$$

$$I_1 = x^5 \sin x + 5x^4 \cos x - 20x^3 \sin x - 60x^2 \cos x + 120x \sin x + 120 \cos x + C.$$

**Example 2.** **The product of an exponential function by a trigonometric function.**

$$I_2 = \int e^{ax} \sin bx dx.$$

**Let $f = e^{ax}$ and $g = \sin bx$. Proceed as in Example 1, but continue to the row in which the product $f^{(n)} g_n$ is a constant multiple of $I_2$.**

$$\begin{array}{lcl}
e^{ax} & & \sin bx \\
 & + & \\
ae^{ax} & & -\dfrac{1}{b} \cos bx \\
 & - & \\
a^2 e^{ax} & + & -\dfrac{1}{b^2} \sin bx
\end{array}$$

$$I_2 = e^{ax}\left(-\frac{1}{b}\cos bx + \frac{a}{b^2}\sin bx\right) - \frac{a^2}{b^2} I_2$$

$$I_2 = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$$

EXAMPLE 3.

$$I_3 = \int xe^x \sin 2x dx.$$

$$\begin{array}{ccc}
xe^x & & \sin 2x \\
 & + & \\
(x+1)e^x & & -\frac{1}{2}\cos 2x \\
 & - & \\
(x+2)e^x & + & -\frac{1}{4}\sin 2x
\end{array}$$

$$I_3 = -\frac{1}{2}xe^x \cos 2x + \frac{1}{4}e^x(x+1)\sin 2x - \frac{1}{4}I_3 - \frac{1}{2}\int e^x \sin 2x dx.$$

The final integral is a special case of Example 2. If we substitute its value and solve for $I_3$, we obtain

$$I_3 = \frac{e^x}{25}\left[(3+5x)\sin 2x + (4-10x)\cos 2x\right] + C.$$

The justification for the preceding method follows from the continued application of the formula for integration by parts.

$$\int fgdx = fg_1 - \int f'g_1dx = fg_1 - f'g_2 + \int f''g_2dx = \cdots$$

$$= fg_1 - f'g_2 + \cdots + (-1)^{n-1}f^{(n-1)}g_n + (-1)^n\int f^{(n)}g_ndx,$$

where

$$f^{(i+1)} = \frac{d}{dx}f^{(i)} \quad \text{and} \quad g_{i+1} = \int g_idx.$$

## A SIMPLIFICATION OF THE SECOND DERIVATIVE TEST

L. S. LAWS, University of Minnesota

This note concerns a simplification of the second derivative test for maximum-minimum in the case where $y=\phi(x)/f(x)$. Obtain $y'=N(x)/[f(x)]^2$ by the usual method for differentiating a quotient of functions, and find a value $x_1$ such that $N(x_1)=0$ and $f(x_1)\neq 0$.

To prove: $N'(x_1)$ has the same sign as $y''(x_1)$.

*Proof*:

$$y'' = \frac{[f(x)]^2N'(x) - N(x)[2f(x)f'(x)]}{[f(x)]^4},$$

but since $f(x_1)\neq 0$ and $N(x_1)=0$, the quantity $[f(x_1)]^2$ is positive and $N'(x_1)$ will

have the same sign as $y''(x_1)$. Therefore, $y$ will have a relative maximum at $x=x_1$ if $y'(x_1)=0$ and $N'(x_1)<0$; or $y$ will have a relative minimum at $x=x_1$ if $y'(x_1)=0$ while $N'(x_1)>0$.

## VECTOR DERIVATION OF THE SINE AND COSINE LAWS IN SPHERICAL TRIGONOMETRY

J. F. HEYDA, Indianapolis, Indiana

The usual derivation of these formulas from a study of the relationships involved in the proper plane triangles frequently leaves the student in a three-dimensional fog, part of this fog arising as a result of the instructor's poor diagrams. The present derivation requires little artistic ability on the part of the instructor, providing that he has spent a session or two explaining the fundamental operations with vectors including dot and cross multiplication.

Assume an arbitrary spherical triangle with vertices at $A$, $B$, $C$ with opposite sides $a$, $b$, $c$ and let $\mathbf{e}_A$, $\mathbf{e}_B$, $\mathbf{e}_C$ denote unit vectors directed from the center of the sphere to the vertices $A$, $B$, $C$, respectively, Also let $\mathbf{t}_{AB}$ denote the unit tangent vector, tangent to arc $AB$ at $A$ and directed toward $B$, with equivalent interpretations for $\mathbf{t}_{AB}$, $\mathbf{t}_{BC}$, *etc.* The unit normals

$$\mathbf{n}_{AB} = \mathbf{e}_A \times \mathbf{t}_{AB}, \qquad \mathbf{n}_{BC} = \mathbf{e}_B \times \mathbf{t}_{BC}, \qquad \mathbf{n}_{AC} = \mathbf{e}_C \times \mathbf{t}_{CA}$$

to the planes of arcs $AB$, $BC$, $AC$, respectively, are then directed toward the interior of the solid angle $O$-$ABC$, our sphere in question having unit radius.

To prove the cosine law we note first that

$$\mathbf{e}_C = \cos b\, \mathbf{e}_A + \sin b\, \mathbf{t}_{AC}$$

and also that

$$\mathbf{t}_{AC} = \cos A\, \mathbf{t}_{AB} + \sin A\, \mathbf{n}_{AB},$$

whence we have

$$\mathbf{e}_C = \cos b\, \mathbf{e}_A + \sin b \cos A\, \mathbf{t}_{AB} + \sin b \sin A\, \mathbf{n}_{AB}.$$

Dotting both sides of the latter equation with $\mathbf{e}_B$, we obtain

$$\mathbf{e}_B \cdot \mathbf{e}_C = \cos a = \cos b \cos c + \sin b \sin c \cos A$$

as desired.

The proof of the sine law follows immediately upon dotting both sides of the identity

$$\mathbf{e}_C = \cos b\, \mathbf{e}_A + \sin b\, \mathbf{t}_{AC} = \cos a\, \mathbf{e}_B + \sin a\, \mathbf{t}_{BC}$$

with $\mathbf{n}_{AB}$, observing that

$$\mathbf{n}_{AB} \cdot \mathbf{e}_A = \mathbf{n}_{AB} \cdot \mathbf{e}_B = 0,$$

so that the desired result is

$$\sin b \sin A = \sin a \sin B.$$

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 791. *Proposed by G. W. Walker, Buffalo, N.Y.*

The court mathematician once received his salary for a year's service all at one time, and all in silver "dollars," which he proceeded to arrange in nine unequal piles, making a magic square. The king looked, and admired, but complained that there was not a single prime number in any of the piles. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime." They investigated, and found that this was indeed true. The king was about to give him nine "dollars" more, when the court jester said, "Wait!." Then the jester subtracted one coin from each pile instead; and they found in this case also a magic square with every element a prime number. The jester kept the nine "dollars." How much salary must the mathematician have been receiving?

E 792. *Proposed by N. S. Mendelsohn, University of Manitoba*

Show that a pack of $2n$ cards is brought back to its original position in at most $2n-2$ perfect riffle shuffles. (A perfect riffle shuffle is one which sends the cards from the arrangement $1, 2, 3, 4, \cdots, 2n-1, 2n$ to the arrangement $1, n+1, 2, n+2, \cdots, n, 2n$.)

E 793. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

With straight edge alone construct a hexagon which can possess both an inscribed and a circumscribed conic.

E 794. *Proposed by Huan-Ting Kuo, National Wuhan University, China*

Show that

$$\begin{vmatrix} \binom{r+1}{r} & \binom{r+1}{r+1} & 0 & 0 & \cdots & 0 \\ \binom{r+2}{r} & \binom{r+2}{r+1} & \binom{r+2}{r+2} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \binom{n}{r} & \binom{n}{r+1} & \binom{n}{r+2} & \binom{n}{r+3} & \cdots & \binom{n}{n-1} \end{vmatrix} = \binom{n}{r}.$$

E 795. *Proposed by N. A. Court, University of Oklahoma*

The pairs of points $U'$, $U''$; $V'$, $V''$; $W'$, $W''$ are marked, respectively, on the edges $DA$, $DB$, $DC$ of a tetrahedron $ABCD$. Determine three points $U$, $V$, $W$ on the edges $BC$, $CA$, $AB$, respectively, so that the three lines joining the vertices of each of the triangles $UV'W''$, $VW'U''$, $WU'V''$, $UVW$ to the corresponding vertices of the triangles $DCB$, $DAC$, $DBA$, $ABC$, respectively, shall have a point in common.

## SOLUTIONS

### A Problem in Stability

E 761 [1947, 163]. *Proposed by C. R. Perisho, Nebraska Wesleyan University.*

An object with a smooth lower plane surface, and center of gravity $h$ units above this surface, is balanced on a sphere of radius $R$. Find the relation between $h$ and $R$ which insures stability of the object under small displacements.

I. *Solution by E. S. Keeping, University of Alberta.* If the object of mass $M$ rocks through a small angle $\theta$, its gravitational potential energy, relative to the center of the sphere, is

$$V = Mg[R\theta \sin \theta + (R + h) \cos \theta],$$

provided that no slipping occurs. Therefore

$$\begin{aligned} dV/d\theta &= Mg(R\theta \cos \theta - h \sin \theta) \\ &= 0 \quad \text{at} \quad \theta = 0, \end{aligned}$$

so that equilibrium exists. Also

$$\begin{aligned} d^2V/d\theta^2 &= Mg[(R - h) \cos \theta - R\theta \sin \theta] \\ &= Mg(R - h) \quad \text{at} \quad \theta = 0. \end{aligned}$$

Hence the equilibrium is stable if $h<R$, unstable if $h>R$.

If $h=R$, then $d^3V/d\theta^3=0$ and $d^4V/d\theta^4=-2Mgh$ at $\theta=0$, so that equilibrium is unstable in this case.

The problem states that the object has a "smooth" under surface. If this means that there is no friction whatever, the equilibrium is bound to be unstable, since the slightest disturbance will cause the object to slide off.

If there is to be a restoring force at angle $\theta$, it is necessary that $\tan \theta \leqq \mu$; and hence the permissible displacement (for $h<R$) will vanish with $\mu$.

II. *Solution by R. A. Bradley, University of North Carolina.* Let $N$ be the point on the sphere vertically above its center $O$, $C$ the point of contact of the object when rocked through a small angle $NOC=\theta$, $A$ the position of the center of gravity of the object when in its displaced position, and $B$ the foot of the perpendicular from $A$ onto the tangent plane at $C$. For stable equilibrium $A$ must lie between $ON$ and the vertical line through $C$. That is, we must have

angle $BAC$ larger than $\theta$, whence

$$\tan BAC = BC/BA = R\theta/h > \tan\theta > \theta.$$

This implies that $h<R$. The equilibrium is unstable if $h \geqq R$.

Also solved by L. M. Kelly, J. C. Miller, G. W. Walker, and the proposer.

Rufus Crane pointed out that this problem occurs as problem 275 on page 242 of *Engineering Mechanics* by Timoshenko and Young. The problem is there solved by virtual displacements with the result given above.

### The Arbelos

E 762 [1947, 163]. *Proposed by J. R. Van Andel, Naval Air Experimental Station, Philadelphia, Pa.*

Let $A_1$ and $A_2$ be two circles with radii $a_1$ and $a_2$ and centers $(a_1, 0)$ and $(a_2, 0)$, respectively, with $a_2 > a_1 > 0$. Let $C$ be any circle in the crescent shaped area $M$ between $A_1$ and $A_2$, and tangent to both $A_1$ and $A_2$.

(a) The locus of the center of $C$ as it sweeps out $M$ is an ellipse with semi-axes $(a_1+a_2)/2$ and $\sqrt{a_1a_2}$.

(b) If $C_t$ is a circle of radius $r_t$ and center $P_t(x_t, y_t)$ where

$$\begin{aligned} r_t &= a_1a_2(a_2 - a_1)\phi_t, \\ x_t &= a_1a_2(a_2 + a_1)\phi_t, \\ y_t &= 2tr_t \\ \phi_t^{-1} &= a_1a_2 + t^2(a_2 - a_1)^2, \end{aligned}$$

then, for any real value of $t$, $C_t$ lies in $M$ and is tangent to $A_1$, $A_2$, and $C_{t-1}$.

*Solution by Norman Anning, Ann Arbor, Michigan.* (a) This part is elementary. A glance at a figure shows that the center of $C$ is always in such a position that the sum of its distances from $(a_1, 0)$ and $(a_2, 0)$ is $a_1+a_2$. The rest follows.

(b) If $(X, Y)$ is a point on $C_t$, then

$$(1) \qquad (X^2 + Y^2)\phi_t^{-1} - 2a_1a_2(a_1 + a_2)X - 4ta_1a_2(a_2 - a_1)Y + 4a_1^2a_2^2 = 0.$$

Apply to this circle the inversion

$$X = 4a_1a_2x/(x^2 + y^2), \qquad Y = 4a_1a_2y/(x^2 + y^2).$$

Then (1) becomes

$$x^2 + y^2 - 2(a_1 + a_2)x - 4t(a_2 - a_1)y + 4\phi_t^{-1} = 0,$$

which may be written as

$$(2) \qquad (x - a_1 - a_2)^2 + (y - 2ta_2 + 2ta_1)^2 = (a_2 - a_1)^2.$$

With $t$ as parameter, (2) is the family of equal circles which touch the

parallel lines $x=2a_1$ and $x=2a_2$. In this family, for every $t$, the circle $C_t$ is tangent to $C_{t-1}$ because the distance between their centers is equal to the diameter of either.

Now invert again. Circle $C_0$ inverts into itself and (2) inverts into (1). The line $x=2a_2$ inverts into the circle $A_1$, the inner boundary of the arbelos. Similarly, $x=2a_1$ inverts into $A_2$, the outer boundary. Since it is well known that inversion turns circles into circles and preserves contacts, the proof of the stated theorem is complete.

One tracing the history of the problem would find it under *arbelos*. See R. Johnson's *Modern Geometry*, for instance. The neatest of the properties, $y_t=2tr_t$, appears in book 4 of Pappus's *Collection*. See Ivor Thomas, *Greek Mathematical Works*, II (Loeb Classical Library, No. 362), p. 578.

Also solved by Paul Brock, Rufus Crane, G. A. Williams, and the proposer.

The proposer pointed out that J. Steiner in *Geometrische Betrachtungen* (1826) discussed, in particular, the chains of circles corresponding to the sequences $t=0, 1, 2, \cdots$ and $t=1/2, 3/2, 5/2, \cdots$. Williams mentioned several additional properties of the figure which easily follow from the inversion. For instance, the line joining the points of contact of $C_t$ with $A_1$ and $A_2$ passes through the fixed point $(2a_1a_2/(a_1+a_2), 0)$; the common internal tangent of $C_t$ and $C_{t-1}$ passes through this same point; the four points consisting of the origin, the centers of $A_1$ and $A_2$, and the above point form a harmonic set.

If the diameter of $A_1$ is taken as two-thirds that of $A_2$, then $r_1$ is one-seventh the diameter of $A_2$. Of this particular figure Victor Thébault has stated a very pretty property. Let the diameter of $A_2$ taken along the line of centers of $A_1$ and $A_2$ be $OB$, and let $BM$ be the tangent from $B$ to the circle $A_1$. Then the circle on $BM$ as diameter is tangent to the circle $C_1$.

*Editorial Note.* In connection with E 762 see *On a Generalization of the Arbelos*, by M. G. Gaba, this MONTHLY [1940, 19].

### A New Property of the Monge Point

E 763 [1947, 163]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenters of the faces of tetrahedron to the reflections in these faces of the points of intersection of the corresponding altitudes with the circumsphere, are concurrent at the Monge point of the tetrahedron.

*Solution by L. M. Kelly, University of Missouri.* In the tetrahedron $ABCD$ let $M$, $G$, $O$ be the Monge point, centroid, and circumcenter, and let $H_a$, $G_a$, $O_a$ be the orthocenter, centroid, and circumcenter of the face $BCD$. Denote by $A'$, $E$, and $G'$ the feet of the perpendiculars from $A$, $M$, and $G$ on the face $BCD$, and let the altitude $AA'$ cut the circumsphere at $X$. (A figure, with the same lettering, may be found on p. 69 of N. A. Court's *Modern Pure Solid Geometry.*) For convenience let us set $AA'=h$, $A'X=x$, $OO_a=p$. Then we have

$$h/4 = GG' = (ME + p)/2,$$

whence

$$ME = h/2 - p.$$

Again

$$2(p + x) = h + x,$$

or

$$x = h - 2p.$$

Thus, since $E$ is the midpoint of $A'H_a$, the line $H_aM$ will pass through the reflection of $X$ in $A'$. The theorem now follows.

Also solved by the proposer.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

***Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.***

### PROBLEMS FOR SOLUTION

4270. *Proposed by S. H. Gould, Victoria College, Toronto*

Let $b$ be a fixed positive integer, $m=1, 2, \cdots$, and $q$ irrational with $0<q<1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b+q$. Prove that every set of $b$ successive gaps contains exactly one multiple of $1+b/q$.

This is a generalization of problem 3173, [1927, 159].

4271. *Proposed by N. A. Court, University of Oklahoma*

The external bisectors of the three face angles of each trihedron of a given tetrahedron are coplanar. The four planes form a second tetrahedron. Show that the lines joining corresponding vertices of the two tetrahedrons form, in general, a hyperbolic group.

4272. *Proposed by A. L. Epstein, Asbury Park, N. J.*

Given the sequence $n_i$ where $n_1=a$, $n_2=b$, and $n_{i+2}=n_{i+1}+n_i$, $i=1, 2, \cdots$. Show that

$$\sum_{i=1}^{4k-2} n_i = r_k n_{2k+1}$$

and determine the form of $r_k$. (This is suggested by the familiar trick in which the subject selects $n_1$, $n_2$, and computes $n_3$, $n_4$, $\cdots$, $n_7$. He tells the operator the result $n_7$, and continues the calculation through $n_{10}$. When he adds the ten numbers he finds the operator has anticipated his result, which must be $11n_7$.)

4273. *Proposed by I. S. Cohen, University of Pennsylvania*

Prove that for any positive odd integer $n$, $\cos\theta$ and $\sin\theta$ are rational functions of $\cos^n\theta$ and $\sin^n\theta$ with rational coefficients. Find the explicit expressions in the case $n=3$.

4274. *Proposed by R. Bouvaist, Vincelles, Saône-et-Loire, France*

Let $A, B, C, D$ be arbitrary points on an equilateral hyperbola $(H)$, and let $A', B', C', D'$ be the corresponding diameterically opposite points. (1) The isogonal conjugates of $A', B', C', D'$ with respect to the triangles $BCD$, $CDA$, $DAB$, $ABC$, respectively, coincide in the same point $P$. (2) The isogonal conjugates of $A, B, C, D$ with respect to the triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, respectively, coincide in the same point $P'$. (3) $P$ and $P'$ are diametrically opposite on $(H)$.

## SOLUTIONS

### Special Pythagorean Triangles

4205 [1946, 278]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a right triangle has sides of integral lengths and the sum of the sides forming the right angle is a square, then the sum of the cubes of these two sides is the sum of two squares. Can the hypothenuse be a square?

*Solution by Paul Bateman, Philadelphia, Pennsylvania.* If a triple of positive integers $(a, b; c)$ satisfies the equation $a^2+b^2=c^2$, then either just one of the three numbers is even or else all three are even. Hence $\frac{1}{2}(c-b+a)$ and $\frac{1}{2}(c+b-a)$ are integers. Now

$$\begin{aligned} a^2 - ab + b^2 &= \tfrac{1}{2}[a^2 + b^2 + (a-b)^2] \\ &= [\tfrac{1}{2}(c+b-a)]^2 + [\tfrac{1}{2}(c-b+a)]^2, \end{aligned}$$

and hence $a^2-ab+b^2$ is always a sum of two squares. Thus $a^3+b^3=(a+b)(a^2-ab+b^2)$ is the sum of two squares if and only if $a+b$ is a square or the sum of two squares.

The hypothenuse can be a square. There are infinitely many such triangles, the smallest of which was given by Fermat:

$$(4\,565\,486\,027\,761,\quad 1\,061\,652\,293\,520;\ 4\,687\,298\,610\,289).$$

See Carmichael, *Diophantine Analysis*, p. 77, Ex. 1; Dickson, *History of the Theory of Numbers*, V. 2, pp. 620–627; Uspensky and Heaslet, *Elementary Number Theory*, pp. 413–419.

Solved also by Murray Barber, D. H. Browne, Daniel Finkel, C. D. Olds, and the Proposer.

**Four Collinear Orthocenters**

4210 [1946, 397]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If the parallels to the sides of triangle $ABC$ drawn through a point $P$ on the circumcircle meet that circle again at $A'$, $B'$, $C'$, the orthocenter $H$ of $ABC$ and those $\alpha$, $\beta$, $\gamma$ of $A'BC$, $B'CA$, $C'AB$ are on a straight line perpendicular to the Simson line, $\Delta$ of $P$ as to $ABC$; and the center of gravity of $\alpha$, $\beta$, $\gamma$ divides into the ratio 2:1 the distance from $H$ to the circumdiameter parallel to $\Delta$.

I. *Solution by J. H. Butchart, Arizona State College at Flagstaff.* The vector $AH$ equals the vector $A'\alpha$ since each is twice the vector from the circumcenter $O$ to the side $BC$ and perpendicular to $BC$. Hence the vector $H\alpha$ equals the vector $AA'$. If $K$ is the point where the circumcircle meets the perpendicular from $P$ to $BC$, it is known that $AK$ is parallel to $\Delta$. Since $A'$ and $K$ are ends of a diameter, $AA'$ is perpendicular to $AK$, and hence $H\alpha$ is perpendicular to $\Delta$. By the same argument $H\beta$ and $H\gamma$ are also perpendicular to $\Delta$, and the first statement of the proposal is proved.

For the second part, note that the diameter parallel to $\Delta$ bisects $AA'$, $BB'$ and $CC'$. Since the vector sum of $OA$, $OB$ and $OC$ is $OH$, the projection of this vector sum on $H\alpha$ is $DH$, where $D$ is the point in which $H\alpha$ meets this diameter. Then if the vectors $AA'$, $BB'$ and $CC'$ are laid off from $H$, their sum is the vector $2HD$. Hence the center of gravity of $\alpha, \beta, \gamma$ is the end of the vector $2HD/3$.

II. *Solution by the Proposer.* Let $ABC$ be the base circle in a system of complex coördinates, $\boldsymbol{P}$ being the unit point. Denote by $t_1$, $t_2$, $t_3$ the coördinates of $A$, $B$, $C$ and by $s_1$, $s_2$, $s_3$ their symmetric functions

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_1t_2 + t_2t_3 + t_3t_1, \qquad s_3 = t_1t_2t_3.$$

Then $s_1$ is the coördinate of $H$. The equation of $\Delta$ is

$$x - s_3\bar{x} = \tfrac{1}{2}(1 + s_1 - s_2 - s_3)$$

and that of the circumdiameter $\delta$ parallel to $\Delta$ is

$$x - s_3\bar{x} = 0;$$

hence the image $H'$ of $s_1$ in $\delta$ is $s_2$.

The coördinate of $A'$ being $t_2t_3$, the point $\alpha$ has as coördinate

$$t_2 + t_3 + t_2t_3,$$

and $\alpha$ is on the line

$$x - s_1 + s_3(\bar{x} - \bar{s}_1) = 0,$$

which contains $H$ and is perpendicular to $\Delta$.

Further, the center of gravity of $\alpha$, $\beta$, $\gamma$ is $(2s_1+s_3)/3$, the point dividing $HH'$ into the ratio 1:2.

Solved also by J. W. Clawson, H. E. Fettis, and Ou Li.

**The Quadrilateral**

4214 [1946, 397]. *Proposed by V. Thébault, Tennie, Sarthe, France*

On the sides $AB$, $CD$ of an arbitrary quadrangle $ABCD$ isosceles triangles are constructed with the same sense $A'AB$, $C'CD$, with the base angle $\theta$; and on the sides $BC$, $DA$ the isosceles triangles $B'BC$, $D'DA$ with the base angle $\pi/2-\theta$ and in the same sense as the first. Prove that (1) The lines $A'C'$, $B'D'$ are perpendicular and that the lengths of the segments are in the ratio $\tan\theta$. (2) The centroid of the quandrangle is on the straight line joining the midpoints of $A'C'$, $B'D'$ which it divides in the ratio $\tan^2\theta$. (3) Find the locus of the midpoints of the sides and diagonals of the quadrangle $A'B'C'D'$ and the envelope of all these lines.

*Solution by R. Goormaghtigh, Bruges, Belgium*

(1) Let $a$, $b$, $c$, $d$ be the complex coordinates of $A$, $B$, $C$, $D$, the origin being $O$; then, if $\tan\theta$ is denoted by $k$, the points $A'$, $C'$, $B'$, $D'$ are

$A'$: $[a+b+ik(a-b)]/2$, $\qquad$ $C'$: $[c+d+ik(c-d)]/2$,

$B'$: $[b+c+i(b-c)/k]/2$, $\qquad$ $D'$: $[d+a+i(d-a)/k]/2$.

The segments $A'C'$ and $B'D'$ are equipollent to those having $O$ for origin and the points

$$[c+d-a-b+ik(c-d-a+b)]/2,$$
$$[d+a-b-c+i(d-a-b+c)/k]/2$$

for extremities; since the first expression is equal to the second multiplied by $-ik$, the segments $A'C'$ and $B'D'$ are perpendicular and in the ratio $k$. (2) The midpoints $M$ and $N$ of $A'C'$ and $B'D'$ are

$$(a+b+c+d)/4+ik(a-b+c-d)/4,$$
$$(a+b+c+d)/4-i(a-b+c-d)/4k.$$

Hence the centroid $G$ of $A$, $B$, $C$, $D$, the coordinate of which is $(a+b+c+d)/4$ and the points $M$, $N$ are collinear and $MG$ and $GN$ are in the ratio $k^2t$.

The same proof applies to the following *generalization*:

*On the sides $A_1A_2, A_3A_4, \cdots, A_{2n-1}A_{2n}$ of a $2n$-sided polygon $A_1A_2\cdots A_{2n}$ are constructed in the same sense isosceles triangles $A_1'A_1A_2$, $A_3'A_3A_4, \cdots$, $A_{2n-1}'A_{2n-1}A_{2n}$ with the base angle $\theta$; on the sides $A_2A_3$, $A_4A_5, \cdots, A_{2n}A_1$ of the same polygon are constructed in the same sense triangles $A_2'A_2A_3$, $A_4'A_4A_5, \cdots$, $A_{2n}'A_{2n}A_1$ with the base angle $\phi$. Then the centroid $G$ of the vertices of the polygon is on the straight line joining the centroids $M$ and $N$ of the two groups of points $A_1', A_3', \cdots, A_{2n-1}'$ and $A_2', A_4', \cdots, A_{2n}'$ and $MG$ and $GN$ are in the ratio* $\tan\theta/\tan\phi$.

(3) The point-sets described by $A'$, $B'$, $C'$, $D'$ on the axes of $AB$, $BC$, $CD$, $DA$ being similar, their joins envelope parabolas tangent to those axes and the midpoints of all the segments between any two of them have as loci straight lines.

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.*

*Preparatory Business Mathematics.* By L. L. Smail. New York, The Ronald Press Co., 1947, 10+240 pages. $2.75.

Briefly this is a good book, though somewhat condensed, with the proper emphasis on the mathematical preparation for business and enough applications in later chapters to sustain interest. The somewhat skeletonized presentation found in the early chapters of the book is justified by the opening sentences: "It is assumed that the reader has had a course in elementary algebra. The first chapter (and a large part of the next few chapters) is intended to give merely a brief review of some of the topics of elementary algebra . . . . "

Teachers generally will approve of the generous use of italicized letters and bold face type to call attention to definitions, fundamental procedures and formulas. The material is broken up into convenient assignment units interspersed with ample sets of exercises, the answers to which are all given. The figures are well drawn with two notable exceptions:

(1) Fig. 11 is intended to show the approximations to the real roots of a cubic, but the curve quite obviously does not pass through any of the three designated points.

(2) Fig. 28, the graph of the parabola is clearly not symmetrical to its axis.

The most serious criticism of the book seems to be the treatment of the number system. Rational, irrational, and imaginary numbers are not defined, and except for one reference to irrational exponents, these words do not appear in the Index.

It may be argued that a knowledge of the classification of the number system is superfluous for business students, hence it is unnecessary to define the various kinds of numbers. In the interest of consistency then all references to rational, irrational, and imaginary numbers should be avoided. This unfortunately has not been done as noted below:

(1) *p.* 50 " . . . if there is no positive or negative $n$th root, any one of the $n$th roots may be taken as the principal root."

(2) *p.* 53 "The preceding definitions of powers will not apply if the expression is an irrational number, as for example $3^{\sqrt{2}}$."

(3) *p.* 54 "The irrational number $\sqrt{2}$ may be represented by the unending sequence of rational numbers 1, 1.4, 1.41, 1.414, 1.4142, . . . . "

(4) *p.* 64 "Every positive number has a logarithm which is a real number; but the logarithm of a negative number is an imaginary or a complex number."

There are good reasons for extending the definitions of the arithmetic and geometric means to include means of more than two numbers, which the author failed to do, and to introduce the concept of weighted means considering their

extensive use in the study of index numbers in statistics. While on the subject of averages, the reviewer still entertains the hope that some elementary textbook writer, especially one who is writing a book of this type, will recognize the statistical implications of writing the formula for the sum of $n$ terms of an arithmetical progression as,

$$S = n \frac{a + l}{2}$$

rather than,

$$S = \frac{n}{2} (a + l).$$

It is gratifying to have a paragraph on "Standard (or Scientific) Notation for Numbers in Decimal Form" which is a very useful tool as indicated on *p.* 59. Yet it is surprising that the advantage of this notation is not recognized on *p.* 69, by giving the following rule for finding the characteristic of the logarithm of $N$: The characteristic of the logarithm of $N$ is the exponent of 10 used when $N$ is expressed in the scientific notation.

Since the idea of the minimum value of a function is so fundamental in the study of statistics the author did well to include a determination of the minimum value of a quadratic function as given in Article 134. However it would have been preferable, pedagogically, to have used the same procedure in completing the square as was used earlier in Article 42 and have written,

$$y = K + \frac{1}{A} (Ax + B)^2,$$

from which both of the theorems on maximum and minimum values of $y$ would have followed immediately and the rôle of $A$ have been more clearly and readily appreciated.

The author is to be commended for attempting to give business students a more adequate understanding of analytical geometry, especially curve fitting by the method of least squares, than is generally done in books of this type, and for stressing the uses of logarithmic graph paper. Some doubts exist however as to the advisability of giving such a casual introduction of the concepts of symmetry and asymptotes as is done here.

It is quite generally accepted that the American Experience Table was based on mortality statistics deduced from the experience of the Mutual Life Insurance Company of New York rather than "from the accumulated results of life insurance companies" as stated on *p.* 206.

The three tables:

I Logarithms of Numbers
II Squares and Square Roots of Numbers
III American Experience Table of Mortality

are adequate for the purposes of the author.

The book is a worthy addition to published books of this type and will be found quite teachable for students with a good background in elementary algebra.

F. S. HARPER

*Mathematics as a Culture Clue, and Other Essays.* (Vol. I of *The Collected Works.*) By C. J. Keyser. New York, Scripta Mathematica, 1947. 7+277 pages. $3.75.

In these essays, which are "designed primarily for laymen" (*see e.g., p.* 25) the author makes an energetic attempt to explain the important mathematical concept of a system of postulates, a concept whose range of possible applicarions is very wide, embracing (*p.* 19) "all subject-matters, material or mental, physical or psychical, organic or inorganic."

Part of the reason for expounding the subject so fully is that Hilbert in his *Foundations of Geometry* was guilty, according to Keyser, (*cf. pp.* 15, 107, 115), of "an obfuscating and misleading practice." In setting forth his system of axioms and their implications

Hilbert at the same time applies it to geometric subject-matter—a fact shown by his use of certain geometric figures. Because the reader's attention is thus concentrated upon the rising structure of the geometric doctrine he perceives but dimly or not at all the simultaneously rising structure of the doctrinal function, of which the geometric doctrine is but one of countless values. And his false impression is, unfortunately, confirmed by Hilbert's calling his book *The Foundations of Geometry*, a flagrant misnomer since it has, essentially, no more to do with geometry than with a thousand other matters—a fact exhibited with ample detail in my *Mathematical Philosophy*. . . . It is not very difficult to show, and in my *Mathematical Philosophy* I have shown, that the proper values are of different types, some of them geometric, some of them arithmetical or numerical, and some of them neither the one nor the other.

But even when it was thus amply pointed out to him that his points and lines could be interpreted in other ways, Hilbert did not change the title of his book nor remove its diagrams. He may have felt that the title might stand provisionally until new applications should attain the same importance as the old ones, and as for the diagrams, they made the text easier to follow, an advantage which could be weighed against their obfuscating tendencies and perhaps be regarded as a compensation for the reader of dim perception.

Seven of Keyser's twelve essays are essentially book reviews, of works by such writers as Peirce, Havelock Ellis, Pareto, Spengler and Korzybski. Of Spengler's book *The Decline of the West* he says (*p.* 47) "it is, with perhaps a single exception, the most mind-stirring work of our time"; and of Korzybski's *Science and Sanity* (*p.* 153) "this work, taken as a whole, is beyond comparison the most momentous single contribution that has ever been made to our knowledge and understanding of what is essential and distinctive in the nature of Man."

It is the essay on Spengler's book which provides the title *Mathematics as a Culture Clue*. This essay has nothing to do with mathematics as part of the cul-

ture of an individual, but as Keyser explains (*p.* 45), "my purpose is to submit a certain grave thesis" of Spengler's, which is (*p.* 52) that "each of the expression-forms (mathematics, music, architecture, *etc.*) of a given culture (Old Egyptian, Babylonian, Chinese, etc., *cf.* p. 43) essentially bears the image of each of its fellow forms . . . revealing in and of itself what is distinctive in that Culture's soul."

In illustration of Spengler's thesis, Keyser writes (*p.* 52)

and so, were a Culture completely lost save for a single one of its expression forms, it would be theoretically possible, and in a measure practically possible, to determine from that sole survivor what the lost forms and the total Culture itself were essentially like, much as, since Cuvier, it has been possible to determine from a single fossil bone of an extinct animal, not only its family, genus, and species, but even the external form of the individual "with certainty and precision."

The project is an interesting one, with many applications; but there seems to have been some delay in carrying them out.

One of the essays, with the title *Three Great Synonyms*: *Relation*, *Transformation*, *Function*, is slightly more mathematical than the others. Its painstaking explanations are somewhat marred by printer's lapses whereby, for example, the three words which in the beginning are synonymous and possess synonymy (*p.* 218) become in the end *synonomous* and possess *synonomy* (*p.* 234). The familiar linguistic principle of dissimilation makes these errors far harder for the proof-reader to detect than a misspelling like *anonomous*. The same principle is at work in the phrases "Human Curosity" (*p.* 263), "irratating slovenliness" (*p.* 274) *etc.* Misprints in general are not rare throughout the book, *e.g.* "summitless heirarchy" (*p.* 28), "our outreachings for Diety" (*p.* 42) *etc.* Assimilation has been at work, too, on the word 'deduction,' for example, (*pp.* 80, 84): "had Gauss known . . . that mathematics and natural science are separated by a chasm as deep and unbridgeable as that which sunders Logical Deducation from Experimental Observation. . . . "

In the course of the book the lay reader is made acquainted with various mathematical concepts, *e.g.* the notion of *Limit* (*p.* 25), "the subtlety of whose meaning it would not be easy to exaggerate," the "essential" point being (*pp.* 26, 27) that "the variable never reaches its limit." The task of the popularizer is hard (Poincaré "not only failed but failed conspicuously," *p.* 72), so no one will object to this definition, seeing that it is used to "disclose in perfect light" (*pp.* 25, 28)—thereby correcting Plato and many others (*pp.* 24, 27)—the ideals of justice, freedom, beauty, happiness, wisdom, moral good, power, clarity, skill, piety, *etc.* (*p.* 27).

But the purpose of the book as a whole is to make clear the essential nature of all mathematics, which is summed up in many places. Thus (*p.* 113), if we let $P$ denote the set of postulates and $T$ a theorem, then the mathematician will say: "I have not . . . proved $P$ to be true, nor have I proved $T$ to be true; what I have proved is the proposition, $P$ implies $T$." In their preface, the publishing committee says:

when the mind of a scholar, philosopher or just a cultured layman gets confused and be-

wildered by the great complexity of scientific data or of political events he takes out a volume—
(P) any volume— of Keyser's and gets his mind geared to the rhythm of the cleansing, crystal-clear current flowing in every one of Professor Keyser's works.

and a few lines later:

(T) they believe that by making the Collected Works available they . . . are rendering a genuine service to the cause of clear thinking.

Let us denote their first statement by $P$ and what they believe by $T$, and give hearty assent to the proposition, $P$ implies $T$.

S. H. GOULD

## NEW BOOKS RECEIVED

*Applied Industrial Mathematics.* By O. B. Jones. New York, Prentice-Hall, Inc., 1947. 8+342 pages. $3.00.

*Intermediate Algebra for Colleges.* By E. B. Miller. New York, Ronald Press, 1947. 10+361 pages. $2.50.

*The Elements of Aerofoil and Airscrew Theory.* Second Edition. By H. Glauert. Cambridge, at the University Press. New York, The Macmillan Company, 1947. 226 pages. $4.00.

*Calculus.* Revised Edition. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1947. 12+568 pages. $3.75.

*Money, Credit and Finance.* Revised Edition. By G. F. Luthringer, L. V. Chandler and D. C. Cline. Boston, D. C. Heath and Co., 1947. 7+389 pages. $2.75.

*Natural Philosophy of the Science of Physics: Chemistry and Engineering.* By C. A. P. Turner. Columbus, Ohio, 1947. 31+320 pages. $6.00.

*Introduction Mathématique aux Théories Physiques Modernes, Part I.* by M. Morand. Paris, Vuibert, 1947. 140 pages. 350 Fr.

*Tables of the Bessel Functions of the First Kind of Orders Four, Five, and Six.* (Annals of the Computation Laboratory of Harvard University, Vol. 5.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1947. $10.00.

*Tables of the Bessel Functions of the First Kind of Orders Seven, Eight, and Nine.* (Annals of the Computation Laboratory of Harvard University, Vol. 6.) By the Staff of the Computation Laboratory, Harvard University Press, 1947. $10.00.

*Theory of Games and Economic Behavior.* Second Edition. By J. von Neumann and G. Morgenstern. Princeton University Press, 1947. 18+641 pages. $10.00.

*Time, Knowledge, and the Nebulae.* By M. Johnson. New York, Dover Publications, 1947. 189 pages. $2.75.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

### CLUB REPORTS, 1946–47

#### Pi Mu Epsilon, University of Kentucky

The *Kentucky Alpha* Chapter of *Pi Mu Epsilon* held six regular meetings. Faculty members of the mathematics department and graduate students in the department presented the following discussions:

*Pursuit curves*, by Dr. L. W. Cohen

*Boolean algebra*, by Dr. A. A. Grau

*A problem in probability*, by Dr. Casper Goffman

*Certain groups*, by Donald Rose.

The *Kentucky Alpha* Chapter sponsors the *White Mathematics Club.* This year a cash prize was presented to an undergraduate student for outstanding work.

There were two initiations during the year at which twenty-one new members were received into the chapter.

The officers for 1946–47 were: Director, Dr. H. H. Downing; Vice-Director, Virginia Rohde; Secretary-Treasurer, Louise Knifley; Librarian, Dr. C. G. Latimer.

The officers for 1947–48 are: Director, Dr. D. E. South; Vice-Director, S. J. Jasper; Secretary-Treasurer, Dorothy Spragens; Librarian, Dr. A. A. Grau.

#### Kappa Mu Epsilon, Upsala College

A two-front action has been the long-range program of the activities of the *New Jersey Alpha* Chapter of *Kappa Mu Epsilon.* These are: the background of our present mathematics as brought about by men and events and the applications of mathematics in science and industry.

At the special midyear meeting the address was given by Professor Howard Fehr of Montclair State Teachers College. At the annual banquet and initiation meeting, Professor D. R. Davis, also of Montclair State Teachers College, spoke on *Mathematics of the Past and the Future.*

Students presented the following papers:

*Pascal*, by June Davidson

*Abel*, by Marjorie Cohen

*Lobajevski*, by Dagny Heggem

*Certain theorems in non-euclidean geometries*, by William Melchinger.

The officers for 1946–47 were: President, Dagny Heggem; Vice-President, and Treasurer, June Davidson; Secretary, Marjorie Cohen; Historian, Natalie Manno; Faculty Sponsor, Professor M. A. Nordgaard. The officers for 1947–48 are: President, Marjorie Cohen; Vice-President and Treasurer, June Davidson; Secretary, Frances Reischmuller; Historian, William Melchinger.

### Mathematics Club, Mount Mary College

The activities of the *Mathematics Club* of Mount Mary College during the year were directed toward gaining a deeper appreciation of mathematics. The programs consisted of a series of reports on men who made major contributions in the field of mathematics.

After functioning as an independent organization, the club received recognition of its achievements by having four faculty members and sixteen club members inducted into *Kappa Mu Epsilon* on May 11, 1947. Mount Mary has the honor of becoming the *Wisconsin Alpha* Chapter of this fraternity.

The formal installation and initiation was presided over by Joseph J. Urbancek, faculty sponsor of the *Illinois Gamma* Chapter of the Chicago Teacher's College.

Officers for the year were: President, Dorothy Nilles; Secretary-Treasurer, Patricia Farrell; Faculty Advisor, Sister Mary Felice, S.S.N.D.

### Mathematics Club, Cooper Union

After a period of inactivity due to the war, the Mathematics Club of Cooper Union reorganized this year. Students presented the following papers:

*An introduction to the calculus of finite differences*, by David Jagerman
*Linear Diophantine equations*, by Harry Hochstadt
*Topology*, by Eugene Wachspress
*Geometry of the triangle*, by Melvin Stern
*Heaviside operational methods*, by David Jagerman
*Nomography*, by Eugene Wachspress
*Approximate summation of series*, by David Jagerman
*Methods of vector analysis*, by Leon Nemerever.

Printed notes, prepared by the speakers, were distributed for the lecture on Heaviside operational methods and for the series of three lectures on methods of vector analysis. It was decided to continue this procedure at all future meetings. More material is often presented at a lecture than can be thoroughly absorbed by the members, and such notes, besides serving as a ready reference during the lecture, enable the members to review the subject at their leisure.

Officers for the year were: President, Leon Nemerever; Vice-President, David Jagerman; Secretary-Treasurer, Eugene Wachspress. Officers for 1947–48 are: President, David Jagerman; Vice-President, Harry Hochstadt; Secretary-Treasurer, Melvin Stern; Faculty Advisor, Professor J. N. Eastham.

### Zeta Mu Tau, University of Washington

*Zeta Mu Tau*, mathematics honorary at the University of Washington, is composed of undergraduates from the departments of pure and applied mathematics, chemistry, physics, and engineering who have fulfilled a required grade-point average both in the major field and in the general university curriculum. The purpose of the organization is to stimulate interest in mathematical re-

search, particularly in the applied field. In addition to the annual initiation banquet for new members, three general meetings were held this year. Among the more significant speeches were:

*Assumptions in engineering elasticity*, by Marvin Stippes

*A journey to France*, by Professor R. M. Winger.

This year's officers included: President, B. R. Laverty; Vice-President, R. Nelson; Secretary, R. P. Kraft; Treasurer, W. R. Carter.

**Mathematics Study Club, Immaculate Heart College**

The papers delivered at the quarterly meetings were:

*Postulates of arithmetic and application in proving* $2+2=4$, by Miss Catherine Campion

*Classification of numbers and proof of irrationality* of $\sqrt{2}$, by Miss Sara Doherty

*Zero and the use of l'Hospital's rule in* $\frac{0}{0}$, by Miss Phyllis Beerling

*History of theory of numbers*, by Miss Suzanne Toulan

*De Moivre's Theorem*, by Miss Joan Pfisterer

*Linear congruences*, by Miss Sara Doherty.

The following book reviews were given:

*A Mathematician's Apology*, by Godfrey H. Hardy, reviewed by Miss Marian Snyder

*The Magic of Numbers*, by E. T. Bell, reviewed by Miss Mary Dominguez.

Five undergraduate students were guests of the Mathematical Association of America, Southern California Section, at the meeting held in March at Pomona College, Claremont, California. An enthusiastic report was brought back to the *Study Club* meeting which followed.

The year's activities culminated in May with a dinner given by the students in honor of Dr. Myrtie Collier, head of the Mathematics Department, who is retiring after seventeen years of successful service with the college.

This year's program was under the direction of: President, Phyllis Beerling; Vice-President, Lavada Moudy; Secretary, Margaret Wehr; Treasurer, Elizabeth Diedrich. Miss Beerling will serve again next year as President while the other offices will be filled at the first meeting in September.

**Mathematics and Physics Club, University of Alberta**

Papers presented to the *Mathematics and Physics Club* of the University of Alberta during the 1946–47 session were:

*The early history of astronomy*, by Mr. Albert Shaw

*The Laplace transformation*, by Professor Max Wyman

*The theory of the chain-reacting pile*, by Mr. George Kokatailo

*The evaluation of* $\pi$, by Miss Marion Roberts

*The Bell helicopter*, by Mr. K. Korsak

*Bernoulli numbers*, by Mr. Eoin Whitney

*High speed flight*, by Mr. Douglas Baines

*Gas turbines and jet propulsion*, by Mr. Ken Lobb

*X-ray spectra*, by Mr. Arthur Stephenson

*Astro-physics*, by Mr. Don Hall.

The year's activities ended with the annual banquet of the club held on March 17, 1947. The guest speaker was Mr. L. E. Gads, Assistant Professor of Engineering, whose topic was *The ordinary man's impression of mathematics and physics.*

The officers of the club for 1946–47 were: President, Miss Sydney Jones; Secretary-Treasurer, Severan Heiberg; Social Convener, Miss Marion Roberts; Executive members, Douglas Baines and Len Greenberg; Faculty Advisors, Professor G. O. Langstroth and Mr. R. C. Jacka.

The newly elected officers for 1947–48 are: President, Len Greenberg; Secretary-Treasurer, Miss Marion Roberts; Social Convener, Miss Lorna Boon; Executive members, Don Hall and Eoin Whitney; Faculty Advisors, Professor J. W. Campbell and Professor E. H. Gowan.

## NEWS AND NOTICES

EDITED BY HARRY POLLARD, Cornell University

*Readers are invited to contribute to the general interest of this department by sending news items to Harry Pollard, White Hall, Cornell University, Ithaca, New York.*

### INSTITUTE OF NUMERICAL ANALYSIS ESTABLISHED

Plans have been completed for the establishment of one of the newest units of the National Bureau of Standards, The Institute of Numerical Analysis, at the University of California at Los Angeles, according to an announcement by Dr. Edward U. Condon, Director of the Bureau.

One of the giant high-speed electronic computing machines, now under development by the Bureau of Standards, will be installed at the Institute when completed. These computers will solve problems in minutes that now take days to work out, and will solve in days problems that are now out of the reach of scientists. Design specifications call for high memory capacity and automatically sequenced mathematical operations from start to finish at speeds attainable only with electronic equipment.

The machines can conceivably revolutionize the field of applied mathematics. Of particular importance both to the physical sciences and to technical industries will be the fact that the Institute will be able to set up a mathematical counterpart of an actual situation, which permits the situation then to be studied through relatively inexpensive calculating rather than costly experimentation. Great as has been the progress of the past century, the time has come when many problems of great importance, especially in hydrodynamics, aerodynamics, and meteorology, can only be handled by computers working at speeds measured in millionths of a second.

The Institute has two primary functions. The first is research in applied mathematics aimed at developing methods of analysis which will extend the use of the high-speed electronic computers. The second is to act as a service group for western industries, research institutions, and government agencies. The service function will include not only the use of the machines for problem solving but also assistance in the formulation of problems in applied mathematics of the more complex and novel types. Service operations are to be initiated immediately, using the latest types of commercially available computing equipment.

## PERSONAL ITEMS

Professor L. W. Cohen of the University of Kentucky has been appointed to an assistant professorship at Queens College.

Dr. Nancy Cole of Connecticut College has been appointed to an assistant professorship at Syracuse University.

Associate Professor H. A. Davis of West Virginia University has been promoted to a professorship.

Dr. Nathan Fine has been appointed to an assistant professorship at the University of Pennsylvania.

Professor H. D. Larsen of the University of New Mexico has been appointed to a professorship at Albion College.

Associate Professor A. N. Milgram of Notre Dame University has been appointed to an associate professorship at Syracuse University.

Dr. A. K. Mitchell has been appointed to an associate professorship at the University of Maryland.

K. H. Murphy of West Virginia University has been promoted to an assistant professorship.

I. D. Peters of West Virginia University has been promoted to an assistant professorship.

Dr. George Piranian of the University of Michigan has been promoted to an assistant professorship.

Dr. Murray Protter of Brown University has been appointed to an assistant professorship at Syracuse University.

Assistant Professor G. E. Schweigert of Purdue University has been appointed to an associate professorship at the University of Pennsylvania.

Dr. G. Tunell of the Carnegie Institute has been appointed to an associate professorship of geology at the University of California at Los Angeles.

The following appointments to instructorships are announced:

West Virginia University: Thomas Bauserman, Miss Marcia Saile

University of Rochester: Walter Klimzak

Professor Giacomo Albanese of the Normal School of Pisa and the University of Sao Paulo died June 8, 1947.

Professor Cora B. Hennel of Indiana University died on June 26, 1947.

Bernard Mason of Hofstra College died August 25, 1947.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirtieth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on April 18 and 19, 1947. There were three sessions, with Professor Greta Neubauer of the University of Wyoming presiding at each.

There were sixty-four persons in attendance, including the following twenty-five members of the Association: C. F. Barr, D. L. Barrick, J. R. Britton, A. G. Clark, G. S. Cook, A. T. Craig, A. B. Farnell, H. T. Guard, Mrs. Leota C. Hayward, I. L. Hebel, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, A. E. Mallory, W. K. Nelson, K. L. Noble, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, L. C. Snively, V. J. Varineau, Mrs. Lillie C. Walters.

At the business meeting the following officers were elected for the coming year: Chairman, H. T. Guard, Colorado State College of A. and M. A.; Vice-Chairman, I. L. Hebel, Colorado School of Mines; Secretary-Treasurer, J. R. Britton, University of Colorado. Invitations to meet at Colorado State College of A. and M. A. in 1948, and at Colorado School of Mines in 1949 were accepted.

The following papers were presented:

1. *Expansion of an arbitrary function in series of functions associated with Bessel functions*, by Professor Leonard Bristow, University of Wyoming, introduced by Professor C. F. Barr.

The author defined a set of functions by generalizing the Poisson integrals for Bessel and for Struve functions. For a suitable arbitrary function there was obtained an expansion resembling the generalized Schlomilch series.

2. *The solution of an integral equation*, by Professor W. H. Jurney, Colorado School of Mines, introduced by Professor I. L. Hebel.

3. *Note on functions of a matrix*, by Professor Clarence Ross, University of Denver, introduced by A. J. Lewis.

The matrix $e^{kt}$ was expanded into a polynomial in $k$ of degree not greater than $n-1$, where $k$ is an $n\times n$ matrix. An application to the solution of linear homogeneous differential equations was explained.

4. *Bounds for the characteristic roots of a matrix*, by Professor A. B. Farnell, University of Colorado.

A brief history of this subject and related topics was presented. Let $\mathbf{A}=(a_{rs})$ be a square matrix of order $n$ with complex numbers as elements. The equation $|\lambda\mathbf{I}-\mathbf{A}|=0$, where $\mathbf{I}$ is the unit matrix and $\lambda$ is a scalar, is called the characteristic equation of the matrix $\mathbf{A}$, and the roots $\lambda_i$, the characteristic roots. Several

new bounds for the characteristic roots were given. Let

$$\sum_s |a_{rs}| = R_r, \quad \sum_r |a_{rs}| = T_s, \quad \sum_s |a_{rs}| R_r = U_r, \quad \sum_r |a_{rs}| T_s = V_r.$$

Then $|\lambda|$ is not larger than any of the three numbers $\max_r (U_r)^{1/2}$, $\max_r (V_r)^{1/2}$, $\max_r (U_r V_r)^{1/4}$.

5. *A new method of approximating Fourier coefficients*, by G. L. Collins, Colorado School of Mines, introduced by Professor I. L. Hebel.

This speaker presented a simple method for evaluating the Fourier coefficients of a curve plotted to a predetermined scale. The essential idea of the method consisted of the use of a series of specially ruled transparent plastic sheets.

6. *Wallis' product for $\pi$*, by W. W. Mitchell, Jr., University of Colorado, introduced by Professor A. J. Kempner.

It was shown how Wallis determined the value of $\pi$ between ever narrowing upper and lower bounds by a process of interpolation in a sequence of numbers related to the first quadrant areas under the curves $y=(1-x^2)^n$, $n=0, 1, 2, \cdots$.

7. *On complex roots of algebraic equations*, by Professor A. J. Kempner, University of Colorado.

Given an equation $f(z)=a_0z^n+ \cdots +a^n=0$ with real coefficients and roots $z_k=x_k+iy_k$, $k=1, \cdots, n$, one knows how to establish by rational operations equations $G(x)=0$, and $H(y)=0$, each of degree $n$, such that each $x_k$ is among the roots of the first, each $y_k$ among the roots of the second equation. However, this leaves in each equation $n^2-n$ roots unaccounted for. The location of these roots is determined by the theorem: The $n$ roots of $G(x)=0$ are $x_j=\frac{1}{2}(z_k+z_l)$, $k, l=1, 2, \cdots, n$; the $n$ roots of $H(y)=0$ are $y_j=\frac{1}{2}(z_k-z_l)$. A striking geometrical interpretation in the plane of complex numbers is possible.

Results are extended in toto to equations with complex coefficients without raising the degrees of $f(z)$, $G(x)$, $H(y)$ by letting $z=u+v$ with the restriction that with $u+v$, the number $u-v$ is also a root of $f(z)=0$. The function $G(x)$ is of the form $f(x)\cdot K^2(x)$, $K$ being of degree $(n^2-n)/2$; $H(y)$ is of the form $y^nL(y^2)$, $L$ being of degree $(n^2-n)/2$ in $y^2$. Similar results hold for the equation for $r$ and for $e^{i\phi}$, $z=re^{i\phi}$.

8. *Statistical inference*, by Professor A. T. Craig, University of Iowa.

This paper was devoted to an exposition of the construction of a mathematical system adequate to furnish methods for drawing inferences from statistical data. The paper included an introduction to the Neyman-Pearson theory of testing statistical hypotheses.

9. *Is mathematics out of this world?* by Professor A. W. Recht, University of Denver.

The main thesis of this paper is that mathematics as presented in high schools and in colleges of liberal arts is out of this world in the sense that the principles of mathematics are set up in the classical and traditional way instead of in the

way in which they occur in real life. The suggestion is made that textbooks be written with the psychological approach by mathematicians who are also experts in fields of real application of mathematics. Problems should be presented as they occur in real life. It is only in this way that mathematics will be able to maintain the high reputation it has acquired in the atomic age; it is only in this way that students in the high schools and colleges will be kept interested in mathematics of reality, and not dazed by operations in a world of unreality.

10. *General mathematics*, by Professor Fred McCune, Colorado State College of Education.

In this paper the author asks why courses in "general" mathematics should duplicate training given in standard algebra and geometry courses. He believes that training in the fundamental skills of arithmetic is more important for the average secondary school student.

11. *The training of mathematics teachers*, by Professor K. H. Stahl, University of Colorado, introduced by the Secretary.

The attitude developed by students in mathematics has great influence not only on them, but also on us as teachers of mathematics. The teacher controls to a great extent the attitudes developed by members of the class, and it is therefore important that all teachers have a proper influence on their students. If the teacher himself is not well grounded in the material to be presented, it is quite unlikely that his influence will be wholesome. In all probability many persons become certified to teach in the elementary schools with very poor backgrounds in arithmetic. It is recommended that college teachers concern themselves with the mathematical preparation of elementary teachers.

12. *Report on the entrance requirement changes at the University of Colorado*, by Professor A. J. Kempner, University of Colorado.

Professor Kempner reported on the recent changes in entrance requirements for the Colleges of Arts and Sciences at the University of Colorado. All students must now offer three units of high school English, besides nine other units in "academic" subjects. These may not be selected arbitrarily; but students may enter the College without any high school work in any chosen one of the four large fields: foreign language, mathematics, natural sciences, social sciences. Under some arrangements students may even enter without any high school work in any chosen two of these fields.

There is opposition within the faculty to this scheme. Departments were not properly consulted.

In mathematics the situation is aggravated by the fact that a student who offers mathematics on his entrance requirements may substitute "high school arithmetic" and "general mathematics" for high school algebra and high school geometry which were required under the old rule.

Criticism of this last regulation centers around the fact that "general mathematics," as the term is understood in our part of the country, represents mathe-

matics courses which were introduced specifically for students who were either admittedly incapable of carrying the standard algebra and geometry courses, or who did not intend to go on to college training, but who wanted vocational courses in mathematics with a minimum of emphasis on theory and logical development. The department of mathematics refuses to recognize these courses as adequate prerequisites for college mathematics. These courses must not be confused with "unified mathematics" courses, which in some parts of the country go under the name of "general mathematics." For these, a strong case can be made out.

The mathematics department consulted groups of Colorado high school teachers, particularly mathematics teachers. The results were revealing. Over a hundred mathematics teachers of the Denver Section of C. E. A. protested unanimously against the changes. The Grand Junction Section, one of the other two sections in the state, sent a similar protest. The mathematics departments of two of the large Denver high schools, Denver North and Denver East, sent unanimous petitions to the president of the University, and so forth.

High school administrators generally favor the new rules, and regret that they do not go farther than they do. There exists scattered disapproval among them, but it has so far not become organized.

Our experience in Colorado proves that we have powerful allies among the high school teachers; they suffer and chafe under the steady deterioration of the standards and are, at least in Colorado, as a group more aware of the dangers and implications of the situation, and far more willing to fight for its improvement, than are college and university faculties.

In the lively discussion which followed the speaker's remarks, sentiment was opposed overwhelmingly to the elimination of mathematics as an entrance requirement, and as bitterly opposed to the admission of high school arithmetic and "general mathematics" in place of algebra and geometry.

J. R. Britton, *Secretary*

## APRIL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-fourth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Mississippi Southern College, Hattiesburg, Mississippi, on Friday and Saturday, April 25 and 26, 1947. Professor W. V. Parker, Chairman of the Section, presided at the Friday afternoon and Saturday morning sessions. Professor W. L. Johnson, Vice-Chairman for Mississippi, presided at the joint dinner with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics.

The attendance was sixty-five including the following thirty members of the Association: T. A. Bickerstaff, H. E. Buchanan, Margaret R. Davis, W. L. Duren, Jr., L. M. Garrison, F. C. Gentry, A. Gilmore, W. C. Griffith, W. L. Johnson, H. T. Karnes, C. G. Killen, Z. L. Loflin, Dorothy McCoy, A. C. Maddox, B. E. Mitchell, S. B. Murray, I. C. Nichols, W. V. Parker, P. K. Rees, F. A. Rickey,

H. F. Schroeder, Maurice Singer, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, P. M. Tullier, Jr., Maralena White.

The officers elected at the business meeting of the section were as follows: Chairman, W. L. Johnson, Mississippi Southern College; Vice-Chairman for Mississippi, T. A. Bickerstaff, University of Mississippi; Vice-Chairman for Louisiana, A. L. Loflin, Southwestern Louisiana Institute; Secretary-Treasurer, F. C. Gentry, Louisiana Polytechnic Institute. The next meeting will be held in February or March, 1948, at Southwestern Louisiana Institute, Lafayette, Louisiana.

Professor H. E. Buchanan, Tulane University, was invited to give the principal address at the joint dinner. His topic was *Mathematics Teaching—Tulane Brand.* Professor H. L. Smith, Louisiana State University, was invited to speak at the Saturday morning session. His subject was *A Foundation for the Point Calculus of Grassmann.* The Friday afternoon session was devoted to short papers. Abstracts of all these papers follow:

1. *A foundation for the point-calculus of Grassmann,* by Professor H. L. Smith, Louisiana State University.

The Grassmann point calculus may be based on a foundation consisting of seven postulates involving an unspecified field of numbers and the undefined terms point, scalar product $s(p_0, p_1; p_2, p_3)$ of a point pair $(p_0, p_1)$ by a point pair $p_2, p_3)$, and the point $\bar{p}$ conjugate to a point $p$. The resulting theory is valid in Hilbert, as well as in Euclidean, spaces.

2. *Installment buying,* by Professor I. C. Nichols, Louisiana State University.

3. *Mathematics applied to meteorology,* by Mr. J. T. Lee, Mississippi Southern College, introduced by Professor Johnson.

This paper was a short introduction to the application of mathematics to meteorology and weather forecasting. A general development of several of the basic equations, such as the gradient wind formula and the fundamental concepts of meteorology was included.

4. *Problems concerning volumes,* by Professor C. D. Smith, Mississippi State College.

The cone inscribed in a sphere whose volume is equal to the volume of the segment of the sphere cut off by its base was found to have a volume approximately equal to that of the inscribed cone of maximum volume. A similar comparison was made for cones inscribed in ellipsoids of revolution.

5. *Three cubic loci,* by Professor F. C. Gentry, Louisiana Polytechnic Institute.

It was shown that if a variable point $P$ of the plane traces the cubic of Darboux relative to a given triangle, the joins of the vertices of the pedal triangle of $P$ and the corresponding vertices of the given triangle are concurrent in a point $Q$ on the cubic of Lucas. If $P$ traces the 17-point cubic, the perpendiculars from the harmonic associates of $P$ on the corresponding sides of the given triangle are concurrent in a point $R$ on the cubic of Darboux, and the joins of their feet and

the corresponding vertices of the given triangle are concurrent in a point $S$ on the cubic of Lucas.

6. *Discussion of the report of the Association Committee for the Coordination of Studies on Mathematical Education*, by L. M. Garrison, Louisiana Polytechnic Institute; T. A. Bickerstaff, University of Mississippi; Z. L. Loflin, Southwestern Louisiana Institute.

F. C. GENTRY, *Secretary*

## APRIL MEETING OF THE SOUTHWESTERN SECTION

The seventh annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico in Albuquerque on April 4, 1947. Professor E. A. Hazlewood, Chairman of the Section, presided at the afternoon session. Professor H. D. Larsen presided at the banquet at which President W. M. Whyburn of Texas Technological College was the guest speaker.

The attendance was twenty-five including the following thirteen members of the Association: L. M. Bauer, J. H. Butchart, Lincoln La Paz, H. D. Larsen, B. D. Roberts, H. P. Rogers, Arthur Rosenthal, Annie N. Rowland, F. W. Sparks, R. S. Underwood, Earl Walden, R. L. Westhafer, W. M. Whyburn.

At the business meeting the following officers were elected: Chairman, H. D. Larsen, University of New Mexico; Vice-Chairman, R. F. Graesser, University of Arizona; Secretary-Treasurer (four years), B. D. Roberts, Highlands University; Governor (three years), R. S. Underwood, Texas Technological College. These officers also constitute a committee to choose the next traveling lecturer sponsored by the Southwestern Section. It was voted to hold the 1948 meeting of the Section in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science, providing the latter meeting is held within the geographical limits of the Section.

The program consisted of the following papers:

1. *Vector methods in modern geometry*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

Professor Butchart called attention to vector proofs of theorems in modern plane geometry, especially those concerning centroids. He also recommended the use of the scalar product of two vectors to prove theorems in modern geometry involving projections or the squares of distances.

2. *An inverse variation problem*, by Frank Lane, University of New Mexico, introduced by Professor Lincoln La Pas.

Darboux's result that the integral curves of the differential equation $y''=f(x, y, y')$ can be regarded as the extremal system of a non-singular variation problem, $I=\int_{x_1}^{x_2} f(x, y, y')\,dx=$ minimum, cannot be extended to 3-space. The first example of a pair of differential equations

$$(1) \qquad y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z')$$

of which the integral curves are not the extremals of any non-singular variation problem of the form

$$(2) \qquad I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx = \text{minimum},$$

was given by Lincoln La Paz in 1928. Additional isolated examples of such systems of differential equations have since been given by other investigators.

Mr. Lane exhibited classes of differential equations (1), the right members of which involve arbitrary functions of $y'$ and $z'$, no one of which has integral curves that are the extremal system of a non-singular variation problem of the form (2). Because of the freedom of choice thus provided in selecting the functions $F$ and $G$, it was possible to so choose these functions that the resulting system of differential equations could be integtated explicitly. Certain properties of the resulting non-extremal, four-parameter families were discussed.

3. *Extended analytic geometry applied to Diophantine equations*, by Professor R. S. Underwood, Texas Technological College.

Professor Underwood used a method suggested by the three-axes case of his extended analytic geometry (this MONTHLY, May, 1945) to solve certain Diophantine equations. In particular, the speaker showed how the solution of two simultaneous linear equations in three unknowns is reduced readily to the solution of a single equation in three unknowns.

4. *On the determination of optimum flight paths*, by Professor Morris Hendrickson, University of New Mexico, introduced by the Secretary.

Professor Hendrickson discussed the problem of determining the route which should be flown through a wind field varying in time and space in order to get from a point $A$ to a point $B$ in the minimum time, assuming that the flight takes place at a constant altitude and with constant air speed. The problem for a flat surface has been treated in the literature by Zermelo who derived both necessary and sufficient conditions, and for an $n$-dimensional euclidean space by Levi Civita who derived only necessary conditions. The speaker discussed the problem of flight over a spherical surface which he had investigated jointly with G. E. Forsythe and K. J. Arrow. A necessary condition was derived in the form of a differential equation which the minimal path must satisfy, and a practical method for carrying out the approximate integration of this equation was presented.

5. *Derivatives of fractional order*, by C. P. Stroud, Highlands University, introduced by Professor B. D. Roberts.

Mr. Stroud, a student at Highlands University, discussed derivatives obtained from the form for successive differentiation

$$\frac{d^n}{dx^n}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

by extending $n$ to fractional, irrational, and negative values. Various formulas for such generalized differentiation were presented. The speaker also suggested the use of fractional indices for operations other than differentiation.

H. D. LARSEN, *Secretary*

## APRIL MEETING OF THE IOWA SECTION

The thirty-fourth annual meeting of the Iowa Section of the Mathematical Association of America was held at the Iowa State Teachers College in Cedar Falls on Friday and Saturday, April 18 and 19, 1947, in conjunction with the Iowa Academy of Science. Professor L. W. Swanson, Chairman of the Section, presided.

The attendance was forty-five, including the following thirty members of the Association: E. W. Anderson, J. W. Beach, E. L. Canfield, E. W. Chittenden, N. B. Conkwright, W. M. Davis, R. E. Gaskell, B. E. Gillam, Cornelius Gouwens, J. J. L. Hinrichsen, D. L. Holl, L. A. Knowler, O. C. Kreider, R. J. Lambert, R. B. McClenon, J. V. McKelvey, Martha M. McKelvey, C. J. Maloney, E. N. Oberg, H. V. Price, Fred Robertson, W. J. Rusk, W. M. Stone, L. W. Swanson, H. P. Thielman, H. C. Trimble, Henry Van Engen, Roscoe Woods, C. C. Wylie, E. A. Zubay.

The following officers were elected for the coming year: Chairman, Professor H. P. Thielman, Iowa State College; Secretary, Professor Fred Robertson, Iowa State College.

The first six of the following papers, including the invited hour address by Professor Holl, were read at the Friday afternoon session. The remainder were read Saturday morning.

1. *On differential difference equations*, by Professor H. P. Thielman, Iowa State College.

The equation $f'(x+a)=K(x)f(x)$ was considered, when $a$ is a constant and $f(x)$ is an unknown real function. This equation was shown to be equivalent to an integral equation of the Volterra type with a discontinuous kernel. Analytic solutions were given for the particular case in which $K(x)$ is a constant, say $k$. For certain values of $k$ the given solutions were either monotone, or periodic functions.

2. *Some applications of the finite Fourier transformations*, by Professor R. E. Gaskell, Iowa State College.

Use of the finite Fourier transformation in solving boundary value problems is limited to problems leading to special linear differential operators, and involving boundary conditions of a special form. For example, if the finite Fourier sine transformation is to be used, only values of the unknown function and its even derivatives (with respect to the transformed variable) may be involved in the boundary conditions. A problem was given showing how this requirement can be relaxed if more general transformations are used.

3. *On the number of paths in a finite partially ordered set*, by Professor E. W. Chittenden, State University of Iowa.

4. *Dirichlet's problem*, by Professor D. L. Holl, Iowa State College.

The first boundary problem of potential theory, Dirichlet's problem, is to find a function harmonic in a region and taking on a preassigned value on its boundary. By Green's identities many properties of such a function can be established. It was shown that the problem is equivalent to finding Green's function.

Gauss, Kelvin, Dirichlet and Riemann gave faulty proofs of the existence of a harmonic function by considering it as a minimal problem in calculus of variations. Hilbert was first to specify proper conditions for this problem as a minimal problem. Direct methods of Ritz, and Trefftz, approximation by finite differences, and experimental models by soap film surfaces were discussed.

5. *A model for irreducible double modules*, by Professor Bernard Vinograde, Iowa State College, introduced by the Secretary.

A model was given which included the irreducible double module $V=L\times R$, $x$ in $V$, such that $L=\sum l_i L_0$, and $R=\sum R_0 r_i$ with $R_0\cong L_0$, where $L$ and $R$ are division rings. The construction depends on the existence of a representation of $L$ over $R$.

6. *Solution of iterated amplifiers by generalized Laplace transform*, by W. M. Stone, Iowa State College.

A simultaneous system of difference equations arising from an iterated linear amplifier system discussed by Faust and Beck (*Journal of Applied Physics*, vol. 17, pp. 749–756) was solved by means of the generalized Laplace transform. With this method there was no necessity for eliminating arbitrary constants or making assumptions as to the nature of the solution.

7. *On the general theory of functions*, by Professor E. W. Chittenden, State University of Iowa.

This paper was read by title.

8. *Nearly efficient estimates of variance components*, by S. Lee Crump, Iowa State College, introduced by the Chairman.

For statistical data classified into groups of unequal sizes, the efficient (maximum likelihood) estimates of the variance components are extremely difficult to compute. Estimates, based on Newton's method of solving equations, which are very nearly fully efficient, were presented.

9. *A property of the projective cubic*, by C. J. Maloney, Iowa State College.

The set of all tangents to a conic is called a line conic. The lines of the line conic cut any two fixed lines in projective ranges. If a projective relation is set up between a (first order) pencil of lines and the lines of the line conic by means of a projective range on some one line of the line conic, the intersections of corresponding rays will trace a cubic, called from the manner of its generation a projective cubic. It was shown geometrically that the cubic so generated is always unicursal, and that it is crunodal, cuspidal, or acnodal, according as the center of the pencil is outside, on, or inside the conic.

10. *Curve fitting—an art or a science*, by Professor G. W. Snedecor and Professor G. W. Brown, Iowa State College.

This paper was read by title.

11. *Interpretation of "college preparation" by individual teachers of high school mathematics*, by Professor H. C. Trimble, Iowa State Teachers College.

It was contended that the words and actions of college people convince teachers of high school mathematics that: (1) Certain topics are essential for the college preparatory student; (2) The fields of arithmetic, algebra, and geometry

must be taught separately; and (3) Standardized tests are a valid measure of a student's preparation for college mathematics.

After a brief discussion of the consequences of these convictions, three questions were raised, namely: (1) What *do* we in the colleges want? (2) How can we make our wants known to teachers of high school mathematics? and (3) What assurance can we give to high school teachers that their students will get a fair chance to show what they know as they enter college?

The writer's own answers to these questions were stated, and a plea was made for further study and action to clarify the situation.

FRED ROBERTSON, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pa., November 22, 1947
ILLINOIS
INDIANA
IOWA, Fairfield, April 16–17, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., December 6, 1947
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Schenectady, N. Y., May 1, 1948
WISCONSIN, Beloit, May 8, 1948

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

# INDEX TO VOLUME 54, 1947

## THE AMERICAN MATHEMATICAL MONTHLY

By R. F. RINEHART, Case Institute of Technology

## GENERAL MATHEMATICAL PAPERS

### ALGEBRA



### ANALYSIS



### GEOMETRY



### PEDAGOGY, BIOGRAPHY

## MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California at Los Angeles



## CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College



## PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College, and E. P. STARKE, Rutgers University

### CORRECTIONS

Numbers in black face type refer to problems, those in light face to pages.

## AUTHORS

Numbers refer to pages, black face type indicating a problem solved and solution published; italics, a problem solved, but the solution not published; ordinary type, a problem proposed.

## SOLUTIONS

Numbers in black face type refer to problems, those in light face to pages.



# RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

## NEW BOOKS RECEIVED



## REVIEWS

Names of authors are in ordinary type, those of reviewers in capitals.

## CLUBS AND ALLIED ACTIVITIES

Edited by L. F. Ollmann, Hofstra College

### ARTICLES



### ACTIVITIES

## NEWS AND NOTICES

EDITED BY B. W. JONES AND HARRY POLLARD, Cornell University

### GENERAL INFORMATION



### PERSONAL INFORMATION

Personal Mention. This section contains the names of officers of the Association and its Sections, persons mentioned in the Department of News and Notices, and those conducting the business of the Association. The list does not include names of new members or of attendants at meetings.

## NECROLOGY



# REPORTS OF THE MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

## MEETINGS OF THE ASSOCIATION

## MEETINGS OF ITS SECTIONS

EDITED BY N. B. CONKWRIGHT, UNIVERSITY OF IOWA

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 10


PART I

CONTENTS




DECEMBER 1947

# The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)





EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions and advertising in the MONTHLY, should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer before the 20th of the month preceding that in which the change becomes effective.





Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

# MATHEMATICS FOR LIBERAL ARTS STUDENTS*

C. B. ALLENDOERFER, Haverford College

At a time when all colleges and universities are reviewing their curricula it is highly appropriate that the Mathematical Association of America consider the role of mathematics in a liberal education. My talk today is only one of a long series of similar papers which have been read before the Association, and I think that this is very healthy. For we need to submit our course offerings to constant examination and evaluation if we are to meet the needs of our students most effectively. In this talk I shall deal only with the question of mathematics for liberal arts students, but I am equally anxious that the Association examine the parallel problem of mathematics for the specialist, for there is much to be done in that direction as well.

It is no easy task to find a definition of the "liberal arts student" which is acceptable to everyone today. Many colleges of "liberal arts" are rapidly becoming schools for specialists, while others are in a frenzy to introduce "general education" of which there are as many definitions as there are advocates. In order to avoid the problem of defining a "liberal education" and yet to be specific as to the subject of my talk today, let me state that I plan to discuss mathematics for those students who now take one year of mathematics at most and who do so without regarding it as a tool subject essential to further studies in other fields. Clearly I am excluding the engineers, the physical scientists, the mathematics majors, and the commerce and agricultural students. The people I am talking about take mathematics in college because of their own interest in it, or more often because of some form of pressure through limited elective requirements or the like by the faculty of their university. For many of these students the traditional course is a chore to be accomplished without real motivation or lasting value. In my opinion it is essential that we reexamine the needs of these students and that we redesign our courses to meet these needs without regard to tradition or our own personal preferences.

In thinking of these students we should first consider those values with which mathematics is supposed to provide them. First we have utility as a chief reason for their study of mathematics. Even the non-specialist meets numbers at every turn and must deal with them, and he hopes to improve his understanding of numbers by taking our freshman courses. Second, mathematics is traditionally known as a logical subject, and its pursuit is supposed to improve the capacity of the mind for reasoning. And finally, the student may hope to attain some understanding of the nature of mathematics and of its contribution to our culture. The liberally educated man is presumably acquainted with the triumphs and limitations of the major fields of knowledge and with their effects upon his daily life. So with this in mind (or in the mind of his university) he takes freshman mathematics.

* Presented before the twenty-ninth Summer Meeting of the Association, New Haven, Sept. 2, 1947.

Let us now examine the standard freshman course to see how well it meets these aims. By the standard course I mean the practically universal year of algebra, trigonometry, analytic geometry, and occasionally calculus. As to utility, I have serious doubts. This year's work is traditionally designed for the specialist as preparation for the calculus, and meets the needs of the "liberal arts" student very imperfectly. The mathematical needs of these students are limited (with one major exception) to a real competence in arithmetic together with a speaking acquaintance with logarithms and elementary trigonometry. These should have been acquired in high school, though we all know they frequently are not. The remainder of the standard course with its quadratic formula, theory of equations, solution of triangles, and analysis of curves is promptly and joyously forgotten, often before the course is over. One genuine need of these students, however, is almost universally neglected. This is an understanding of the fundamentals of probability and statistics. The abbreviated treatment of probability frequently included in algebra courses does not, in my opinion, make any contribution toward fulfilling this need. One does not have to read more deeply than the daily newspaper (not to mention the racing form) to realize how frequently common man is faced with questions of odds, statistical charts, index numbers, sampling techniques, and calculated risks. That he deals badly with these is evidenced by his persistent losses on the stock market and in other games of chance, and by the ease with which fraudulent statistics are foisted upon the public. If we plan our course strictly on the needs of the students, I think we must include a very generous dose of probability and statistics. Steps in this direction have already been taken by a few far-sighted universities, and I strongly encourage others to follow their lead.

The next goal of our course in mathematics is an increase in the students' reasoning power. The ability to construct a sound mathematical argument is popularly supposed to increase our reasoning powers in other fields of endeavor. I insist that this position is unsound, and furthermore that our freshman course does not even sponsor sound mathematical reasoning. In my opinion our standard textbooks train the students in a limited number of routine processes and rarely call upon them to carry out original logical thought processes. Indeed some of the most widely accepted texts are little more than cookbooks in which the student learns how to find the answer in the book by following Step One, Step Two, and Step Three. Also witness the deterioration of even the "worded problems" in many texts to type forms similar to worked examples in the book. Though these methods of teaching quickly enable the student to pass standardized examinations, they do not teach him anything about mathematical reasoning. But even the attainment of modest skill in mathematical reasoning does not assure a similar skill in other types of thought. Any belief to the contrary should be exploded by considering the arguments advanced by some professional mathematicians and other mathematically trained scientists when they speak or write on non-professional questions. President Conant recently said in

this university: "My own observations lead me to conclude that as human beings scientific investigators are statistically distributed over the whole spectrum of human folly and wisdom much as other men."

It is indeed a difficult task to teach men to think, and I have no magic formula for doing it. I submit, however, that more progress can be made by a frontal attack than by a flanking attack through freshman mathematics. Traditional courses in logic are too formalized to fill this bill, and moreover are too difficult for freshman students. Further by their emphasis on deductive reasoning they fail to present a balanced view of the ways men think.

It may be too much to ask freshmen to master a course in "Methods of Thinking" (even if such a course could be devised), but I think a more limited objective can be attained. My suggestion is that a course be designed to teach "Methods of Quantitative Thinking." Such a course was suggested to me by my war-time experience in Operations Research, which may be described as an attempt to analyze the quantitative aspects of conduct of military operations. It was disturbing to me to observe how many well-educated officers found it difficult to state quantitative problems in a fashion which could lead to their solution or to use elementary mathematics as a native language for expressing their ideas. It is remarkable to realize how many problems of business, industry, and education are wholly quantitative or else involve numbers in an essential way. Since most of these problems can be solved by relatively simple mathematical tools, the objective of such a course would be to develop a feeling for quantitative reasoning rather than to teach complex techniques.

Before considering the content of such a course, let us mention the kinds of questions which must be answered before one can obtain a solution to such a problem. First one must state the objective of the investigation, and this must be in quantitative terms. Often this appears as the statement that some variable is to be maximized or minimized. Examples of such variables are: profit, output, cost, or to take a military example, the ratio of friendly losses to enemy losses. This involves no mathematics, but demands a clear understanding of the numerical aspects of the problem. It is remarkable how helpful it is in a general discussion to have this objective firmly agreed upon. Then the other assumptions of the problem have to be introduced: what is to be held constant, what variables are to be considered as independent, and how realistic are these assumptions? Finally an equation is developed, the coefficients determined by statistical methods, and a course of action is determined as a result of this equation. Another type of problem concerns a decision to be made in which at least one of the important factors is very uncertain; indeed this is true to some extent in most of the decisions we have to make every day. The quantitative approach in this case is to estimate the probabilities associated with the uncertain variable, and the gains or losses associated with the possible eventualities. The reasonable man will then proceed in a fashion which will maximize his *expected return*, rather than play for a long shot. The popular failure to respect

this precept accounts not only for the success of gambling houses and the ruin of their regular customers, but also for many of the tragedies in our personal and business lives.

A course in "Methods of Quantitative Thinking" should then include an introduction to deductive logic, axioms, and abstract thinking; a thorough review of elementary algebra and arithmetic; a substantial treatment of probability and statistics; together with numerous practical applications to everyday quantitative problems. This is essentially a course in mathematics, and I believe it could be taught to our freshman students of liberal arts.

Turning now to the last objective, namely, explaining the role of mathematics in our culture, I feel that it is only the rare teacher who takes any steps in this direction. The educated public is almost totally unaware of what mathematics has done or how a mathematician thinks. I am sure many of you have told a new acquaintance that you are a mathematician, and have been treated with a most dubious "Oh!" as a reply, followed by an immediate change of the subject. From our own personal points of view this is an unhealthy situation, for it makes it difficult for mathematics to develop popular support and recognition. But it is even more serious for our culture as a whole when educated persons think of mathematics as a set of manipulative tricks, idle formalism, or black magic. There have been numerous attempts at various levels to explain mathematics "to the million." Although a few of these are excellent, the majority are quite misleading, and I think that there is room for much further progress in this direction. In particular there is a great scarcity of textbooks of this nature which are suitable for freshmen, and although those in print are excellent as pioneers, I do not feel that the ideal course has yet been developed. In constructing a course in "Mathematics in Our Civilization" there are several points to bear in mind. We must be sure that the content of the course *is* mathematics and is not just *about* mathematics. I do not think that it is possible to understand our subject without actually entering into its details and solving some of its problems. A mere acquaintance with fancy terms such as "Non-Euclidean Geometry," "transfinite numbers," and "group" is not enough. We must not teach a veneer of mathematical terms which will merely enable our students to dazzle their contemporaries in a polite conversation. We must also avoid the temptation to overemphasize mathematical recreations. Amusing as these may be, they are not the solid fabric of our subject, and they must not become the center of our instruction.

It is my position, therefore, that our standard freshman course needs radical revision if it is to meet the real needs of liberal arts students. Unfortunately, however, it is far easier to attack the present curriculum than it is to develop a sound alternative to it. The preparation of a novel course, the publication of a suitable text, and its acceptance by our university authorities are no mean hurdles. I come before you as one who is still in the first phase of this process; I have no pat course or textbook to peddle. It will require the combined efforts and talents of most of us before a new, widely accepted freshman curriculum

is developed and adopted. My own ideas are in a formative stage, but I suggest that progress will come along some of the following lines, or a synthesis of them:

(1) Replacement of much of analytic geometry and trigonometry by probability and statistics.

(2) Incorporation into our course of material adequate to explain to the non-specialist the nature of mathematics and its contribution to our civilization.

(3) Development of a new course in "Methods of Quantitative Thinking" along the lines sketched above.

### Bibliography

Since the above remarks are highly personal, I have made no attempt to gather an extensive bibliography. A few recent papers are listed below, several of which contain other references.

1. Conant, J. B.: On Understanding Science. Yale University Press, 1947.

2. Kittel, Charles: Nature and Development of Operations Research; Science, Vol. 105, pp. 150–153 (1947).

3. Northrop, E. P.: Mathematics in a Liberal Education. This MONTHLY, Vol. 52, pp. 132–135 (1945).

4. Ore, Oystein: Mathematics for Students of the Humanities. This MONTHLY, Vol. 51, pp. 453–458 (1944).

5. Richardson, M.: The Place of Mathematics in a Liberal Education. Nat. Math. Mag. Vol. 19, pp. 1–10 (1945).

### DISCUSSION OF THE PAPER

Dean W. L. Ayres, Purdue University: In considering this proposed curriculum we must keep clearly in mind the group of students for whom it is planned, that is, the group remaining after all engineers and majors in mathematics and physical science have been removed. Most of us will admit our complete failure at teaching the traditional course to this remaining group. Thus any proposed change is not likely to make the situation worse. This group is composed of those who have taken mathematics only because it has been required and have always hated it. They have been taught the technique of algebra in high school but the inoculation has not taken. There is little hope that a new exposure to the traditional techniques will improve them. I would consider the proposed course a success if it merely aroused the interest of this group and convinced them that mathematics was not the dull and dreary subject they had always believed it to be.

Professor J. S. Frame, Michigan State College: In connection with the suggestion that one half of the freshman mathematics course be devoted to statistics, a serious staff problem would be confronted in some of the larger institutions. During the winter term, 1947, at Michigan State College there were 1157 students who completed courses in statistics in addition to 1618 who completed college algebra, trigonometry, and analytical geometry, and a thousand more were enrolled in more elementary courses. All our teachers who could and would teach statistics were assigned to the regularly scheduled statistics courses.

Whether or not it would be pedagogically desirable, we could not have offered statistics to the other 1600 freshmen without an elaborate teacher training program.

Professor W. B. Carver, Cornell University: Are you able at the beginning of the freshman year to separate those students with the kind of general cultural interest at which this course is aimed from the students who will want a thorough training in mathematics for applications to the sciences, economics and so forth; if not, do you think that the course would be at all suited to the latter group of students?

Reply by Professor Allendoerfer: I think that a fairly satisfactory separation of the students into the two groups mentioned can be achieved at the beginning of their freshman year by considering their answers to the question: "Are your reasonably sure that you will take only one year of mathematics?" Of course, some students will answer "yes" to this question, will take the course I have outlined, and then will change their minds and wish more mathematics. These students will then be chronologically behind in their technical mathematics, but this will be balanced by their greater appreciation of the relation between mathematics and the subjects to which they will apply it. Indeed in the long run they will be better off then if they had started out with the traditional course.

Professor E. R. Ott, Rutgers University: It is significant that the four or five mathematicians who have spoken are in surprising agreement with the major premise of this paper, namely, that there are numerous topics which should be worked together to replace many of the topics now taught in trigonometry and analytic geometry. It seems agreed that there is an abundance of topics in algebra which are suitable for a first semester course. Then for a second semester, it is urged that a great deal of time be devoted to topics from applied statistics. It is doubtful if any existing text meets these specifications.

Professor Allendoerfer is concerned primarily with a terminal course in mathematics. However, I should like to point out that the inclusion of topics from applied statistics would be welcomed in the preparation of our science and engineering students. Engineers from industry and research, meeting at the Annual Convention of the American Society for Quality Control (Chicago, June 5–7, 1947), urged that every effort be made to include many concepts and techniques from applied statistics in the regular undergraduate engineering curriculum. In order to provide competent instruction, it was proposed that a program of staff study and discussion be instituted to be led by a person experienced in the techniques and applications of applied statistics.

# GEOMETRY OF THE KASNER TRIANGLE

G. H. F. GARDNER, Trinity College, Dublin, Ireland

**1. Introduction.** This paper intends to develop somewhat the geometry of the triangle in the Kasner plane, showing an extension of the parabolic analogue of the nine-point circle, a hyperbolic analogue of the nine-point circle, and an analogue of the Simpson line.

The *measure* of the *horn angle* between two curves with first order contact is defined by $(\Gamma_2-\Gamma_1)^2/(d\Gamma_2/ds_2-d\Gamma_1/ds_1)$ where $\Gamma$ denotes the curvature of the curve at the point of contact and $s$ denotes the arc measured along the curve. A fundamental theorem of Kasner states that the measure of the horn angle, thus defined, is invariant under the group of conformal transformations. The distance between two points $(x_1, y_1)$, $(x_2, y_2)$ in the Kasner plane is defined by the quantity $(x_2-x_1)^2/(y_2-y_1)$, which is the measure of the horn angle between the curves corresponding to the points $x=\Gamma$, $y=\Gamma/ds$. Thus the metric of the plane is $ds=dx^2/dy$. This is equivalent to the following change of axes transformation:

$$G_3: \quad x = mX + h, \qquad y = m^2Y + k.$$

The lines $x=$ a constant are called zero lines, and the lines $y=$ a constant are called infinite lines. The angle between two lines $y=p_1x+r_1$, $y=p_2x+r_2$ is defined as $\alpha_{12}=p_2/p_1$ since this is invariant under $G_3$. The distance with slope $P$ from the point $(x_0, y_0)$ to the line $y=px+r$ is $D(P)=(y_0-Px_0-r)/P(p-P)$. This has a relative minimum $P/p=1/2$, so that an angle $\alpha=1/2$ may be called a right angle. When $\alpha_{12}=\alpha_{21}$, it follows that $\alpha=\pm 1$, so $\alpha=-1$ may be called a quasi-right angle (a symmetrical definition); $\alpha=+1$ gives parallel lines.

If the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is divided in the ratio $l:m$ by the point $(x_3, y_3)$ then it is readily seen that

$$x_3 = \frac{lx_2 + mx_1}{l+m}, \qquad y_3 = \frac{ly_2 + my_1}{l+m}.$$

By defining a circle as the locus of a point $(x, y)$ at a constant distance $r$ from a fixed point $(a, b)$, we obtain $(x-a)^2/(y-b)=r$. This is called a parabolic circle, with its center at $(a, b)$ and with the radius $r$. We take as a standard parabolic circle, the unit circle $x^2=y$ with the center $O_p=(0, 0)$. Its parametric equations are $x=t$, $y=t^2$.

**2. Standard position of a triangle.** Clearly a triangle may be taken quite generally as any three points on a unit parabolic circle $(t_i, t_i^2)$, $i=1, 2, 3$. By writing

$$\begin{aligned} S_1 &= t_1 + t_2 + t_3, \\ S_2 &= t_2t_3 + t_3t_1 + t_1t_2, \qquad S^2 = t_1^2 + t_2^2 + t_3^2 = S_1^2 - 2S_2, \\ S_3 &= t_1t_2t_3, \end{aligned}$$

the inscribed parabolic circle is easily seen to be $(x+S_1)^2=4(y+S_2)$, having the center $I=(-S_1, -S_2)$.

**3. Nine-point parabolic circle.** The parabolic circle $(2x-S_1)^2=-(2y-S^2)$, with the center $N_p=(\frac{1}{2}S_1, \frac{1}{2}S^2)$, passes through the midpoints of the sides and the feet of the zero lines through the vertices of the standard triangle. The centroid $G$ of the triangle is $(S_1/3, S^2/3)$. The points $O_p$, $G$ and $N_p$ lie on $S^2x=S_1y$, which may thus be called an Euler line. Consider now the point $H_p$ on this Euler line which would correspond to the orthocenter of the triangle $(H_p=(S_1, S^2))$, so that $ON_p=N_pH_p$. Any point $Q$ on the circumparabolic circle is $(t, t^2)$. Therefore the midpoint of $H_pQ$ is

$$x=\frac{t+S_1}{2}, \qquad y=\frac{t^2+S^2}{2}\cdot$$

This midpoint lies on $(2x-S_1)^2=2y-S^2$, a parabolic circle with center $N_p=(\frac{1}{2}S_1, \frac{1}{2}S^2)$. The nine-point circle of a Euclidean triangle passes through the midpoints of the sides and the feet of the altitudes, and bisects all lines joining the orthocenter to the circumcircle. In the Kasner triangle this splits into two parabolic circles, concentric, equal, but oppositely oriented. Their equations are $(2x-S_1)^2=\pm(2y-S^2)$.

**4. Hyperbolic circles.** Consider the definition of a circle specified by having the angle in a segment constant. Let two fixed points be $(x_1, y_1)$, and $(x_2, y_2)$. Then the angle $\alpha$ subtended at a variable point $(x, y)$ is given by

$$\frac{y-y_2}{x-x_2}\cdot\frac{x-x_1}{y-y_1}=\alpha.$$

Hence if $\alpha$ is constant, $(x, y)$ lies on the curve,

$$xy(1-\alpha)-x(y_2-\alpha y_1)-y(x_2-\alpha x_1)+y_1x_2-\alpha y_2x_1=0. \tag{1}$$

This may then be called a hyperbolic circle, and its asymptotes are always parallel to the zero and infinite lines. In the case where

$$\frac{x_1+x_2}{2}=\frac{y_1+y_2}{2}=0, \qquad \alpha=-1 \text{ (a quasi-right angle)},$$

we obtain $xy=$a constant, with its center at $(0, 0)$. Its parametric equations are $x=u$, $y=c/u$. It is easily seen that the angle subtended by points $u_1$, $u_2$ is $u_2/u_1$, and that the tangent is quasi-perpendicular to the radius.

**5. Analogue of the Simpson line.** Consider the triangle with vertices $(t_i, t_i^2)$ in the unit parabolic circle. Let us find the locus of a point $P(x, y)$ such that the feet of the lines drawn from $P$, making constant angles $\alpha$ with the three sides of the triangle, are collinear. We will see that it is the hyperbolic circumcircle of the triangle. Let two such lines meet the sides $(t_1t_3)$, $(t_1t_2)$ in the points $(x_4, y_4)$, $(x_5, y_5)$. Then we have for the slope of the line joining these two points,

$$\frac{y_4 - y_5}{x_4 - x_5} = \frac{\alpha(t_1 + t_2)(t_1 + t_3)(x - t_1)}{y - t_1^2}.$$

Hence if the three feet are collinear, $(x, y)$ must lie on the curve,

$$\frac{\alpha(t_1 + t_2)(t_1 + t_3)(x - t_1)}{y - t_1^2} = \frac{\alpha(t_2 + t_3)(t_2 + t_1)(x - t_2)}{y - t_2^2},$$

which reduces to

$$xy + xS_2 - yS_1 - S_3 = 0. \tag{2}$$

This is a hyperbolic circle with center $O_H \equiv (S_1, -S_2)$. Solving with the circum-parabolic circle $y = x^2$, we obtain $(x - t_1)(x - t_2)(x - t_3) = 0$; hence we do have the hyperbolic circumcircle of the triangle $(t_i, t_i^2)$.

THEOREM. *If from any point on the hyperbolic circumcircle of a triangle lines are drawn making equal angles with the sides of the triangle, their feet are collinear and the line may be called the generalized Simpson line of the triangle.*

In the case where $\alpha = -1$ the line may be called the *Simpson line* of the triangle. The parametric equations of (2) are

$$x = S_1 + u, \qquad y = -S_2 + (S_3 - S_1S_2)/u.$$

The equation of the generalized Simpson line of the point $(u)$ is then, on reduction,

$$(1 - \alpha)\{y + \alpha ux\} = -(S_1 + u)\{\alpha^2 u + \alpha S_1 + S_2/u\} + S_3/u. \tag{3}$$

The slope of the generalized Simpson line of the point $(u)$ is $-\alpha u$. From this we have the following familiar theorems:

(a) *The generalized Simpson lines of diametrically opposite points are quasi-perpendicular.*

(b) *The triangle formed by the generalized Simpson lines of three points is similar to the triangle formed by the three points.*

**6. Nine point hyperbolic circle.** We have seen above that hyperbolic circles offer an analogue for the Simpson line. Let us consider the corresponding hyperbolic nine-point circle. We take the triangle, as usual, in the unit parabolic circle. The hyperbolic circle

$$xy - \tfrac{1}{2}x(S_1^2 - S_2) + \tfrac{1}{4}(S_1S_2 - S_3) = 0, \tag{4}$$

with center $N_H = [0, \tfrac{1}{2}(S_1^2 - S_2)]$, goes through the midpoints of the sides and the feet of the quasi-perpendiculars through the vertices. Also the quasi-perpendiculars meet in the point $H = (-S_1, S_1^2)$ called the orthocenter. Any point $P$ on the hyperbolic circumcircle is $[S_1 + u, -S_2 + (S_3 - S_1S_2)/u]$. Then the midpoint of $HP$ is given by

$$x = \frac{u}{2}, \qquad y = \tfrac{1}{2}[S_1^2 - S_2 + (S_3 - S_1S_2)/u],$$

which is easily seen to lie on (4).

THEOREM. *The hyperbolic circle, $xy - \frac{1}{2}x(S_1^2 - S_2) - \frac{1}{4}(S_3 - S_1S_2) = 0$, passes through the midpoints of the sides and the feet of the quasi-perpendiculars, and bisects all lines joining the orthocenter to the hyperbolic circumcircle.*

Such a hyperbolic circle may be called the hyperbolic nine-point circle of the triangle, and offers a complete analogue of the nine-point circle.

THEOREM. *The Simpson line ($\alpha = -1$) of the point $P$ bisects the line $PH$ and meets the hyperbolic nine-point circle there.*

The coördinates of the midpoint are

$$x = \frac{u}{2}, \qquad y = \tfrac{1}{2}(S_1^2 - S_2) + (S_3 - S_1S_2)/2u,$$

which are easily seen to lie on (3) with $\alpha = -1$ and (4).

THEOREM. *The centroid, hyperbolic circumcenter, hyperbolic nine-point center and the orthocenter are collinear.*

The four points

$$G \equiv \left\{\tfrac{1}{3}S_1, \tfrac{1}{3}S^2\right\}, \qquad O_H \equiv \left\{S_1, -S_2\right\}, \qquad N_H \equiv \left\{0, \tfrac{1}{2}(S_1^2 - S_2)\right\},$$

and $H \equiv \{-S_1, S_1^2\}$ all lie on the line given by

$$2yS_1 + x(S_1^2 + S_2) - S_1(S_1^2 - S_2) = 0, \tag{5}$$

which may be called the hyperbolic Euler line. Moreover,

$O_HG{:}GH_N{:}N_HH = 2{:}1{:}3$, just as

$O_PG{:}GH_P{:}\ N_PH_P = 2{:}1{:}3$ in the parabolic case.

THEOREM. *The hyperbolic nine-point circle touches the hyperbolic inscribed circle.*

This is awkward to show analytically as the equation of the inscribed hyperbolic circle has coefficients which are not rational in the parameter $t$. However we may see that such a conclusion follows as a particular case of the following:

THEOREM. *A conic through the midpoints of the sides of a triangle touches a conic tangent to the sides of the triangle if the conics have parallel asymptotes.*

This is easily seen by projecting the two common points of intersection of the conics with the line at infinity into the circular points at infinity. The conics then become simultaneously the nine-point and inscribed circles of the triangle, which do touch.

In particular, the inscribed parabolic circle touches the nine-point parabolic circle, and the inscribed hyperbolic circle touches the nine-point hyperbolic circle.

**References**

1. Kasner, Conformal Geometry, Proceedings of the Fifth International Congress of Mathematicians, Vol. 2, pp. 81–90, 1912.

2. Kasner and Comenetz, Conformal geometry of horn angles, Proceedings of the National Academy of Sciences, Vol. 22, pp. 303–309, 1936.

3. Kasner, Trihornometry: A new chapter in conformal geometry, Proceedings of the National Academy of Sciences, Vol. 23, pp. 337–341, 1937.

4. De Cicco, The analogue of the Moebius group of circular transformations in the Kasner plane, Bulletin of the American Mathematical Society, Vol. 45, pp. 936–943, 1939.

5. De Cicco, Analog of the nine-point circle in the Kasner plane, this MONTHLY, Vol. 46, pp. 627–643, 1943.

6. De Cicco, Differential geometry in the Kasner plane, this MONTHLY, Vol. 53, pp. 305–313, 1946.

# NONSEPARABLE CONVEX SYSTEMS

H. HADWIGER, Berne, Switzerland

A system of $n$ closed convex bodies $K_i$ $(i=1, 2, \cdots, n)$ in a plane will be called *separable* if there is a straight line $G$ which intersects no $K_i$ and which divides the plane into two half planes each containing at least one $K_i$. In the contrary case we shall call the system *nonseparable.*

In a recent note A. W. Goodman and R. E. Goodman* showed that a nonseparable system of circles $K_i$ having radii $R_i$ $(i=1, 2, \cdots, n)$ can always be covered by a circle $K$ of radius

$$R = \sum_{i=1}^{n} R_i.$$

In this note we shall prove in a very simple way an analogous result concerning general nonseparable convex systems.

THEOREM. *Let $K_0$ denote the convex hull of the nonseparable convex system $K_i$ $(i=1, 2, \cdots, n)$. Let $L_i$, $D_i$, and $R_i$ denote the circumference, the diameter, and the radius of the circumcircle, respectively, of the body $K_i$ $(i=0, 1, \cdots, n)$. Then*

$$L_0 \leqq \sum_{i=1}^{n} L_i, \qquad D_0 \leqq \sum_{i=1}^{n} D_i, \qquad R_0 \leqq \sum_{i=1}^{n} R_i. \tag{1}$$

Proof: Let $G(\phi)$ be any line which with a fixed reference line $G_0$ forms an

* A. W. Goodman and R. E. Goodman, A circle covering theorem, this MONTHLY, Vol. LII, No. 9, 1945.

angle $\phi$ $(0 \leqq \phi < \pi)$. Let $S_i(\phi)$ be the length of the projection of the convex region $K_i$ $(i=0, 1, \cdots, n)$ on the line $G(\phi)$. The quantity $S_i(\phi)$ is just the so called breadth of the body $K_i$ in the direction determined by $\phi$.

Since each perpendicular to $G(\phi)$ which intersects the convex hull $K_0$ must, by the assumption of nonseparability of the system, intersect at least one of the convex bodies $K_i$, the segments $S_i(\phi)$, $(i=1, 2, \cdots, n)$ completely cover the segment $S_0(\phi)$, so that for all $\phi$ one has

$$S_0(\phi) \leqq \sum_{i=1}^{n} S_i(\phi). \tag{2}$$

By a well known formula of Cauchy

$$\int_0^{\pi} S_i(\phi)d\phi = L_i,$$

so that an integration of the inequality (2) gives the first of the inequalities (1). If one next selects the angle $\phi = \phi_0$ so that $S_i(\phi_0) = D_0$ our inequality gives

$$D_0 \leqq \sum_{i=1}^{n} S_i(\phi_0)$$

and since $S_i(\phi_0) \leqq D_i$, it follows that

$$D_0 \leqq \sum_{i=1}^{n} D_i.$$

Finally consider the circumcircles of the convex bodies $K_i$ $(i=1, 2, \cdots, n)$. Obviously these form a nonseparable system of circles. By the theorem of A. W. Goodman and R. E. Goodman these circles can be covered by a circle of radius $R = \sum_{i=1}^{n} R_i$, and since $R_0 \leqq R$, we have

$$R_0 \leqq \sum_{i=1}^{n} R_i.$$

The corresponding result for spherically convex systems on the surface $E$ of the unit sphere are inherently simpler, since in this case no reference to a convex hull is needed, in the formulation of a necessary condition for nonseparability.

A convex body on the surface of the unit sphere $E$ is a closed set in $E$, which lies entirely in some hemisphere $E'$ and which contains, along with every pair of points of the body, all the points of the arc of the great circle lying in $E'$ and joining the two points.

A convex system $K_i$ $(i=1, 2, \cdots, n)$ on $E$ will be called *separable* if there is a great circle $G$ which intersects no $K_i$, and which divides the sphere into two hemispheres each containing at least one $K_i$. Otherwise the system is called *nonseparable*. We prove now the somewhat more general result.

THEOREM. *Let $K_i$ $(i=1, 2, \cdots, n)$ be a system of convex bodies on the surface of the unit sphere E. Let $N(G)$ denote the number of bodies intersected by a great circle G. Suppose further that for all G*

$$\alpha \leqq N(G) \leqq \beta.$$

*Then*

$$2\pi\alpha \leqq \sum_{i=1}^{n} L_i \leqq 2\pi\beta,$$

*where $L_i$ is the circumference of the convex body $K_i$.*

As a corollary we have:

COROLLARY. *If the convex system on the unit sphere is nonseparable and is not contained in any hemisphere, then*

$$2\pi \leqq \sum_{i=1}^{n} L_i.$$

Proof: Let $dG$ denote the (spherical) kinematic density on $E$ of the moving great circle $G$; then

$$\int dG = 8\pi^2$$

where the integral is taken over all motions of $G$.

By a formula from integral geometry due to L. A. Santaló* one has for the integral of $N(G)$

$$\frac{1}{4\pi}\int N(G)dG = \sum_{i=1}^{n} L_i.$$

Taking into consideration the hypothesis $\alpha \leqq N(G) \leqq \beta$, one can easily obtain from the above integral formula the inequality (3) which was to be proved.

The corollary concerning nonseparable convex systems is an easy consequence of the remark that in this case $\alpha \geqq 1$.

* L. A. Santaló, Algunos valores medios y desigualdedes referentes a curvas situadas sobre la superficie esferica; Revista de la Unión Matemática Argentina, Vol. VIII, 1942.

# AN INVERSION FORMULA FOR AN INTEGRAL RELATED TO DIRICHLET SERIES

W. C. G. FRASER, Rutgers University

In this article a real inversion formula is obtained for an integral of the form

$$f(s) = \int_1^\infty t^{-s} d\alpha(t). \tag{1}$$

This is the Stieltjes integral form of the Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s}, \tag{2}$$

where we define $\alpha(t)$ as follows:

$$\alpha(1) = 0, \qquad \alpha(t) = \sum_{n<t} a_n, \qquad (t > 1).$$

While complex inversion formulas for (2) can be found in textbooks, the author has not seen the following real inversion formula, which is the analogue of the Phragmén inversion formula [1] for the Laplace transform.

It can be shown [2, 35 *et seq.*] that there is a real number $\sigma_c$ such that the integral on the right side of (1) converges if $R(s)$, the real part of $s$, is greater than $\sigma_c$, and diverges if $R(s)$ is less than $\sigma_c$. It can also be shown that the function $\alpha(t)$ satisfies the inequalities

$$\alpha(t) \begin{cases} = o(t^{(\sigma_c+\epsilon)}) & (\sigma_c \geqq 0), \\ \text{is bounded} & (\sigma_c < 0). \end{cases} \tag{3}$$

THEOREM. *If $\alpha(t)$ is a function of bounded variation in every finite interval $(1, R)$ with $\alpha(1) = 0$, and*

$$f(s) = \int_1^\infty t^{-s} d\alpha(t), \qquad R(s) > \sigma_c, \tag{4}$$

*then*

$$\lim_{s\to+\infty} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} t^{ns} f(ns) \tag{5}$$

*equals $\alpha(t)$ where $\alpha(t)$ is continuous, or equals*

$$\frac{1}{e}\alpha(t-) + \left(1 - \frac{1}{e}\right)\alpha(t+)$$

*where $\alpha(t)$ is discontinuous, with limits $\alpha(t-)$ and $\alpha(t+)$ on the left and right, respectively.*

It is not necessary to assume that $\alpha(1)=0$, but no generality is lost by doing so, and the details are somewhat simplified.

We have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} t^{ns} f(ns) \\
&= \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} t^{ns} \int_1^{\infty} \tau^{-ns} d\alpha(\tau) \\
&= \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} t^{ns} \left\{ \left[\alpha(\tau)\tau^{-ns}\right]_1^{\infty} + ns \int_1^{\infty} \alpha(\tau)\tau^{-ns-1} d\tau \right\}.
\end{aligned} \tag{6}
$$

Using the inequalities (3), and the fact that $\alpha(1)$ equals zero, the integrated part on the right side of (6) is seen to vanish. Consequently the expression (6) may be written in the form

$$
\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} t^{ns} ns \int_1^{\infty} \alpha(\tau)\tau^{-ns-1} d\tau. \tag{7}
$$

The expression (7) converges if all terms are replaced by their moduli. Thus we may invert the order of summation and integration to obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} t^{ns} ns \int_1^{\infty} \alpha(\tau)\tau^{-ns-1} d\tau \\
&= \int_1^{\infty} \alpha(\tau) \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} ns \cdot t^{ns}\tau^{-ns-1} d\tau \\
&= \int_1^{\infty} \alpha(\tau) \frac{d}{d\tau} \left\{ \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\frac{t}{\tau}\right)^{ns} \right\} d\tau \\
&= \int_1^{\infty} \alpha(\tau) \frac{d}{d\tau} \left\{ e^{-(t/\tau)^s} \right\} d\tau \\
&= \int_1^{\infty} \alpha(\tau) K_s\left(\frac{t}{\tau}\right) d\tau,
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
K_s\left\{\frac{t}{\tau}\right\} &= \frac{d}{d\tau} \left\{ e^{-(t/\tau)^s} \right\} \\
&= \frac{s}{\tau}\left(\frac{t}{\tau}\right)^s e^{-(t/\tau)^s} \geqq 0, \quad \text{when} \quad s \geqq 0.
\end{aligned}
$$

Also

$$
\int_1^{\infty} K_s\left(\frac{t}{\tau}\right) d\tau = \left[e^{-(t/\tau)^s}\right]_1^{\infty}
$$

$$= 1 - e^{-t^s}$$
$$\to 1 \quad \text{as} \quad s \to +\infty.$$

Thus the factor $K_s(t/\tau)$ is positive and its integral tends to 1. Furthermore,

$$\int_1^{t-\epsilon} K_s\left(\frac{t}{\tau}\right) d\tau = e^{-(t/(t-\epsilon))^s} - e^{-t^s}$$
$$\to 0 \quad \text{as} \quad s \to +\infty,$$

and

$$\int_{t+\epsilon}^{\infty} K_s\left(\frac{t}{\tau}\right) d\tau = 1 - e^{-(t/(t+\epsilon))^s}$$
$$\to 0 \quad \text{as} \quad s \to +\infty.$$

The order relations (3) and the fact that $K_s(t/\tau)$ is positive, are sufficient to ensure that both

$$\int_1^{t-\epsilon} \alpha(\tau) K_s\left(\frac{t}{\tau}\right) d\tau \to 0, \qquad \int_{t+\epsilon}^{\infty} \alpha(\tau) K_s\left(\frac{t}{\tau}\right) d\tau \to 0, \quad \text{as} \quad s \to +\infty.$$

Thus it follows that if $\alpha(\tau)$ is continuous at $\tau = t$, then the expression (8) tends to the limit $\alpha(t)$ as $s \to +\infty$.

However, where $\alpha(t)$ is discontinuous, we have

$$\int_t^{t+\epsilon} \alpha(\tau) K_s\left(\frac{t}{\tau}\right) d\tau$$
$$= \alpha(t+) \int_t^{t+\epsilon} K_s\left(\frac{t}{\tau}\right) d\tau + \int_t^{t+\epsilon} \{\alpha(\tau) - \alpha(t+)\} K_s\left(\frac{t}{\tau}\right) d\tau.$$

The first term on the right

$$= \alpha(t+) \left[e^{-(t/(t+\epsilon))^s} - e^{-1}\right]$$
$$\to \alpha(t+) \left(1 - \frac{1}{e}\right) \quad \text{as} \quad s \to +\infty.$$

The absolute value of the second term on the right is less than $\delta(1 - 1/e)$ for any arbitrary positive real number $\delta$, as $s \to +\infty$, provided $\epsilon$ is sufficiently small. The integral $\int_{t-\epsilon}^{t} \alpha(\tau) K_s(t/\tau) d\tau$ may be treated similarly, leading to the result that where $\alpha(t)$ is discontinuous, the expression (8) tends to the limit

$$\frac{1}{e}\alpha(t-) + \left(1 - \frac{1}{e}\right)\alpha(t+) \quad \text{as} \quad s \to +\infty.$$

## References

1. E. Phragmén, Sur une extension d'un théorème classique de la Théorie des Fonctions, Acta Mathematica, Vol. 28 (1904), pp. 351–368.

2. D. V. Widder, The Laplace Transform, Princeton, N. J., 1941.

# MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

*Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.*

## NEW PROOF OF A MINIMUM PROPERTY OF THE REGULAR $n$-GON

L. F. TOTH, Budapest, Hungary

J. Kurschák gives in his paper *Über dem Kreis ein- und umgeschriebene Vielecke** among others a complete and entirely elementary geometrical proof of the well known fact according to which the regular $n$-gon $\overline{P}_n$ has a minimal area among all $n$-gons $P$ circumscribed about a circle $c$. In this proof $P_n$ is carried, after a dismemberment and a suitable reassembly, in $n-1$ steps into $\overline{P}_n$ so that the area increases at every step.

In this note we give an extremely simple proof,† which appears to be new, showing immediately that if $P_n$ is not regular, then $P_n > \overline{P}_n$, where the area is denoted by the same symbol as the domain.

Consider the circle $C$ circumscribed about $\overline{P}_n$. We show that already for the part $P_n \cdot C$ of $P_n$ lying in $C$ we have

$$P_n \cdot C > \overline{P}_n.$$

We have $P_n \cdot C = C - ns + (s_1 s_2 + s_2 s_3 + \cdots + s_n s_1)$, where we denote by $s_1, s_2, \cdots, s_n$ the circular sections of $C$ cut off by the consecutive sides of $P_n$, and by $s$ the circular section of $C$ cut off by a tangent to $c$. Hence

$$P_n \cdot C \geqq C - ns.$$

Then

$$P_n \geqq P_n \cdot C \geqq C - ns = \overline{P}_n.$$

Equality holds in $P_n \geqq P_n C$ resp. in $P_n C \geqq \overline{P}_n$ only if no vertex of $P_n$ lies in the outside resp. in the inside of $C$; this completes the proof.

## BINOMIAL COEFFICIENTS MODULO A PRIME

N. J. FINE, University of Pennsylvania

The following theorem, although given by Lucas in his *Theorie des Nombres* (pp. 417–420), does not appear to be as widely known as it deserves to be:

THEOREM 1. *Let $p$ be a prime, and let*

$$M = M_0 + M_1 p + M_2 p^2 + \cdots + M_k p^k \qquad (0 \leqq M_r < p),$$
$$N = N_0 + N_1 p + N_2 p^2 + \cdots + N_k p^k \qquad (0 \leqq N_r < p).$$

* Mathematische Annalen 30 (1887), pp. 578–581.

† As P. Szász remarked [Bemerkung zu einer Arbeit von K. Kürschák, Matematikai es Fizikai Lapok XLIV (1937), p. 167, note 3] Kürschák's proof is independent of the axiom of parallels. This advantage is preserved in the present proof.

*Then*

$$\binom{M}{N} \equiv \binom{M_0}{N_0}\binom{M_1}{N_1}\binom{M_2}{N_2}\cdots\binom{M_k}{N_k} \pmod{p}.$$

We offer a short proof of the above theorem:

$$\sum_{N=0}^{M}\binom{M}{N}x^N = (1+x)^M = \prod_{r=0}^{k}\left\{(1+x)^{p^r}\right\}^{M_r},$$

$$\equiv \prod_{r=0}^{k}(1+x^{p^r})^{M_r} \pmod{p},$$

$$= \prod_{r=0}^{k}\left\{\sum_{s_r=0}^{M_r}\binom{M_r}{s_r}x^{s_rp^r}\right\},$$

$$= \sum_{N=0}^{M}\left\{\sum\prod_{r=0}^{k}\binom{M_r}{s_r}\right\}x^N,$$

where the inner sum is taken over all sets $(s_0, s_1, \cdots, s_k)$ such that $\sum_{r=0}^{k} s_rp^r = N$. But $0 \leqq s_r \leqq M_r < p$, so there is at most one such set, given by $s_r = N_r$ $(0 \leqq r \leqq k)$ if every $N_r \leqq M_r$; if not, the sum is zero. The theorem follows by equating coefficients of $x^N$, since

$$\binom{M_r}{N_r} = 0 \quad \text{for} \quad N_r > M_r.$$

THEOREM 2. *Let $T(M)$ be the number of integers $N$ not exceeding $M$ for which*

$$\binom{M}{N} \not\equiv 0 \pmod{p}.$$

*Then*

$$T(M) = \prod_{r=0}^{k}(M_r + 1).$$

Proof: Since $M_r < p$, there are $M_r + 1$ values of $N_r$, given by $0 \leqq N_r \leqq M$, for which

$$\binom{M_r}{N_r} \not\equiv 0 \pmod{p},$$

and these are the only ones.

THEOREM 3. *A necessary and sufficient condition that all the binomial coefficients*

$$\binom{M}{N}, \qquad 0 < N < M,$$

*be divisible by $p$ is that $M$ be a power of $p$.*

Proof: The function $T(M)$ takes the value 2 if and only if one of the $M_r$ is 1 and all the others are 0.

In the opposite direction, we may ask for what values of $M$ *none* of the binomial coefficients

$$\binom{M}{N}, \qquad 0 \leqq N \leqq M,$$

are divisible by $p$.

THEOREM 4. *A necessary and sufficient condition that none of the binomial coefficients of order $M$, with*

$$M = M_0 + M_1 p + \cdots + M_k p^k \qquad (0 \leqq M_r < p;\ M_k > 0)$$

*be divisible by $p$ is that $M_r = p-1$ for $r<k$.*

Proof: Let $M^* = M - M_k p^k$. Suppose first that $T(M) = M+1$. Then

$$\begin{aligned} M_k p^k + M^* + 1 = M + 1 = T(M) &= (M_k + 1)T(M^*) \leqq (M_k + 1)(M^* + 1) \\ &= M_k(M^* + 1) + M^* + 1 \leqq M_k p^k + M^* + 1. \end{aligned}$$

From this chain of inequalities it follows that $M^* = p^k - 1$. Conversely, if $M^* = p^k - 1$, then

$$T(M) = (M_k + 1)p^k = M_k p^k + M^* + 1 = M + 1.$$

Our last theorem deals with the "probability" that a binomial coefficient chosen "at random" will be divisible by $p$. More precisely, consider the $\frac{1}{2}(m+1)(m+2)$ binomial coefficients

$$\binom{M}{N}, \qquad 0 \leqq N \leqq M \leqq m,$$

and let $Q(p; m)$ be the fraction of these which are not divisible by $p$.

THEOREM 5. *For every prime $p$,* $\lim_{m\to\infty} Q(p; m) = 0$.

Proof: For $k \geqq 0$, let

$$G(k) = \tfrac{1}{2}p^k(p^k + 1)Q(p; p^k - 1) = \sum_{M=0}^{p^k-1} T(M).$$

Clearly $G(0) = 1$. Using the notation introduced in the proof of the preceding theorem, we have

$$\begin{aligned} G(k + 1) &= \sum_{M=0}^{p^{k+1}-1} T(M) \\ &= \sum_{M_k=0}^{p-1} \sum_{M^*=0}^{p^k-1} (M_k + 1)T(M^*) \end{aligned}$$

$$= \left\{\sum_{M_k=0}^{p-1} (M_k + 1)\right\} \left\{\sum_{M^*=0}^{p^k-1} T(M^*)\right\}$$
$$= \tfrac{1}{2}p(p + 1)G(k).$$

It follows immediately that $G(k) = (\frac{1}{2}p(p+1))^k$. Now suppose that $p^k \leqq m < p^{k+1}$. Then

$$Q(p; m) \leqq \frac{2}{(m + 1)(m + 2)} G(k + 1) < 2p^{-2k}G(k + 1) = 2p^{-2k}(\tfrac{1}{2}p(p + 1))^{k+1}$$
$$= p(p + 1)\left(\frac{1 + 1/p}{2}\right)^k,$$

which tends to 0 with increasing $m$.

By an obvious extension it follows that, given an arbitrary finite set of primes, it is "virtually certain" that a binomial coefficient chosen at random will be divisible by all the primes in the set.

## VOLUME OF AN $n$-DIMENSIONAL SPHERE

H. P. EVANS, University of Wisconsin

1. **Introduction.** That certain definite integrals may be evaluated from probability considerations is well known [1]. It will be shown that the evaluation of the multiple integral for the volume of an $n$-dimensional sphere may be obtained from the probability distribution of the sum of squares of $n$ independent and normally distributed random variables, all having the same standard deviation $\sigma$ and mean zero. Since the study of this distribution is a standard topic for courses in both probability theory [1] and mathematical statistics [2, 3, 4], and since the formula for the volume of an $n$-dimensional sphere is derivable by the methods of advanced calculus [5, 6], a consequence of the present note is the establishment of a further connection between the methods of probability theory and the methods of advanced calculus.

2. **Analytical development.** Let $x_i$ $(i=1, 2, \cdots, n)$ be $n$ independent and normally distributed random variables, each having the standard deviation $\sigma$ and the mean value zero. If we put $x=\sum x_i^2$, where the summation is from 1 to $n$, it is known from probability theory that the probability density function for $x$ is

$$(1) \qquad \begin{cases} f(x) = \dfrac{(2\sigma^2)^{-n/2}}{\Gamma\left(\dfrac{n}{2}\right)} x^{n/2-1}e^{-x/2\sigma}, & x > 0, \\ f(x) = 0, & x \leqq 0. \end{cases}$$

From the given distribution of the $x_i$'s it follows that the distribution function for $x=\sum x_i^2$ is

$$P\left(\sum x_i^2 \leqq t\right) = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \underbrace{\int\int \cdots \int}_{R} e^{-\Sigma x_i^2/2\sigma^2} dx_1 dx_2 \cdots dx_n, \tag{2}$$

where $R$ is the region in the space $R_n$ for which $\sum x_i^2 \leqq t$, that is, $R$ is the region interior to the $n$-dimensional sphere of radius $r = \sqrt{t}$ and center at the origin. By means of (1) we have also,

$$P\left(\sum x_i^2 \leqq t\right) = P(x \leqq t) = \frac{2^{-n/2}\sigma^{-n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^t x^{n/2-1} e^{-x/2\sigma^2} dx. \tag{3}$$

Now let $F(x)$ be any function whose expected value $E\{F(x)\}$ exists. Using the probability density functions which appear in (2) and (3), respectively, it follows that

$$E\left\{F\left(\sum x_i^2\right)\right\} = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \underbrace{\int\int \cdots \int}_{R_n} e^{\Sigma x_i^2/2\sigma^2} F\left(\sum x_i^2\right) dx_1 dx_2 \cdots dx_n, \tag{4}$$

where $R_n$ denotes the entire $n$-space, and

$$E\left\{F\left(\sum x_i^2\right)\right\} = E\{F(x)\} = \frac{2^{-n/2}\sigma^{-n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty x^{n/2-1} e^{-x/2\sigma^2} F(x) dx. \tag{5}$$

Next we define

$$\begin{cases} F(x) = e^{x/2\sigma^2}, & x \leqq t, \\ F(x) = 0, & x > t. \end{cases} \tag{6}$$

Then

$$\begin{cases} F\left(\sum x_i^2\right) = e^{\Sigma x_i^2/2\sigma^2}, & \sum x_i^2 \leqq t, \\ F\left(\sum x_i^2\right) = 0, & \sum x_i^2 > t. \end{cases} \tag{7}$$

Since by (7) we have $F(\sum x_i^2) = 0$ for all points exterior to $R$, equation (4) becomes

$$E\left\{F\left(\sum x_i^2\right)\right\} = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \underbrace{\int\int \cdots \int}_{R} dx_1 dx_2 \cdots dx_n = (\sqrt{2\pi}\,\sigma)^{-n} V_n, \tag{8}$$

where $V_n$ is the volume of the $n$-dimensional sphere of radius $r = \sqrt{t}$ which bounds the region $R$. Similarly, substitution of (6) in (5) gives

$$E\{F(\sum x_i^2)\} = \frac{2^{-n/2}\sigma^{-n}}{\Gamma\left(\frac{n}{2}\right)}\int_0^t x^{n/2-1}dx = \frac{2^{-n/2}\sigma^{-n}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}t^{n/2}. \tag{9}$$

From (8) and (9), since $r=\sqrt{t}$ and $n/2\Gamma(n/2)=\Gamma(n/2+1)$, there results

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}r^n. \tag{10}$$

If the surface area $S_n$ of the $n$-dimensional sphere is defined by the relation $dV_n=S_n dr$, one obtains also

$$S_n = \frac{n\pi^{n/2}r^{n-1}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{11}$$

**References**

1. J. V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, pp. 279, 331.
2. H. Cramér, Mathematical Methods of Statistics, Princeton University Press, pp. 121, 233.
3. C. E. Weatherburn, A First Course in Mathematical Statistics, Cambridge University Press, p. 152.
4. P. G. Hoel, Introduction to Mathematical Statistics, John Wiley & Sons, p. 135.
5. R. Courant, Differential and Integral Calculus, Vol. II, Nordeman Publishing Co., p. 302.
6. P. Franklin, Treatise on Advanced Calculus, John Wiley & Sons, p. 580.

## CLASSROOM NOTES

Edited by C. B. Allendoerfer, Haverford College

*All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.*

### AN ORNITHOLOGICAL NOTE

R. A. Johnson, Brooklyn College

In the old Granville Calculus (p. 380, ex. 19) is proposed the problem of finding "the volume of the *egg* generated by the revolving the curve

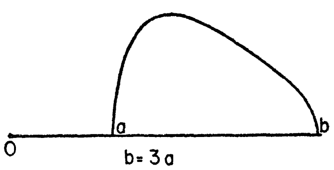
$$x^2y^2 + (x-a)(x-b) = 0, \qquad a < b$$

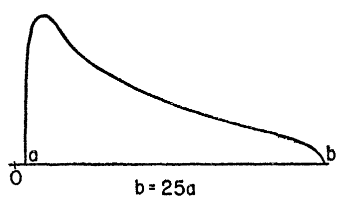
about $OX$." By implication $0<a$; and the second term needs a coefficient $c^2$ to render the equation homogeneous. The problem, as stated, presents no difficulty; but the determination of the *area* bounded by the curve is a good exercise for

honor students in a first course, and for all students in a second course. With a fair amount of labor the result comes out in the neat form

$$A = \pi c(b^{1/2} - a^{1/2})^2.$$

Using elementary principles of curve-tracing, one casually deduces that the term "egg" is justified. But if one takes more pains with the sketching of the curve, the results are surprising. It is easy to set up a geometric construction for points of the curve; if $Y = [(x-a)(b-x)]^{1/2}$ is recognized as the ordinate of a circle, then $y/c = Y/x$. Thus for any chosen $a$ and $b$, the curve may be drawn rapidly. It turns out that if $a$ and $b$ are nearly equal, the curve does resemble the profile of an ordinary egg; but if $a$ is small as compared to $b$, the form of the curve is remarkably modified, as shown in the figures. It resembles no egg that Dr. Granville ever saw.




The nature of the curve may better be studied by translating the origin to a point midway between $a$ and $b$, and changing the notation. The equation may then be written

$$y = c(r^2 - x^2)^{1/2}/(x + h) \qquad h > 0.$$

From this it is easy to find that the maximum occurs where $hx + r^2 = 0$, namely on the polar of $(-h, 0)$ with regard to the base circle; and the inflections are given by

$$2hx^3 + 3r^2x^2 + (h^2 - 2r^2)r^2 = 0.$$

It follows that there are no real inflections if $h > r\sqrt{2}$; if $h = r\sqrt{2}$ there are coincident inflections at $x = 0$; if $h$ is less than this value there are real inflections for two values of $x$, almost equidistant from $x = 0$. This agrees with the evidence of the curves as drawn.

My colleague, Professor Edward Fleisher, an enthusiastic student of bird life, points out that the intermediate forms of the curve (not the extreme ones) are approximated by the eggs of certain cliff-nesting birds. As an illustration, one may observe the murre's nest in *The Book of Birds*, National Geographic Society, volume I, page 326.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 796. *Proposed by Henry Scheffé, University of California at Los Angeles.*

Describe a coin-tossing experiment in which the probability of success is one third.

E 797. *Proposed by C. O. Hines, University of Toronto.*

If ellipses are described on diameters of a given circle as major axis, and such that they all pass through a given point (within, or on the boundary of, the circle), then they will also all pass through a second point, symmetrical about the center to the first, and the locus of their foci will be an ellipse having the two fixed points as foci and the common diameter as major axis.

### SOLUTIONS

#### Diabolic Squares

E 764 [1947, 163]. *Proposed by R. J. Walker, Cornell University*

If $a_{ij}$ is the number in the $i$th row and $j$th column of a diabolic (pandiagonal magic) square of order five, show that

$$\sum_{i=1}^{5}\left(\prod_{j=1}^{5} a_{ij}\right) = \sum_{j=1}^{5}\left(\prod_{i=1}^{5} a_{ij}\right).$$

This relationship also holds for magic squares of order three. Does it, or any similar relation, hold for magic, or diabolic, squares of any other order—in particular, for diabolic squares of order seven?

*Solution by the Proposer.* We first obtain a form of the general diabolic square more symmetric than the one given by Schell (this MONTHLY [1946, 395]). Add together the elements of the row, column, and diagonals passing through $a_{11}$, add to this the similar sums for $a_{44}$, and subtract the similar sums for $a_{23}$ and $a_{32}$. The result reduces to

$$5(a_{11} + a_{44} - a_{23} - a_{32}) = 0,$$

or

$$a_{11} - a_{23} = a_{32} - a_{44}.$$

Since diabolicity is preserved under cyclic permutations of rows and columns we

can say in general that

$$a_{ij} - a_{i+1,j+2} = a_{i+2,j+1} - a_{i+3,j+3}, \tag{1}$$

where we define

$$a_{i+5n,j+5m} = a_{i,j}.$$

To get the general square we assign general values to the elements $a_{11}$, $a_{23}$, $a_{35}$, $a_{42}$, $a_{54}$, $a_{32}$, $a_{53}$, $a_{24}$, $a_{45}$, writing these values in the forms $a+\alpha$, $a+\beta$, $a+\gamma$, $a+\delta$, $a+\epsilon$, $b+\alpha$, $c+\alpha$, $d+\alpha$, $e+\alpha$, respectively. By (1) the rest of the elements are then determined, and the general square is

$$\begin{array}{ccccc} a+\alpha & d+\epsilon & b+\delta & e+\gamma & c+\beta \\ e+\delta & c+\gamma & a+\beta & d+\alpha & b+\epsilon \\ d+\beta & b+\alpha & e+\epsilon & c+\delta & a+\gamma \\ c+\epsilon & a+\delta & d+\gamma & b+\beta & e+\alpha \\ b+\gamma & e+\beta & c+\alpha & a+\epsilon & d+\delta \end{array}$$

Then

$$\begin{aligned} \Sigma_i(\Pi_j a_{ij}) = {} & 5abcde + 5\alpha\beta\gamma\delta\epsilon \\ & + \text{all possible terms (25 of them) of the type } wxyz\xi \\ & + \text{all possible terms of the type } w\xi\eta\zeta\phi \\ & + 50 \text{ terms of the type } wxy\xi\eta \\ & + 50 \text{ terms of the type } wx\xi\eta\zeta. \end{aligned} \tag{2}$$

Since

$$\Sigma_j(\Pi_i a_{ij}) \tag{3}$$

has a similar expansion we need worry only about terms of the last two types. The ten terms of the type $wxy\xi\eta$ obtained from the first row are found among the terms of the expansion of (3). Since the square preserves its form under cyclic permutation of the rows the same will be true of each of the rows. Since the terms of the type $wx\xi\eta\zeta$ can be treated similarly the required equality is proved.

This relationship between the rows and columns implies a similar relationship between the two sets of diagonals, since it is known* that a diabolic square of odd order can be rearranged into another diabolic square whose rows and columns are the diagonals of the original square and vice versa.

As regards squares of other orders, the general magic square of order three has the relationship between its rows and columns but not between its diagonals.

---

* See Barkley Rosser and R. J. Walker, *The algebraic theory of diabolic magic squares*, Duke Mathematical Journal, 5 (1939), Theorem 3.3.

The square

$$\begin{array}{cccc} a+\alpha & b+\gamma & c+\alpha & d+\gamma \\ c+\beta & d+\delta & a+\beta & d+\delta \\ a+\gamma & b+\alpha & c+\gamma & d+\alpha \\ c+\delta & d+\beta & a+\delta & d+\beta \end{array}$$

which becomes the general diabolic square on imposing the further relations

$$a+c=b+d, \qquad \alpha+\gamma=\beta+\delta,$$

has the relation between its diagonals. That the relation does not hold for the rows and columns of a diabolic square of order four is shown by a numerical example.

In the magic square of order five

| | | | | |
|---|---|---|---|---|
| 17 | 24 | 1 | 8 | 15 |
| 23 | 5 | 7 | 14 | 16 |
| 4 | 6 | 13 | 20 | 22 |
| 10 | 12 | 19 | 21 | 3 |
| 11 | 18 | 25 | 2 | 9 |

the relation does not hold for either rows and columns or diagonals. This is most easily seen by considering only the last digits. The product of each row and column is divisible by ten with the one exception of the third column. A similar situation holds for the diagonals.

In the same way it is seen that the relation does not hold for either rows and columns or diagonals of the diabolic square

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 47 | 38 | 35 | 24 | 20 | 9 |
| 26 | 16 | 8 | 6 | 46 | 42 | 31 |
| 49 | 39 | 33 | 23 | 15 | 12 | 4 |
| 19 | 11 | 7 | 45 | 41 | 30 | 22 |
| 37 | 29 | 27 | 17 | 14 | 3 | 48 |
| 10 | 5 | 44 | 36 | 34 | 25 | 21 |
| 32 | 28 | 18 | 13 | 1 | 43 | 40 |

There exists, however, a large class of diabolic squares of order seven call "regular" squares,* which have structures similar to that of the diabolic square of order five. There are six types of these, and it is not difficult to show that in three of these types the relation holds for the rows and columns but not for the diagonals, whereas the opposite is true for the other three.

Beyond this point things get complicated, but it seems likely that no rela-

* *Loc. cit.*, p. 709.

tions holds for magic squares of orders greater than three or for diabolic squares, even when regular, of orders greater than five.

Also solved (partially) by D. H. Browne.

### Wheatstone Bridges

E 765 [1947, 164]. *Proposed by Harold Becker, Omaha, Nebraska*

(a) What is the joint resistance of the general Wheatstone bridge as a function of the five component resistors? (b) What are forms for the components so that they and the resultant will all be integers?

*Solution by the Proposer.* (a) There are at least five methods of solution, employing: (A) Kirchhoff's equations, (B) Maxwell's mesh-current determinant, (C) Kennelly's star-delta transformation, (D) Kennelly's delta-star transformation, (E) series-parallel synthesis. These are in the chronological order of historical appearance, and also in the order of increasing simplicity. We here offer (D) and (E).

(D) Both the topology (drawing) and joint resistance of the circuit may be represented by the expression

$$R = (a + A) \parallel (b + B) \perp \beta,$$

where $+$, $\parallel$, and $\perp$ denote the series, parallel, and bridge connections respectively, with $a$ and $b$ at one line terminal, $A$ and $B$ at the other.

Then $A$, $B$, and $\beta$ form a delta, which may be replaced by an equivalent star:

$$(1) \qquad A' = A\beta/(A + B + \beta), \quad B' = B\beta/(A + B + \beta), \quad \beta' = AB/(A + B + \beta).$$

(See, *e.g.*, pp. 55–60, *A.C. Bridge Methods*, B. Hague, London, 1938.) The circuit thus transforms to

$$R = (a + A') \parallel (b + B') + \beta' = \beta' + (a + A')(b + B')/(a + A' + b + B').$$

Eliminating $A'$, $B'$, $\beta'$ by means of relations (1) we find, after simplifying,

$$(2) \qquad \begin{aligned} R = [\beta(a + A)(b + B) + ab(A + B) + AB(a + b)]/[\beta(a + A + b + B) \\ + (a + b)(A + B)] = D/N. \end{aligned}$$

(E) If we short the bridger, $\beta = 0$, $R$ becomes

$$\begin{aligned} R_0 &= a \parallel b + A \parallel B = ab/(a + b) + AB/(A + B) \\ &= [ab(A + B) + AB(a + b)]/(a + b)(A + B) = D_0/N_0. \end{aligned}$$

If we open the bridger, $\beta = \infty$, $R$ becomes

$$R_\infty = (a + A) \parallel (b + B) = (a + A)(b + B)/(a + A + b + B) = D_\infty/N_\infty.$$

Multiplying the latter by $\beta/\beta$, and synthesizing with $R_0$, we have, agreeing with (2),

$$(3) \qquad R = (\beta D_\infty + D_0)/(\beta N_\infty + N_0).$$

This expresses the desired joint resistance of $\beta=0$ or $\infty$, obviously; and for all intermediate values of $\beta$, by combinatory considerations. The form of (3) may be generalized to $2^n$ addends, for solving an extensive class of higher bridge circuits involving $n$ bridgers.

Practically, it is perhaps more important to know the individual branch currents rather than their sum. Each branch transfer conductance (b.t.c.) is readily found by breaking down the numerator of the joint conductance $G=1/R=N/D$ and remembering that in the numerator of a b.t.c. the branch resistance value does not appear. That is, $G=G_a+G_b=G_A+G_B$, $G_\beta=G_B-G_b=G_A-G_a$, and

$$(4)\qquad \begin{aligned} G_a &= [\beta(b+B)+b(A+B)]/D, & G_A &= [\beta(b+B)+B(a+b)]/D,\\ G_b &= [\beta(a+A)+a(A+B)]/D, & G_B &= [\beta(a+A)+A(a+b)]/D, \end{aligned}$$

$$(5)\qquad G_\beta = (aB-Ab)/D.$$

Of course, if $E$ is the voltage across the circuit, the current through branch $A$ is $EG_A$, *etc.* When the bridge is in balance (that is, $A=ka$, $B=kb$), (2) and (5) reduce to $R=k(a+b)/(k+1)$ and $G_\beta=0$.

Incidentally, when the circuit is studied as a laboratory experiment and put near or at balance, the value of $\beta$ makes little or no difference in the value of $R$. On the other hand, in the case of extreme unbalance ($a=B=\infty$, $A=b=0$) the value of $\beta$ makes all the difference. For given values of the bridge arms, the difference in $R$ as $\beta$ is varied from $\infty$ to 0 is given by

$$(6)\qquad R_\infty - R_0 = (aB-Ab)^2/(a+A+b+B)(a+b)(A+B),$$

and thus is a function of the balance discriminant.

(b) This seems to be a difficult problem. As some simple examples we can offer

$$(5+5)\parallel(25+5)\perp 5 = 7 = (5+10)\parallel(10+5)\perp 5,$$
$$(7+14)\parallel(14+7)\perp 14 = 10 = (9+15)\parallel(12+6)\perp 21.$$

*Editorial Note.* For an indication of the proposer's *circuit algebra* see abstract 377, p. 1010, Nov. 1946, *Bulletin of the American Mathematical Society.*

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4275. *Proposed by Raymond Redheffer, Massachusetts Institute of Technology, Cambridge.*

Let $a_i$ represent any set of points in the complex plane, while $b_i$ are any complex numbers. Prove there exists an integral function $f(z)$ such that $f(a_i)=b_i$.

4276. *Proposed by P. A. Pizá, San Juan, P. R.*

Let the integers ${}_nK_c$ be defined by the relations

$$ {}_nK_1 = 1; \qquad {}_nK_m = 0, \quad m > n; \qquad {}_{n+1}K_c = c({}_nK_c + {}_nK_{c-1}), \quad c > 1. $$

Prove the following summations:

$$ x^n = \sum_{j=1}^{n} {}_nK_j\binom{x}{j}, \qquad \sum_{a=1}^{x-1} a^n = \sum_{j=1}^{n} {}_nK_j\binom{x}{j+1}. $$

### SOLUTIONS

#### A Definite Integral

4212 [1946, 397]. *Proposed by H. F. Sandham,* Trinity College

Evaluate

$$ \int_0^\infty \frac{e^{-x^2}dx}{(x^2+1/2)^2}. $$

I. *Solution by A. B. Farnell, University of Colorado.* Applying the formula for integration by parts twice, we find

$$ \begin{aligned} \int \frac{e^{-x^2}dx}{(x^2+1/2)^2} &= \frac{e^{-x^2}}{2x}\cdot\frac{-1}{x^2+1/2} - \int \frac{e^{-x^2}dx}{x^2} \\ &= \frac{-e^{-x^2}}{2x(x^2+1/2)} + \frac{e^{-x^2}}{x} + 2\int e^{-x^2}dx \\ &= \frac{xe^{-x^2}}{x^2+1/2} + 2\int e^{-x^2}dx. \end{aligned} $$

The value of the definite integral corresponding to the last term is well known, and thus we have

$$\int_0^\infty \frac{e^{-x^2}dx}{(x^2+1/2)^2} = \sqrt{\pi}.$$

II. *Solution by J. B. Rosser, Cornell University.* The proposed formula is only one of several similar formulas which can be evaluated by starting from the relation

$$\frac{\sqrt{\pi}}{2} - \int_0^z e^{-x^2}dx = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-z^2(1+x^2)}dx}{1+x^2}. \tag{1}$$

When $z=0$, (1) is easily verified, and by aid of the standard integral formula

$$\int_0^\infty e^{-z^2x^2}dx = \frac{\sqrt{\pi}}{2z},$$

we verify that both sides of (1) have the same derivative with respect to $z$. A variant of (1), obtained by putting $y=zx$, is

$$e^{z^2}\left(\frac{\sqrt{\pi}}{2} - \int_0^z e^{-x^2}dx\right) = \frac{z}{\sqrt{\pi}}\int_0^\infty \frac{e^{-y^2}dy}{y^2+z^2}. \tag{2}$$

See also Bromwich, *Theory of Infinite Series*, First Edition, 1908, p. 352, Ex. 15. Suppose we denote the left side of (2) by $F(z)$. One easily verifies

$$\frac{d}{dz}F(z) = 2zF(z) - 1, \qquad \frac{d}{dz}\frac{F(z)}{z} = \frac{(2z^2-1)F(z) - z}{z^2}.$$

So by (2),

$$\frac{2z}{\sqrt{\pi}}\int_0^\infty \frac{e^{-y^2}dy}{(y^2+z^2)^2} = -\frac{d}{dz}\frac{F(z)}{z} = \frac{z-(2z^2-1)F(z)}{z^2}.$$

Hence

$$\frac{z}{\sqrt{\pi}}\int_0^\infty \frac{2z^2e^{-y^2}dy}{(y^2+z^2)^2} = z - (2z^2-1)F(z) = z - (2z^2-1)\frac{z}{\sqrt{\pi}}\int_0^\infty \frac{e^{-y^2}dy}{y^2+z^2}$$

by (2). So

$$\int_0^\infty \frac{2z^2e^{-y^2}dy}{(y^2+z^2)^2} = \sqrt{\pi} - (2z^2-1)\int_0^\infty \frac{(y^2+z^2)e^{-y^2}dy}{(y^2+z^2)^2}.$$

Transposing gives

$$\int_0^\infty \frac{[(2z^2-1)y^2+(2z^2+1)z^2]e^{-y^2}dy}{(y^2+z^2)^2} = \sqrt{\pi}.$$

Put $z^2=w$.

$$\int_0^\infty \frac{[(2w-1)y^2+(2w^2+w)]e^{-y^2}dy}{(y^2+w)^2}=\sqrt{\pi}. \tag{3}$$

Sandham's result is the special case where $w=1/2$.

We may again differentiate (3) with respect to $w$. There results

$$\int_0^\infty \frac{[2y^4+(2w+3)y^2-w]e^{-y^2}dy}{(y^2+w)^3}=0. \tag{4}$$

But by (3)

$$\int_0^\infty \frac{[(2w-1)y^4+4w^2y^2+2w^3+w^2]e^{-y^2}dy}{(y^2+w)^3}=\sqrt{\pi}. \tag{5}$$

So

$$\int_0^\infty \frac{[(4w^2-4w+3)y^2+(4w^3+4w^2-w)]e^{-y^2}dy}{(y^2+w)^3}=2\sqrt{\pi}. \tag{6}$$

Putting $w=\frac{1}{2}+i\sqrt{\frac{1}{2}}$ or $w=(\frac{1}{2}\sqrt{2}-1)$, for example, in (6) gives

$$\int_0^\infty \frac{e^{-y^2}dy}{(y^2+\frac{1}{2}+i\sqrt{\frac{1}{2}})^3}=\frac{-\sqrt{\pi}(1+i\sqrt{\frac{1}{2}})}{3}.$$

or

$$\int_0^\infty \frac{y^2e^{-y^2}dy}{[y^2+\frac{1}{2}(\sqrt{2}-1)]^3}=\frac{\sqrt{\pi}(2+\sqrt{2})}{4}.$$

Further, one can differentiate (6) and combine with (6) to get a relation in which $(y^2+w)^4$ occurs in the denominator under the integral sign.

Also solved by H. E. Fettis, N. J. Fine, Daniel Finkel, J. B. Kelly, D. J. Newman, A. S. Peters, M. R. Spiegel, M. Wyman, and the Proposer.

### Tetrahedron, Spheres, Sums of Powers

4213 [1946, p. 397]. *Proposed by V. Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and an arbitrarily chosen point $M$: (1) The sum of the powers of the vertices, respectively, with respect to three spheres on $(MB, MC, MD)$, $(MC, MD, MA)$, $(MD, MA, MB)$, $(MA, MB, MC)$, as diameters, is equal to the sum of the squares of the edges. (2) Construct the point $M$ when the sums of the powers of the vertices $A$, $B$, $C$, $D$ relative to the corresponding set of three spheres, are proportional to given numbers $\alpha$, $\beta$, $\gamma$, $\delta$. Consider the case where these last four numbers are equal.

*Solution by R. Goormaghtigh, Bruges, Belgium.* (1) The power of $A$ as to the sphere having $MB$ as diameter and $B'$ as center is

$$\overline{AB'}^2 - \tfrac{1}{4}\overline{MB}^2 = \tfrac{1}{2}(\overline{MA}^2 - \overline{MB}^2 + \overline{AB}^2).$$

The sum $s_a$ of the powers of $A$ with respect to the spheres having $MB$, $MC$, $MD$ as diameters is therefore

$$s_a = \tfrac{1}{2}(3\overline{MA}^2 - \overline{MB}^2 - \overline{MC}^2 - \overline{MD}^2 + \overline{AB}^2 + \overline{AC}^2 + \overline{AD}^2)$$

and the sum of $s_a$ and the similar ones is equal to the sum of the squares of the edges.

(2) If $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ are the expressions

$$\alpha/(\alpha + \beta + \gamma + \delta), \cdots,$$

and if $G_a$ is the centroid of $BCD$, we have

$$\begin{aligned}\overline{MB}^2 + \overline{MC}^2 + \overline{MD}^2 &= 3\overline{MG_a}^2 + \overline{BG_a}^2 + \overline{CG_a}^2 + \overline{DG_a}^2\\ &= 3\overline{MG_a}^2 + (\overline{BC}^2 + \overline{CD}^2 + \overline{DB}^2)/3\end{aligned}$$

and

$$\begin{aligned}\overline{MA}^2 - \overline{MG_a}^2 &= \tfrac{1}{3}(2\alpha' - 1)(\overline{AB}^2 + \overline{AC}^2 + \overline{AD}^2)\\ &\quad + \tfrac{1}{3}(2\alpha' + \tfrac{1}{3})(\overline{BC}^2 + \overline{CD}^2 + \overline{DB}^2) \equiv \sigma_a.\end{aligned}$$

Hence, $M$ belongs to the plane perpendicular to the median $AG_a$, at the distance $\sigma_a/2AG_a$ from its midpoint and $M$ is at the intersection of that plane and the three similar ones, corresponding to $B$, $C$, $D$.

When $\alpha=\beta=\gamma=\delta$, another construction is derived from the fact that, in that case, $J$ being the midpoint of $CD$,

$$2(\overline{MA}^2 - \overline{MB}^2) = \overline{JB}^2 - \overline{JA}^2.$$

The required point $M$ belongs therefore to the plane perpendicular to $AB$ and passing through the point $N$ such that, if $J'$ is the projection of $J$ on $AB$ and $K$ the midpoint of $AB$, $KN$ is half $J'K$; hence, in that case, *$M$ is the image of the centroid in the center of the circumsphere.*

*Editorial Note.* See 4064, [1944, 168] for a similar problem solved by vector algebra.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### NEW MEMBERS

Professor W. B. Carver, Secretary-Treasurer, announces that the following one hundred and twenty-six persons have been elected to membership on applications duly certified:

R. G. ALBERT, B.S.(Brooklyn College) Instr., Sampson College, Samspon, N. Y.

T. R. BACHILLER, Dr.Sc.(Univ. of Madrid) Catedratico de la Fac. de Ciencas de la Universidad de Madrid, Madrid, Spain

J. E. BEARMAN, M.A.(Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn.

LOUIS BERKOFSKY, A.B.(Brooklyn) Instr., Brooklyn Evening Coll., Brooklyn, N. Y.

SISTER ADA M. BOEHM, B.A.(Coll. of St. Catherine) Teacher, Coll. of St. Catherine, St. Paul 1, Minn.

F. R. BROWN, M.A.(Columbia) Asst. Prof., James Millikin Univ., Decatur, Ill.

H. E. BROWN, A.B.(Univ. of Maine) Teacher, Morse High School, Bath, Me.

J. W. BROWN, B.S.(North Carolina State) Instr., Clemson Coll., Clemson, S. C.

M. D. BROWN, M.A.(Geo. Peabody Coll.) Asst. Prof., Macalester Coll., St. Paul 5, Minn.

K. A. BUSH, M.A.(Columbia) Asso. Prof., Mohawk Coll., Utica, N. Y.

MARGARET C. BYRNE, A.M.(Columbia) Teacher, St. Joseph's Coll. for Women, Brooklyn, N. Y.

J. W. CALKIN, Ph.D.(Harvard) Asso. Prof., Rice Institute, Houston, Tex.

E. L. CANFIELD, A.M.(Northwestern) Instr., Drake Univ., Des Moines, Iowa

R. E. CARR, Ph.D.(Iowa State) Instr., Iowa State Coll., Ames, Iowa

ELLA F. CASEY (Mrs. N. E.), M.Ed.(Univ. of Rochester) Instr. Sampson Coll., Sampson, N. Y.

C. W. CASSEL, Jr., A.B.(Wittenburg Coll.) Instr. Univ. of Dayton, Dayton, Ohio

L. L. CAULUM, M.A.(Columbia) Prof., Sampson Coll., Sampson, N. Y.

VIRGINIA L. CHATELAIN, B.S.Ed. (Kansas State Teachers Coll.) Grad. Student, State Teachers Coll., Emporia, Kan.

F. E. CLARK, A.M.(Duke) Grad. Student, Duke Univ., Durham, N. C.

CHARLES CLOS, C.E.(N. Y. Univ.) Instr.' Pratt Inst., Brooklyn, N. Y.

R. N. COBB, A.M.(Harvard) Asst. Prof., Worcester Poly. Inst., Worcester, Mass.

V. L. COLLINS, A.M.(Alabama) Head of Dept., State Teachers Coll., Troy, Ala.

FRANCO CROCI, 3992 White Plains Ave., New York, N. Y.

DON CUDE, A.M.(S.W. Texas State Teachers Coll.) Prof., S.W. Texas State Teachers Coll., San Marcos, Tex.

VICTOR DANA, Bachelier es-Mathematiques (Le Claire) Student, Fuad Ist Univ., Cairo, Egypt

ROBERT DAVIES, Ph.M.(Wisconsin) Grad. Student, Univ. of Wisconsin, Madison, Wis.

ROLAND DEAUX, D.Sc.(Univ. of Gand) Prof., Faculte Polytechnique, Mons, Belgium.

M. A. DERNHAM, LL.B.(California) Attorney-at-Law, 405 Montgomery St., San Francisco, Calif.

J. G. DEUTSCH, A.M.(Columbia) Chm. of Dept., Andrew Jackson High School, Queens, N. Y.

G. M. DILLON, B.A.(Long Island Univ.) Student, Columbia Univ., New York, N. Y.

J. W. DOBBINS, Student, St. Mary's Univ., San Antonio, Tex.

LUCIEN DROUSSENT, Ingenieur(Ecole Centrale des Arts et Manufactures), 36 Rue de la Cartoucherie, Clermont-Ferrand, Puy-de-Dome, France

HAZEL E. EGGETT, A.M.(California) Teacher, Union High School, Ukiah, Calif.

R. L. ERICKSON, M.S.(Wisconsin) Instr. Univ. of Wisconsin, Madison, Wis.

I. K. FEINSTEIN, M.A.(Northwestern) Asst. Prof., Univ. of Illinois, Evanston, Ill.

R. B. FOLSOM, A.M.(Columbia) Asst. Prof., The Citadel, Charleston, S. C.

J. W. FORMAN, B.A.(Omaha) Grad. Asst., Univ. of Iowa, Iowa City, Iowa

EVELYN FRANK, Ph.D.(Northwestern) Asst. Prof. Univ. of Illinois, Evanston, Ill.
JACK FRIERSON, M.S.(South Carolina) Assoc. Prof., Coker Coll., Hartsville, S. C.
L. E. FULLER, M.S.(Wisconsin) Grad. Asst. Univ. of Wisconsin, Madison, Wis.
LANDIS GEPHART, B.S.(Dayton) Instr., Univ. of Dayton, Dayton, Ohio
K. S. GHENT, Ph.D.(Chicago) Asso. Prof., Univ. of Oregon, Eugene, Ore.
W. M. GILBERT, B.A.(Oregon) Student, University of Oregon, Eugene, Ore.
HAROLD GLANDER, B.S.(Milwaukee State Teachers Coll.) Student, Marquette Univ. Grad. School, Milwaukee, Wis.
ABRAHAM GOLUB, A.B. (Brooklyn) Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Aberdeen, Md.
E. G. GOMAN, B.S.(Oregon State Coll.) Grad. Asst., Oregon State Coll., Corvallis, Ore.
S. T. GORMSEN, B.S.(Ohio State) Instr., Univ. of Florida, Gainesville, Fla.
G. E. GOURRICH, B.S.(California) Teaching Asst., Univ. of California at Los Angeles, Los Angeles, Calif.
W. W. GRAHAM, Ph.D.(Geo. Peabody Coll.) Asso. Prof., Vanderbilt Univ., Nashville, Tenn.
B. F. HADNOT, Student, Univ. of Alabama, University, Ala.
RALPH HAFNER, M.A.(Michigan) Instr., Univ. of Dayton, Dayton, Ohio
W. B. HAMMER, A.M.(Iowa) Dean, Junior Coll., Estherville, Iowa
H. M. HARDY, B.S.(Sam Houston State Teachers Coll.) Grad. Asst., Sam Houston State Teachers Coll., Huntsville, Tex.
ERIK HEMMINGSEN, Ph.D.(Pennsylvania) Asst. Prof., Univ. of Georgia, Athens, Ga.
J. S. HILL, (Asso. Actuarial Society of Amer.) Minnesota Mutual Life Ins. Co., St. Paul, Minn.
JANICE M. HOFFMAN, B.S.Ed.(Bowling Green State Univ.) Teacher, High School, Delray Beach, Fla.
I. M. HOSTETTER, Ph.D.(Washington) Asso. Prof., Oregon State Coll., Corvallis, Ore.
C. A. HUCK, M.A.(Geo. Peabody Coll.) Peru State Teachers Coll., Peru, Nebr.
S. J. JASPER, M.A.(Ohio State) Instr., Univ. of Kentucky, Lexington, Ky.
O. B. JOHNSON, Jr., B.S.(Chicago Tech. Coll.) Student, Roosevelt Coll., Chicago, Ill.
L. E. LAIRD, A.B.(Kansas State Teachers Coll.) Grad. Fellow, State Teachers Coll., Emporia, Kan.
R. J. LAMBERT, A.B.(Drake) Instr., Iowa State Coll., Ames, Iowa
R. J. LEVIT, Ph.D.(California) Asst. Prof., Univ. of Georgia, Athens, Ga.
R. L. MADOR, A.M.(Trinity Coll.) Asso. Instr., U. S. Merchant Marine Cadet School, San Mateo, Calif.
J. M. MAGUIRE, Supervisor, Tractor Eng. Dept., Ford Motor Co., Dearborn, Mich.
E. W. MARCHAND, M.S.(Washington) Instr., Univ. of Rochester, Rochester, N. Y.
A. M. MARK, Ph.D.(Cornell) Instr., Univ. of Wisconsin, Madison, Wis.
REV. NORBERT MARTIN, B.A.(Catholic Univ.) Instr., St. Francis Prep. School, Spring Grove, Pa.
P. E. MARTIN, Ph.D.(Michigan) Prof., Wheaton Coll., Wheaton, Ill.
SISTER MARY CANISIA, B.S.(De Paul Univ.) Teacher, Holy Family Academy, Chicago, Ill.
C. W. MATHEWS, Ph.D.(Illinois) Instr., Washington Univ., St. Louis, Mo.
G. H. MAY, A.M.(South Carolina) Asst. Prof., Wofford Coll., Spartanburg, S. C.
J. W. MCCLIMANS, Ph.D.(Peabody) Prof., Southeastern Louisiana Coll., Hammond, La.
R. D. MCKNELLY, M.A.(Oklahoma) Instr. Oklahoma Univ., Norman, Okla.
W. H. MCKENZIE, M.A.(California) Instr., San Francisco Jr. Coll., San Francisco, Calif.
HARLAN C. MILLER, Ph.D.(Texas) Asso. Prof., Texas State Coll. for Women, Denton, Tex.
A. G. MONTGOMERY, M.A.(Minnesota) Instr., Coll. of St. Thomas, St. Paul, Minn.
MABEL D. MONTGOMERY, B.A.(Houghton Coll.) Instr., Univ. of Buffalo, Buffalo, N. Y.
LEO MOSER, M.A.(Toronto) Lecturer, Univ. of Manitoba, Winnipeg, Manitoba, Can.
S. S. MYERS, M.Ed.(Cincinnati) Teacher, Western State High School, Kalamazoo, Mich.
D. S. NATHAN, Ph.D.(Cincinnati) Instr., Coll. of the City of New York, New York, N. Y.

VIVIAN NEMECEK, M.A.(Oklahoma) Teacher, Oklahoma Military Academy, Claremore, Okla.

M. L. NORDEN, B.S.(Massachusetts Institute of Technology) Statistician, Aberdeen Proving Ground, Md.

H. J. OSNER, M.A.(California) Teaching Asst., Univ. of California, Berkeley, Calif.

A. J. OWENS, M.S.(Iowa) Grad. Student, Univ. of Florida, Gainesville, Fla.

ARTHUR PANCOE, B.S.(Wisconsin) Vice-Pres., Standard Stationary Co., Wilmette, Ill.

EUGENE PARK, A.M.(Lehigh) Instr., Univ. of Georgia, Athens, Ga.

A. C. PATTON, M.A.(Yale) Asso. Prof., Clark Univ., Worcester, Mass.

MARY H. PAYNE (Mrs. W. T.), Ph.D.(Brown) Instr., Univ. of Detroit, Detroit, Mich.

FRANCISCO PEREZ, Ph.D.(Chicago) Teacher, Univ. of the Philippines, Manila, P. I.

MARY A. PETERS, M.S.(Iowa) Chm. of Dept., High School, Elgin, Ill.

COSTAS PLETHIDES, degree from Univ. of Athens, Greece, 2705 Marshall Ave., Newport News, Va.

R. B. PLYMALE, M.A.(Columbia) Associate, Mercer Univ., Macon, Ga.

HARRY POLLARD, Ph.D.(Harvard) Asst. Prof., Cornell Univ., Ithaca, N. Y.

S. W. PROCTOR, S.M.(Chicago) Instr., Vashon High School, St. Louis, Mo.

O. M. RASMUSSEN, M.S.(Kansas State Teachers Coll.) Instr., Univ. of Kansas, Lawrence, Kan.

FLORENCE V. ROHDE, M.A.(Miami) Instr., Univ. of Kentucky, Lexington, Ky.

BEATRICE A. ROHR, A.M.(Columbia) Instr., Hofstra Coll., Hempstead, Long Island, N. Y.

W. E. ROTH, Ph.D.(Wisconsin) Sampson Coll., Sampson, N. Y.

HAZEL M. ROTHLISBERGER, B.A.(Iowa State Teachers Coll.) Asso. Prof., Univ. of Dubuque, Dubuque, Iowa

SALLY I. ROTHSTEIN, A.B.(Hunter) Asst., Univ. of Wisconsin, Madison, Wis.

ERWIN SCHMID, B.S.(Haverford) Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.

EDITH R. SCHNECKENBURGER, Ph.D.(Michigan) Asst. Prof., Univ. of Buffalo, Buffalo, N. Y.

AUGUSTA L. SCHURRER, A.M.(Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison, Wis.

H. W. SCHWARTZ, Sc.M.(New York Univ.) Meteorology Inst., A.A.F. Tech. Training Command, Paxton, Ill.

F. C. SEVIER, M.S. in Ed.(Pennsylvania) Instr., Penn State College Center, Swarthmore, Pa.

DAVID SINGER, B.C.E.(New York Univ.) Tutor, The City Coll. School of Tech., New York, N. Y.

C. B. SOLLOWAY, Student, Indiana State Teachers Coll. Terre Haute, Ind.

R. D. SPECHT, Ph.D.(Wisconsin) Asst. Prof., Univ. of Wisconsin, Madison, Wis.

A. H. SPELTZ, B.A.(St. Mary's Coll.) Instr., Coll. of St. Thomas, St. Paul, Minn.

Capt. J. P. SPICKELMIER, c/o The Military Attache, Amer. Embassy, London, England

Rev. R. J. SWORDS, M.A.(Harvard) Instr., Weston Coll., Weston, Mass.

A. H. TAUB, Ph.D.(Princeton) Prof., Univ. of Washington, Seattle, Wash.

R. A. TAUSSIG, Lt. (j.g.), U.S.M.S., M.B.A. (California) Asso. Instr., U. S. Merchant Marine Academy, San Mateo, Calif.

H. E. TAYLOR, M.S.(California Inst. of Tech.) Asst., Rice Institute, Houston, Tex.

BRYANT TUCKERMAN, Ph.D.(Princeton) Instr., Cornell Univ., Ithaca, N. Y.

P. M. TULLIER, Jr., M.S.(Louisiana State) Instr., Loyola Univ., New Orleans, La.

FRANCES E. WALSH, B.S.(Mt. St. Scholastica Coll.) Student, St. Louis Univ., St. Louis, Mo.

CHIH-YI WANG, B.S.(Yenching) Grad. Student, Univ. of Minnesota, Minneapolis, Minn.

BETTY R. WEBER, A.B.(Columbia) Instr., Univ. of South Carolina, Columbia, S. C.

T. J. WHITE, A.B.(Rice) Asst., Rice Inst., Houston, Tex.

G. N. WOLLAN, M.S.(Iowa) Asst. Prof., Sampson Coll., Sampson, N. Y.

MARIE A. WURSTER, Ph.D.(Chicago) Instr., Temple Univ., Philadelphia, Pa.

F. H. YOUNG, B.A.(Oregon State College) Instr., Oregon State Coll., Corvallis, Ore.

JEAN-PIERRE ZAHLER, Student, The Univ. of Nancy, France. (Director of the Mathematico-pedagogical periodical Pythagore,

member of the Societe Mathematique de France).

R. K. ZEIGLER, M.S.(Nevada) Grad. Asst., State Univ. of Iowa, Iowa City, Iowa

## THE TWENTY-NINTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-ninth summer meeting of The Mathematical Association of America was held at Yale University, New Haven, Connecticut, on Monday and Tuesday, September 1–2, 1947, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. About seven hundred and thirty persons were in attendance at the meetings, including the following two hundred and ninety-seven members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, Cornell University
C. B. ALLENDOERFER, Haverford College
R. C. ARCHIBALD, Brown University
H. A. ARNOLD, New York City
H. E. ARNOLD, Wesleyan University
L. A. AROIAN, Hunter College
H. T. R. AUDE, Colgate University
SILVIO AURORA, Columbia University
H. G. AYRE, Western Illinois State College
FRANK AYRES, Dickinson College
W. L. AYRES, Purdue University

H. M. BACON, Stanford University
FRANCES E. BAKER, Vassar College
D. H. BALLOU, Middlebury College
JOSHUA BARLAZ, Rutgers University
W. D. BATEN, Brooklyn College
HELEN P. BEARD, Newcomb College
H. M. BEATTY, Ohio State University
E. G. BEGLE, Yale University
H. A. BENDER, Rhode Island State College
A. A. BENNETT, Brown University
GARRETT BIRKHOFF, Harvard University
A. H. BLACK, Lebanon Valley College
C. R. BLYTH, Queens University
R. P. BOAS, JR., Brown University
JULIA W. BOWER, Connecticut College
J. G. BOWKER, Middlebury College
A. T. BRAUER, University of North Carolina
RICHARD BRAUER, University of Toronto
J. R. BRITTON, University of Colorado
FOSTER BROOKS, Kent State University
R. C. BUCK, Brown University
L. H. BUNYAN, Rutgers University
SISTER LEONARDA BURKE, Regis College, Massachusetts
F. J. H. BURKETT, Union College
J. HOBART BUSHEY, Hunter College
JEWELL H. BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota

S. S. CAIRNS, Syracuse University
B. H. CAMP, Wesleyan University
R. E. CARR, Michigan State College
W. B. CARVER, Cornell University
W. B. CATON, University of Maine
W. F. CHENEY, JR., University of Connecticut
M. I. CHERNOFSKY, Junior College of Commerce, Connecticut
D. E. CHRISTIE, Bowdoin College
HELEN CLARKSON, Creighton University
A. B. COBLE, University of Illinois
A. C. COHEN, JR., University of Georgia
NANCY COLE, Syracuse University
R. H. COLE, University of Western Ontrario
E. P. COLEMAN, Aberdeen Proving Ground, Maryland
T. F. COPE, Queens College
LENNIE P. COPELAND, Wellesley College
W. H. H. COWLES, Pratt Institute
H. S. M. COXETER, University of Toronto
C. C. CRAIG, University of Michigan
E. L. CROW, U. S. Naval Ordnance Test Station, Calif.
H. B. CURRY, Pennsylvania State College

MARGUERITE D. DARKOW, Hunter College
D. R. DAVIS, New Jersey State Teachers College
F. F. DECKER, Syracuse University
C. E. DIMICK, Gales Ferry, Connecticut
BERNARD DIMSDALE, Aberdeen Proving Ground, Maryland
L. L. DINES, Carnegie Institute of Technology
MARY P. DOLCIANI, Cornell University
H. L. DORWART, Washington and Jefferson College

ARNOLD DRESDEN, Swarthmore College
NELSON DUNFORD, Yale University
W. L. DUREN, JR., Tulane University
W. H. DURFEE, Dartmouth University

SAMUEL EILENBERG, Columbia University
H. P. EVANS, University of Wisconsin

WILLIAM FELLER, Cornell University
F. A. FICKEN, University of Tennessee
J. N. FIELD, Queens College
N. J. FINE, University of Pennsylvania
W. W. FLEXNER, Cornell University
L. R. FORD, Illinois Institute of Technology
R. M. FOSTER, Polytechnic Institute of Brooklyn
A. H. FOX, Union College
J. S. FRAME, Michigan State College
ORRIN FRINK, JR., Pennsylvania State College
K. G. FULLER, Teachers College of Connecticut

H. M. GEHMAN, University of Buffalo
B. H. GERE, Hamilton College
B. P. GILL, College of the City of New York
R. E. GILMAN, Brown University
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
M. J. GOTTLIEB, Institute for Advanced Study, New Jersey
G. E. GOURRICH, University of California
F. G. GRAFF, Amherst College
EDISON GREER, Kansas State College
V. G. GROVE, Michigan State College
H. T. GUARD, Colorado A & M
THEODORE HAILPERIN, Lehigh University
D. W. HALL, University of Maryland
P. R. HALMOS, Institute for Advanced Study
O. H. HAMILTON, Oklahoma A. & M. College
R. W. HAMMING, Bell Telephone Laboratories
BERTHA I. HART, Aberdeen Proving Ground
KATHARINE E. HAZARD, New Jersey College for Women
G. A. HEDLUND, University of Virginia
M. H. HEINS, Brown University
R. G. HELSEL, Ohio State University
FRITZ HERZOG, Michigan State College
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
R. H. HOSKINS, John Hancock Mutual Life Insurance Co.
A. S. HOUSEHOLDER, Clinton Laboratory, Tennessee
W. A. HURWITZ, Cornell University
L. C. HUTCHINSON, Polytechnic Institute of Brooklyn

E. D. JENKINS, Kent State University
S. A. JENNINGS, University of British Columbia
R. H. JOHANSON, Boston University
ROBERTA F. JOHNSON, Wilson College
W. L. JOHNSON, Mississippi Southern College

AIDA KALISH, Queens College
IRVING KAPLANSKY, University of Chicago
L. M. KELLY, University of Missouri
A. J. KEMPNER, University of Colorado
EVELYN M. KENNEDY, Johns Hopkins University
E. C. KIEFER, Millikin University
W. J. KLIMCZAK, Yale University

W. D. LAMBERT, Lime Rock, Connecticut
R. E. LANGER, University of Wisconsin
LEO LAPIDUS, University of Maine
J. P. LASALLE, University of Notre Dame
H. L. LEE, University of Tennessee
MARGUERITE LEHR, Bryn Mawr College
WALTER LEIGHTON, Washington University
F. C. LEONE, Purdue University
W. J. LEVEQUE, Harvard University
D. C. LEWIS, University of Maryland
F. A. LEWIS, University of Alabama
S. B. LITTAUER, Columbia University
Z. L. LOFLIN, Southwestern Louisiana Institute
W. R. LONGLEY, Yale University
C. I. LUBIN, University of Cincinnati

L. A. MACCOLL, Bell Telephone Laboratories
H. F. MAC NEISH, University of Miami
JOHN MANDEL, National Bureau of Standards
MORRIS MARDEN, University of Wisconsin
M. H. MARTIN, University of Maryland
W. T. MARTIN, Massachusetts Institute of Technology
V. O. MCBRIEN, Holy Cross College
E. D. MCCARTHY, University of Detroit
R. B. MCCLENON, Grinnell College
J. W. MCCLIMANS, Southeastern Louisiana College
DOROTHY MCCOY, Belhaven College
N. H. MCCOY, Smith College
EDITH A. MCDOUGLE, University of Delaware
W. H. MCEWEN, University of Manitoba

A. E. MEDER, JR., Rutgers University
G. M. MERRIMAN, University of Cincinnati
J. S. MIKESH, The Lawrenceville School
JOSEPH MILKMAN, U. S. Naval Academy
E. J. MILES, Yale University
A. N. MILGRAM, Syracuse University
KNOX MILLSAPS, Ohio State University
H. J. MISER, Williams College
B. E. MITCHELL, Millsaps College
J. M. MITCHELL, Oklahoma A & M
E. B. MODE, BOSTON University
C. N. MOORE, University of Cincinnati
T. W. MOORE, U. S. Naval Academy
E. M. MORENUS, Cleveland, N. Y.
R. K. MORLEY, Worcester Polytechnic Institute
D. S. MORSE, Union College
F. C. MOSTELLER, Harvard University
W. R. MURRAY, Franklin & Marshall College
J. R. MUSSELMAN, Western Reserve University

A. L. NELSON, Wayne University
P. F. NEMENYI, Whiteoak, Maryland
C. V. NEWSOM, Oberlin College
M. L. NORDEN, Ballistic Research Laboratory
P. B. NORMAN, Pratt Institute
E. P. NORTHROP, University of Chicago
L. R. NORWOOD, Yale University

E. R. OTT, Rutgers University

A. C. PATTON, Clark University
MARY H. PAYNE, University of Detroit
D. K. PEASE, Brown University
P. M. PEPPER, University of Notre Dame
R. I. PEPPER, Winthrop College
H. R. PHALEN, William & Mary College
GEORGE PIRANIAN, University of Michigan
HARRY POLLARD, Cornell University
J. C. POLLEY, Wabash College
GEORGE POLYA, Stanford University
ADRIEN POULIOT, Laval University, Quebec
G. B. PRICE, University of Kansas
A. L. PUTNAM, University of Chicago

M. A. RADER, Moravian College & Theological Seminary
G. Y. RAINICH, University of Michigan
J. F. RANDOLPH, Oberlin College
ANATOL RAPAPORT, University of Chicago
RUTH B. RASMUSSEN, Chicago City Júnior College
C. H. RAWLINS, U. S. Naval Postgraduate School
G. E. RAYNOR, Lehigh University
A. W. RECHT, University of Denver
L. J. REED, Johns Hopkins University
C. J. REES, University of Delaware
MINA S. REES, Office of Naval Research
EDGAR REICH, Massachusetts Institute of Technology
W. T. REID, Northwestern University
B. P. REINSCH, Florida Southern College
C. N. REYNOLDS, West Virginia University
C. E. RHODES, Alfred University
R. G. D. RICHARDSON, Brown University
P. R. RIDER, Washington University
JOHN RIORDAN, Bell Telephone Laboratories
J. F. RITT, Columbia University
L. V. ROBINSON, University of South Carolina
JOSEPH ROSENBAUM, The Milford School
J. B. ROSSER, Cornell University
M. F. ROSSKOPF, John Burroughs School
W. E. ROTH, Sampson College
E. H. ROTHE, University of Michigan
LEILA R. RUBASHKIN, Cornell University

CHARLES SALTZER, Brown University
HANS SAMELSON, University of Michigan
ARTHUR SARD, Queens College
MAX SASULY, Robinson Foundation
E. D. SCHELL, Bureau of Labor Statistics
O. F. G. SCHILLING, University of Chicago
I. J. SCHOENBERG, University of Pennsylvania
K. C. SCHRAUT, University of Dayton
ABRAHAM SCHWARTZ, Pennsylvania State College
W. T. SCOTT, Northwestern University
C. E. SEALANDER, Ohio State University
L. W. SHERIDAN, College of St. Thomas
D. T. SIGLEY, Johns Hopkins University
M. M. SLOTNICK, Humble Oil & Refining Company
M. F. SMILEY, Northwestern University
C. V. L. SMITH, Raytheon Manufacturing Company
F. C. SMITH, College of St. Thomas
R. E. SMITH, College of William & Mary
W. M. SMITH, Lafayette College
ERNST SNAPPER, University of Southern California
ANDREW SOBCZYK, Watson Laboratories
VIVIAN E. SPENCER, U. S. Bureau of Census
HILLEL SPITZ, Naval Research Laboratory
C. E. SPRINGER, University of Oklahoma
MARION E. STARK, Wellesley College

E. P. STARKE, Rutgers University
R. C. STEPHENS, Knox College
RUTH W. STOKES, Syracuse University
MARY C. SUFFA, Elmira College
MILDRED M. SULLIVAN, Queens College
J. L. SYNGE, Carnegie Institute of Technology
GABOR SZEGO, Stanford University

A. H. TAUB, University of Washington
J. S. TAYLOR, University of Pittsburgh
FEODOR THEILHEIMER, Trinity College
P. D. THOMAS, U. S. Department of Commerce
J. E. THOMPSON, Pratt Institute
R. M. THRALL, University of Michigan
LEONARD TORNHEIM, University of Michigan
MARIAN M. TORREY, Goucher College
J. I. TRACEY, Yale University
BRYANT TUCKERMAN, Cornell University
J. W. TUKEY, Princeton University
LOUISE C. TURNER, Nichols, Connecticut

H. S. UHLER, Yale University

E. P. VANCE, Oberlin College
C. H. VEHSE, West Virginia University
JOHN VON NEUMANN, Institute for Advanced Study

HELEN M. WALKER, Teachers College, Columbia University
R. J. WALKER, Cornell University
J. L. WALSH, Harvard University
R. M. WALTER, Rutgers University
JEAN B. WALTON, University of Pennsylvania
C. W. WATKEYS, University of Rochester
G. C. WEBBER, University of Delaware
B. A. WELCH, Manhattan College
D. W. WESTERN, Brown University
A. L. WHITEMAN, U. S. Navy
P. M. WHITMAN, Tufts College
D. V. WIDDER, Harvard University
L. R. WILCOX, Illinois Institute of Technology
S. S. WILKS, Princeton University
BERYL W. WILLIAMS, A & T College, Greensboro, N. C.
JOHN WILLIAMSON, Queens College
G. M. WING, University of Rochester
R. M. WINGER, University of Washington
FLORENCE A. WIRSCHING, Purdue University
H. A. WOOD, Chance Vought Aircraft
FRANCES M. WRIGHT, Triple Cities College

BERTRAM YOOD, Cornell University
J. W. T. YOUNGS, Indiana University

OSCAR ZARISKI, Harvard University

The members of the organizations and their families were housed in the dormitories of the University, and meals were served in the cafeteria of the University Dining Hall.

Yale University was host at a tea served in the President's Room, Woolsey Hall, Tuesday afternoon, and a very pleasing organ recital was given that evening in Woolsey Hall by H. Frank Bozyan, University Organist and Professor of Organ Playing. The schedule for Wednesday afternoon included a visit to the Aetna and Connecticut Mutual Life Insurance Companies at Hartford, and a picnic supper at Hammonasset State Park on the shore of Long Island Sound.

A joint dinner for the three organizations was served on Thursday evening in the University Dining Hall. After an excellent meal, the toastmaster, Professor W. R. Longley, introduced Professor E. W. Sinnott of the Sheffield Scientific School who welcomed the visitors to the Yale campus. This was followed by short talks by Professors William Feller, L. R. Ford, and Einar Hille, the presidents respectively of the Institute, the Association, and the Society. Resolutions were presented by Professor P. R. Rider, and adopted by the assemblage, expressing the thanks of the visitors to the Administration of Yale University, to Professor E. G. Begle and the other members of the Committee on Arrangements, to the ladies of the Yale Department of Mathematics, to Miss A. Margaret Bowers, Director of the Dining Halls, and her staff, to Professor Bozyan

for his organ recital, and to the Connecticut Mutual and Aetna Life Insurance Companies for the privilege of the visit to their offices.

The American Mathematical Society held its sessions between noon Tuesday and noon Friday. The twenty-ninth Colloquium consisted of four lectures on *Abstract algebraic geometry* by Professor Oscar Zariski of Harvard University. Professor S. S. Wilks of Princeton University gave an address by invitation on *Sampling theory of order statistics.*

The Institute of Mathematical Statistics held sessions on Tuesday, Wednesday, and Thursday.

A program committee consisting of Professors Garrett Birkhoff, chairman, Lillian R. Lieber, and C. E. Springer arranged the very interesting program for the two sessions of the Association which were held Monday afternoon and Tuesday morning.

### First Session of the Association

"Atomic energy," by Dr. R. C. Bacher, U. S. Atomic Energy Commission.

"How to solve it": Symposium on mathematical problems at the college level. Chairmen, Professors George Polya, H. F. MacNeish, J. L. Synge; additional contributors, Professors H. S. M. Coxeter, A. M. Gleason, P. G. Helsel, Tibor Rado, L. M. Kelly, and E. P. Starke.

### Second Session of the Association

"Mathematics for the liberal arts student" by Professor C. B. Allendoerfer, Haverford College.

"The mathematics program in the College of the University of Chicago" by Professor E. P. Northrop, University of Chicago.

Symposium on computing machines: Professor H. H. Aiken, Harvard University, and Professor John von Neumann, Institute for Advanced Study.

### Meeting of the Board of Governors

The Board met on Monday at 8:00 P.M. in the Faculty Lounge of Saybrook College, with thirteen members of the Board present. The following are some of the more important items of business transacted:

Assistant Professor Harry Pollard of Cornell University was elected Associate Secretary for the balance of 1947, and Assistant Professor Edith R. Schneckenburger of the University of Buffalo was elected Associate Secretary for a term of five years beginning January 1, 1948.

The Board confirmed the appointment of G. E. Hay and Harry Pollard as Associate Editors of the MONTHLY.

On recommendation of the Committee on Slaught Memorial Papers, the Board voted approval of the publication of the sixth edition of the Outline of the History of Mathematics by R. C. Archibald as a Slaught Memorial Paper.

The Board approved the proposal to make the Committee on Publicity of the American Mathematical Society a joint-committee of the Society and Association, and to ask the present members of that committee to serve as such a joint-committee.

The Board voted a further appropriation of $250.00 in cash for the work of the Committee on Aid to Devastated Libraries.

W. B. CARVER,
*Secretary-Treasurer*

**CALENDAR OF FUTURE MEETINGS**

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, Spring, 1948.
ILLINOIS
INDIANA
IOWA, Fairfield, April 16–17, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA
ROCKY-MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Schenectady, N. Y., May 1, 1948
WISCONSIN, Beloit, May 8, 1948

## RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, 27, N. Y. and not to any of the other editors or officers of the Association.*

*Multiple Factor Analysis.* By L. L. Thurstone. University of Chicago Press, 1947. 19+535 pages. $7.50.

This book will be of direct interest to teachers of linear algebra, since it is a thorough and comprehensive account of the best-known approaches to the leading application of real matrix theory. The book begins with a 50 page mathematical introduction, and uses the ideas of (real) vector and matrix theory throughout its entire course. Plane figures and photographs of three-dimensional models do more than in any other book known to the reviewer to stimulate the reader's geometrical intuition. It should be of great value as supplementary reading for courses dealing with vectors and matrices, and with its consideration of practical and well-tested computational methods, it might well serve as one of the texts. It can be recommended for independent reading.

Any mathematician interested in the methodology of quantitative science, in the application of mathematics to science, or in the basic problems of educational measurement, will find time spent in reading the carefully reasoned discussions and pondering over the ideas developed to be time well spent. The author makes his points carefully, and with painstaking thoroughness. The reader with these interests will do well to combine the reading of Thurstone's book with that of Godfrey H. Thomsen's second edition of *The Factorial Analysis of Human Ability*. Between them there is a great store of careful and wise thought about the nature of scientific method and the wise application of mathematics in science. This is the philosophy of scientific method in general, and not that of psychology alone.

Regarding *Multiple Factor Analysis* as a second edition of *The Vectors of Mind*, one finds, in addition to the expected features of greater coherence, greater scope and inclusion of modern advances in theory and practice, two important improvements:

(*i*) a greatly improved and seasoned philosophy underlying the method,

(*ii*) an exceedingly careful and thorough exposition.

As an example of the latter, we quote from page 411: "We shall describe these two factorial domains in terms of a literal notation, a physical example, a diagrammatic interpretation, a geometrical example, and the matrix equations relating the two domains." This promise is duly carried out. It is only fair to notice, that, although the explanations are well and carefully done, many readers will find the language and presentation a little dull.

Nearly all approaches to factor analysis start out by representing a set of measurements on an individual (often the performance of the individual in a set of assigned tasks) as the components of a vector. If there are $N$ measurements, and only $n$ properties of the individual are important in linearly determining the results of the measurements, then, in the $N$-dimensional space, the vectors corresponding to different individuals will all lie near the $n$-dimensional plane determined by the *n factors*. The deviations of the individual vectors from this $n$-dimensional factor plane are due to two causes, first the inevitable errors of measurement, and second the individuality (technically called *specificity*) of particular measurements: What constitutes specificity will depend on the battery of measurements in which a particular measurement is used. The first computational problem, then, is to approximate the $N$-dimensional matrix of correlations between measurements by a matrix of rank $n$ (This corresponds to locating the $n$-dimensional factor plane). Approximation is only relevant for off-diagonal elements, with the side conditions that diagonal elements shall not exceed unity and the matrix shall be positive semi-definite. Many computational solutions are explained in this book.

Once the $n$-dimensional plane has been selected, the next step is to select a coördinate system within it—to determine the factors. It is now important to notice that the $N$ measurement axes may be projected, each onto that vector in the $n$-space which predicts it most closely in the sample taken. The directions

and sizes of these projections define an invariant ellipsoid in the $n$-space. The ellipsoid will change if, for example, half the measurements in the battery are each replaced by three equivalent measurements (although $n$ will not change, and there is a natural correspondence between the two $n$-dimensional spaces).

Here various approaches diverge rapidly. Some feel that all coördinate systems are equivalent, and that selection of particular axes is illusory. Others prefer to use the principal axes of the invariant ellipsoid. This procedure was originally due to Thurstone—the theory has been worked out by Hotelling. The main contribution of Thurstone to this problem is the idea of *simple structure*. This theory and many computational methods, some of very recent origin, are ably exposited. The basic ideas are simple. First, it is assumed that there are factors. Second, while the experimenter cannot be assumed to recognize these factors, else no factor analysis would be needed, it can be assumed that his intuition can recognize their nonappearance—that the experimenter can compose a battery of measurements, each of which fails (or nearly fails) to involve at least one factor. If this is so, then there will be a coördinate system in the $n$-dimensional factor plane such that the projections of the individual measurements lie along the coördinate hyperplanes—this is simple structure.

Thurstone makes the point carefully and cogently that the statistical theory of estimating a simple structure is lacking, and that statisticians should do something about this.

One recurrent mathematical error mars the high polish of this work, even although it is a minor scratch. On pages 93, 183, 343 and possibly elsewhere, the author states that a set of vectors, each of which lies within 90° of each other "must be contained within a cone of 45° generating angle." The author's figure on page 184 shows clearly that this is already false in three dimensions for three vectors near the coördinate axes. The three references above were easily gathered from the magnificent index, extending over 18 pages, which is the best the reviewer has seen for many a day. Like the careful presentation, the careful index sets a mark toward which other authors should strive.

J. W. TUKEY

*An Introduction to Mechanics.* By J. W. Campbell. New York, Pitman Publishing Corporation, 1947. 18+372 pages. $4.50.

This introductory course is designed for students who are taking a beginning course in calculus concurrently with a course in mechanics. The vector concept is explained and used. In addition to the usual topics in an introductory course in mechanics, there is a chapter in which Hooke's Law and the Laws of Friction are discussed, a chapter on flexure and torsion and a chapter on flexible chains and cables. The book ends with a chapter on Lagrange's equations. Great care has evidently been taken to make the explanations clear and precise and the illustrations are good (some of these are photographs that vividly bring home the physics of the subject to the student). Some of the material (centers of mass, moments of inertia, and so on) is more likely to be given in the United States

in a calculus course rather than in a course in mechanics. The demands made on the mathematical maturity of the students for whom the book is intended seem somewhat severe, and we feel that the book would be most successful with students at the Junior level (it being assumed that these students have had calculus in their Sophomore year). We are confident that any student who has worked through the large collection of interesting problems in this book will be well trained in mechanics.

F. D. MURNAGHAN

## NEW BOOKS RECEIVED

*College Algebra.* By E. R. Heineman. New York, The Macmillan Company, 1947. 9+359 pages. $3.25.

*Compléments de Géométrie Plane.* By R. Deaux. Brussels, De Boeck, 1945. 7+150 pages. 45 Fr.

*Introduction de la Géométrie des Nombres Complexes.* By R. Deaux. Brussels, De Boeck, 1947. 163 pages. 100 Fr.

*Elementary Arithmetic. Its Meaning and Practice.* By B. R. Buckingham. Boston, Ginn and Co., 1947. 8+744 pages. $4.50.

*Intermediate Algebra.* By J. R. Britton and L. C. Snively. New York, Rinehart and Co., 1947. 9+337 pages. $2.00.

*Mathematics of Business and Accounting.* By K. L. Trefftzs and E. J. Hills. New York, Harper and Brothers, 1947. 12+267+51 pages. $3.00.

*Money, Credit and Finance.* Revised Edition. By G. F. Luthringer, L. V. Chandler, and D. C. Cline. Boston, D. C. Heath and Co., 1947. 10+389 pages. $2.75.

*Probit Analysis. A Statistical Treatment of the Sigmoid Response Curve.* By D. J. Finney (With a foreword by F. Tattersfield). Cambridge, at the University Press; New York, The Macmillan Company, 1947. 13+256 pages. $3.75.

*Self-Help Geometry Workbook.* By H. D. Welte, F. B. Knight, and L. S. Walker. Chicago, Scott Foresman Co., 1947. 84 pages. $0.72.

# CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

*Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.*

## CLUB REPORTS, 1946–47

### Undergraduate Mathematics Club, University of Rochester

In September 1947, after a wartime lapse of four years, the club was reorganized. Monthly meetings were held at which the following papers were presented:

*Transcendental numbers*, by William Donoghue
*Operational methods*, by Arthur E. Danese
*Analytical methods in the solution of heat problems*, by Lois W. Mann
*Vector spaces*, by Lois W. Mann
*Transfinite cardinal numbers*, by Peter Lyman
*The Dirichlet integral*, by Eugene Trabka.

A picnic was held in Maplewood Park on May 31 at which Messrs. Trabka and Lyman were announced as the winners of the two prizes awarded annually for the best talks given during the year.

Officers for the year were: President, Peter Lyman; Secretary, Althea Blodgett; Faculty Advisor, Arthur K. Lohwater.

**Mathematics Club, College of the Holy Cross**

The Holy Cross *Mathematics Club* held semi-monthly meetings at which members presented the following papers:

*Interpolation formulas using finite differences*, by Fred Wolock
*An algebraic method for finding asymptotes*, by W. J. Winsper
*The trisection problem*, by W. D. Shaughnessy
*Generating certain bicircular quartics*, by M. W. Snow
*Some properties of cross-ratio*, by R. J. Coen
*Asymptotes by methods of the calculus*, by J. E. Madden
*Construction of the regular pentagon*, by R. L. Eisenman
*The nine-point circle*, by R. T. Blinn
*Solution to certain stress problems*, by Jack Hadley
*The addition formulas of trigonometry*, by J. G. Murray.

The officers for 1946–47 were: Presidents, R. J. Coen and R. T. Blinn; Secretary, W. J. Winsper; Faculty Advisors, Professors V. O. McBrien and C. G. Schilling.

**Mathematics Club, Montana State University**

The following papers were presented:

*Philosophy and mathematics*, by Dr. N. J. Lennes
*Infinity*, by Thomas Joyce
*University mathematics courses*, by Dr. A. Noble
*Determinants*, by Noreen Ingle
*Slide rule*, by Robert Willson
*Mathematical applications to engines*, by Robert Line
*Journey toward space*, by Donald Helterline
*Mathematics of long range flying*, by Mr. Tuma
*College mathematics in the northwest*, by Dr. A. S. Merrill.

Officers for 1946–47 were: President, Thomas Joyce; Vice-President, Frank Reed; Secretary-Treasurer, Anna Harwick; Co-Social Chairmen, Noreen Ingle and Donald Marshall. Officers for 1947–48 are: President, Donald Marshall, Vice-President, Jean Popham; Secretary-Treasurer, Veronica Kreitel.

**Mathematics Club, Upsala College**

Nine program meetings were held during the year. At six of these, students discussed the following topics:

*The value of mathematics in every day life,* by June Davidson

*Pythagoras,* by June Sandberg

*Anomalous propagation,* by William Melchinger

*Mathematics in the evolution of physics,* by Marjorie Cohen

*Omar Khayyam,* by Martin Monroe

*The cycloid and its properties as demonstrated by Roberval,* by Fern Freedman.

On February 27, the Club joined with *Kappa Mu Epsilon* in the midyear special meeting, where Professor Howard Fehr of Montclair State Teachers College spoke on *the exponential function, its ramifications and applications, scientific and financial.* Refreshments and a social time followed the program.

On April 14, members of the club were guests at the home of its president, Marjorie Cohen, where they enjoyed refreshments and scenes from a television set.

On May 21, the Upsala Club along with the clubs from five neighboring colleges were guests of the *Mathematics Club* of Montclair State Teachers College for a supper and games, followed by a lecture on *Probability* by Professor Edward Molina of Newark College of Engineering.

The officers for 1946–47 were: President, Marjorie Cohen; Vice-President, June Davidson; Secretary-Treasurer, June Sandberg; Faculty Advisor, Dr. M. A. Nordgaard. The officers for 1947–48 are: President, Martin Monroe; Vice-President, June Sandberg; Secretary-Treasurer, Robert Wallace; Faculty Advisor, Professor Norma Gilberts.

**Mathematics Club, Wellesley College**

The club held four meetings during the year. At the first meetings talks were given by faculty members, then followed two student meetings. Among the topics presented at the first student meeting were:

*Non-euclidean geometry*

*The three-point circle*

*Mathematics in architecture.*

At the last meeting students showed the correlation of mathematics with their other subjects in a program entitled *Everything correlates.*

The guest speaker for the year was L. Creighton Buck, Junior Fellow at Harvard University, who gave an interesting lecture on *Probability.*

The officers for 1946–47 were: President, Dorothy Schoenfuss; Vice-President, Mildred Kelton; Treasurer, Lois Wood; Secretary, May Field Manny; Junior Executive, Dawn O'Day; Faculty Advisor, Miss Helen Russell. The following officers were elected for 1947–48: President, Ann Kellogg; Vice-President, Jann Goehner; Treasurer, Barbara Wehle; Secretary, Lindsley Clark; Junior Executive, Mary Ann Berry; Faculty Advisor, Miss Miriam Ayer.

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.*

### NATIONAL APPLIED MATHEMATICS LABORATORIES ESTABLISHED

A federal center of applied mathematics, the National Applied Mathematics Laboratories, has been established as a division of the National Bureau of Standards, according to an announcement by Dr. E. U. Condon, Director of the Bureau.

Organized to conduct research and provide services in the field of applied mathematics, the new organization is oriented around modern mathematical statistics as applied to the physical and engineering sciences and to the development and use of modern high speed computing. The Applied Mathematics Laboratories include four separate laboratories: the Institute of Numerical Analysis, the Computation Laboratory, the Statistical Engineering Laboratory, and the Machine Development Laboratory. Dr. J. H. Curtiss, who has been the Director's Assistant in Applied Mathematics at the National Bureau of Standards, has been named chief of the National Applied Mathematics Laboratories, according to the announcement by Dr. E. U. Condon. The Applied Mathematics Laboratories have been set up to serve primarily as a national center for research and training in applied mathematics for units of the Federal Government and private industry. Among the responsibilities of the new organization is the development of giant high speed electronic computing machines, statistical service to other units of the government and industry aimed at extending the part played by applied mathematics in present day scientific research, and the training of scientists in methods used in this important field.

Dr. Churchill Eisenhart heads the Statistical Engineering Laboratory. This Laboratory plans and advises in the mathematical and statistical phases of scientific research and testing. The Laboratory also offers a consulting service to the laboratories at the Bureau of Standards and other government agencies on particular problems in the field of statistical engineering.

Dr. E. W. Cannon has been appointed chief of the Machine Development Laboratory. This laboratory is responsible for the design of the high-speed electronic computing machines now under construction by the Bureau of Standards.

A. S. Cahn, applied mathematician and meteorologist, has been appointed executive officer of the Institute of Numerical Analysis.

Dr. A. N. Lowan has been appointed chief of the Computation Laboratory. Dr. Lowan's group, which is at present being underwritten by the Office of Naval Research, is providing a computation service and is continuing the work of the Mathematical Tables Project in the preparation of tables used in statistical and mathematical analysis.

# TEXTBOOK NEWS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

# THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

# LIST OF OFFICERS AND MEMBERS

DECEMBER 1947

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND OBERLIN, OHIO

THE AMERICAN
# MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION
OF AMERICA
(INCORPORATED)

VOLUME 54 NUMBER 10

PART II

# LIST OF
# OFFICERS AND MEMBERS



DECEMBER 1947

Price 50 cents

# The Chauvenet Prize

The Chauvenet Prize for a "noteworthy expository paper published in English during the five-year period by a member of the Association" was established in 1925 by a gift from Professor J. L. Coolidge, then president of the Association. Its frequency period was reduced to three years by an additional gift from Professor W. B. Ford during his presidency and by an anonymous gift in 1936.

As determined more recently by the Board of Governors, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The next two awards will be made in December 1947 and December 1950, covering the periods 1944-46 and 1947-49 respectively.

*Seven awards have been made as follows:*

1920-1924. G. A. Bliss, "Algebraic Functions and Their Divisors," *Annals of Mathematics.*

1925-1928. T. H. Hildebrandt, "The Borel Theorem and Its Generalizations," *Bulletin* of the American Mathematical Society.

1929-1931. G. H. Hardy, "An Introduction to the Theory of Numbers," *Bulletin* of the American Mathematical Society.

1932-1934. Dunham Jackson, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums," *Annals of Mathematics;* "The Convergence of Fourier Series," *American Mathematical Monthly.*

1935-1937. G. T. Whyburn, "On the Structure of Continua," *Bulletin* of the American Mathematical Society.

1938-1940. Saunders Mac Lane, "Modular Fields," and "Some Recent Advances in Algebra," both in the *American Mathematical Monthly.*

1941-1943. R. H. Cameron, "Some Introductory Exercises in the Manipulation of Fourier Transforms," *National Mathematics Magazine.*

# The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)



EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions for the Monthly, should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N.Y. Change of address must reach the Secretary-Treasurer before the 20th of the month preceding that in which the change becomes effective.

---

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.
(Devoted to the Interests of Collegiate Mathematics)



---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.

## OFFICERS

*Honorary President*, W. D. CAIRNS, Oberlin College
*President*, L. R. FORD, Illinois Institute of Technology (1947–48)
*First Vice-President*, W. L. AYRES, Purdue University (1946–47)
*Second Vice-President*, C. B. ALLENDOERFER, Haverford College (1947–48)
*Editor-in-Chief*, C. V. NEWSOM, Oberlin College (1947–51)
*Secretary-Treasurer*, H. M. GEHMAN, University of Buffalo (1948–52)
*Associate Secretary*, EDITH R. SCHNECKENBURGER, University of Buffalo (1948–52)

## ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

*Ex-Presidents*

R. W. BRINK, University of Minnesota (1943–48)
W. D. CAIRNS, Oberlin College (1945–50)
C. C. MACDUFFEE, University of Wisconsin (1947–52)

*Governors at Large*

WALTER BARTKY, University of Chicago (1945–47)
H. S. M. COXETER, University of Toronto (1945–47)
DEANE MONTGOMERY, Yale University (1946–48)
C. V. NEWSOM, Oberlin College (1946–48)
D. H. LEHMER, University of California (1947–49)
W. T. MARTIN, Massachusetts Institute of Technology (1947–49)

*Regional Governors (representing the seven remaining regions)*

1. E. S. HAMMOND, Bowdoin College (July 1946–July 1949)
2. C. W. WATKEYS, University of Rochester (July 1946–July 1948)
5. W. L. WILLIAMS, University of South Carolina (July 1946–July 1949)
7. H. M. BEATTY, Ohio State University (July 1946–July 1948)
10. RALPH HULL, University of Nebraska (July 1946–July 1948)
12. A. E. RECHT, University of Denver (July 1946–July 1948)
13. A. S. MERRILL, Montana State University (July 1946–1949)

*Sectional Governors (July 1, 1947–July 1, 1950)*

*Philadelphia*, E. P. STARKE, Rutgers University
*Maryland-District of Columbia-Virginia*, D. W. HALL, University of Maryland
*Louisiana-Mississippi*, W. L. DUREN, Tulane University
*Michigan*, J. W. BRADSHAW, University of Michigan
*Illinois*, E. H. C. HILDEBRANDT, Northwestern University
*Iowa*, H. P. THIELMAN, Iowa State College
*Minnesota*, L. E. BUSH, College of St. Thomas
*Texas*, H. E. BRAY, Rice Institute
*Southern California*, I. S. SOKOLNIKOFF, University of California at Los Angeles

## STANDING COMMITTEES OF THE ASSOCIATION

### Finance Committee

H. M. Gehman, *ex officio*
W. B. Carver (1948–51)
J. F. Randolph (1946–49)

### Editorial Committee on Carus Monographs

Philip Franklin, *Chairman* (1946–51)
A. J. Kempner ( –1948)
Garrett Birkhoff ( –1949)
E. F. Beckenbach ( –1950)
J. L. Synge (1947–52)
H. S. M. Coxeter (1948–53)

### Committee on Slaught Memorial Papers

N. H. McCoy, *Chairman*
L. L. Dines
W. E. Milne
C. V. Newsom

### Committee on the Arnold Buffum Chace Fund

R. C. Archibald
R. W. Brink
W. D. Cairns
W. R. Longley

### Committee for the Coordination of Studies in Mathematical Education

C. C. MacDuffee, *Chairman*
M. S. Knebelman
C. V. Newsom
W. V. Parker

### Committee on Section Meetings

H. M. Gehman, *ex officio*
Ruth Mason Ballard
W. V. Parker

### Committee on the Putnam Prize Competition

Irving Kaplansky (1946–48)
Mark Kac (1947–49)
George Polya (1948–50)

### Representatives of the Association

On the National Research Council—G. T. Whyburn (July 1947–July 1950)
On the Council of the American Association for the Advancement of Science—G. C. Evans (1946–47), W. T. Martin (1947–48)
On the American Council on Education—A. A. Bennett ( –1948), B. H. Brown ( –1949), Tomlinson Fort (1948–50)
On the Cooperative Committee on Science Teaching—Raleigh Schorling
On the American Documentation Institute—E. J. Finan (1946–48)

### Associate Editors of the Official Journal

C. B. Allendoerfer
E. F. Beckenbach
L. M. Blumenthal
N. B. Conkwright
H. S. M. Coxeter
H. P. Evans
Howard Eves
G. E. Hay
N. H. McCoy
W. T. Martin
L. F. Ollmann
Edith R. Schneckenburger
R. F. Rinehart
E. P. Starke
E. P. Vance

## SECTIONS OF THE ASSOCIATION

### Allegheny Mountain

J. B. Rosenbach, Carnegie Institute of Technology, *Chairman*
E. W. Montroll, University of Pittsburgh, *Secretary-Treasurer*

### Illinois

C. N. Mills, Illinois State Normal University, *Chairman*
M. G. Moore, Bradley Polytechnic Institute, *Vice-Chairman*

### Indiana

G. H. Graves, Purdue University, *Chairman*
H. E. Wolfe, Indiana University, *Vice-Chairman*
P. M. Pepper, University of Notre Dame, *Secreatary-Treasurer*

### Iowa

H. P. Thielman, Iowa State College, *Vice-Chairman*
Fred Robertson, Iowa State College, *Secretary-Treasurer*

### Kansas

C. A. Reagan, Friends University, *Chairman*
Sister M. Helen Sullivan, Mt. St. Scholastica College, *Vice-Chairman*
Anna Marm, Bethany College, *Secretary-Treasurer*

### Kentucky

D. W. Pugsley, Berea College, *Chairman*
Sallie E. Pence, University of Kentucky, *Secretary-Treasurer*

### Louisiana-Mississippi

W. L. Johnson, Mississippi Southern College, *Chairman*
F. C. Gentry, Louisiana Polytechnic Institute, *Secretary-Treasurer*
T. A. Bickerstaff, University of Mississippi, *Vice-Chairman for Mississippi*
A. L. Loflin, Southwestern Louisiana Institute, *Vice-Chairman for Louisiana*

### Maryland-District of Columbia-Virginia

W. K. Morrill, Johns Hopkins University, *Chairman*
E. J. Finan, Catholic University, *Secretary-Treasurer*

### Metropolitan New York

H. E. Wahlert, New York University, *Chairman*
W. H. H. Cowles, Pratt Institute, *Vice-Chairman*
Morris Hertzig, Forest Hills High School, *Vice-Chairman*
C. B. Boyer, Brooklyn College, *Secretary*
Aaron Shapiro, Midwood High School, *Treasurer*

### Michigan

G. C. Bartoo, West Michigan College of Education, *Chairman*
L. J. Rouse, University of Michigan, *Secretary-Treasurer*

### Minnesota

K. H. Bracewell, Hamline University, *Chairman*
L. E. Bush, College of St. Thomas, *Secretary-Treasurer*

### Missouri

R. R. Middlemiss, Washington University, *Chairman*
P. R. Rider, Washingron University, *Secretary-Treasurer*

### Nebraska

Ralph Hull, University of Nebraska, *Chairman*
C. B. Gass, Nebraska Wesleyan, *Vice-Chairman*
Lulu L. Runge, University of Nebraska, *Secretary*

### Northern California

George Polya, Stanford University, *Chairman*
G. C. Evans, University of California, *Vice-Chairman*
E. B. Roessler, University of California at Davis, *Secretary-Treasurer*

OHIO

H. S. POLLARD, Miami University, *Chairman*
FOSTER BROOKS, Kent State University, *Secretary-Treasurer*

OKLAHOMA

D. R. SHEVE, University of Tulsa, *Chairman*
W. T. SHORT, Oklahoma Baptist University, *Vice-Chairman*
J. C. BRIXEY, University of Oklahoma, *Secretary*

PHILADELPHIA

C. J. REES, University of Delaware, *Chairman*
W. H. GOTTSCHALK, University of Pennsylvania, *Secretary*

ROCKY MOUNTAIN

H. T. GUARD, Colorado State College of A. and M. A., *Chairman*
I. L. HEBEL, Colorado School of Mines, *Vice-Chairman*
J. R. BRITTON, University of Colorado, *Secretary-Treasurer*

SOUTHEASTERN

J. W. CELL, North Carolina State College, *Chairman*
L. A. DYE, The Citadel, *Vice-Chairman*
H. A. ROBINSON, Agnes Scott College, *Secretary-Treasurer*

SOUTHERN CALIFORNIA

D. V. STEED, University of Southern California, *Chairman*
E. F. BECKENBACH, University of California at Los Angeles, *Vice-Chairman*
P. H. DAUS, University of California at Los Angeles, *Secretary-Treasurer*

SOUTHWESTERN

H. D. LARSEN, University of New Mexico, *Chairman*
R. F. GRAESSER, University of Arizona, *Vice-Chairman*
B. D. ROBERTS, Highlands University, *Secretary-Treasurer*

TEXAS

R. S. UNDERWOOD, Texas Technological College, *Chairman*
F. E. ULRICH, Rice University, *Vice-Chairman*
C. R. SHERER, Texas Christian University, *Secretary-Treasurer*

UPPER NEW YORK STATE

D. S. MORSE, Union College, *Chairman*
E. P. ALLEN, Rensselaer Polytechnic Institute, *Vice-Chairman*
C. W. MUNSHOWER, Colgate University, *Secretary-Treasurer*

WISCONSIN

H. P. EVANS, University of Wisconsin, *Chairman*
P. L. TRUMP, University of Wisconsin, *Secretary*

PACIFIC NORTHWEST

M. S. KNEBELMAN, State College of Washington, *Chairman*
W. E. MILNE, Oregon State College, *Vice-Chairman*
F. S. NOWLAN, University of British Columbia, *Secretary-Treasurer*

## MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

Note. This list gives data for the members of the Association as taken from the last similar list issued in December 1945, unless changes in position, mailing address, etc., have been reported to the Secretary. Every effort has been made to make the information accurate; but since a number of members fail to return the data cards, there are doubtless some errors and omissions.

ABLOW, C. M., A.M., (U.C.L.A.) *785 N.W. 17 Court, Miami 35, Fla.*
ADAMS, B. T., A.M. (Baylor) Instr., Amarillo Army Air Field, Amarillo, Tex. *2172 Ave. I, Wichita Falls, Tex.*
ADAMS, PROF. C. R., Ph.D. (Harvard) Brown Univ., Providence, R. I. *60 Intervale Rd.*
ADAMS, Prof. E. P., Ph.D. (Harvard) Physics, Princeton Univ., Princeton, N.J. *Edgerstoune*
ADAMS, Asst. Prof. LOUISE, A.M. (North Carolina) High Point Coll., High Point, N.C.
ADAMS, L. J., A.M. (Southern California) Head of Dept., Santa Monica City Coll., Santa Monica, Calif. *227–25th St.*
ADAMS, O. S., D.Sc. (Kenyon) Principal Mathematician, Retired, U. S. Coast and Geodetic Survey, Washington 25, D.C. *R.F.D. 4, Mt. Vernon, Ohio*
ADKINS, J. B., Ed.M. (Harvard) Teacher, Phillips Exeter Acad., Exeter, N.H.
ADKINS, Prof. L. K., M.S. (Chicago) Head of Dept., State Teachers Coll., La Crosse, Wis.
ADKISSON, Prof. V. W., Ph.D. (Pennsylvania) Chm. of Dept., Univ. of Arkansas, Fayetteville, Ark. *236 Buchanan St.*
ADLER, Asst. Prof. CLAIRE F., Ph.D. (New York Univ.) New York Univ., New York, N.Y. *189-21 Tioga Drive, St. Albans, N.Y.*
AGARD, Prof. H. L., Ph.D. (Yale) Williams Coll., Williamstown, Mass. *Box 49*
AGNEW, Prof. R. P., Ph.D. (Cornell) Cornell Univ., Ithaca, N.Y. *112 White Hall*
AHLFORS, Prof. L. V., Ph.D. (Helsingfors) Harvard Univ., Cambridge, Mass. *52 Dunster St.*
ALBERT, Prof. A. A., Ph.D. (Chicago) Univ. of Chicago, Chicago 37, Ill. *Asso. Director, Applied Math. Group, Northwestern Univ.*
ALBERT, Asso. Prof. G. E., Ph.D. (Wisconsin) Univ. of Tennessee, Knoxville 16, Tenn. *Dept. of Math.*
ALBERT, Prof. O. W., Ph.D. (Washington) Univ. of Redlands, Redlands, Calif. *629 Buena Vista St.*
ALBERT, R. G., B.S. (Brooklyn) Instr., Sampson Coll. Sampson, N.Y. *Faculty*
ALBERTI, FURIO, B.S. (Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill. *1806 S. Sixth Ave., Maywood, Ill.*
ALDEN, Asst. Prof. H. H., Ph.D. (Ohio State) Ohio State Univ., Columbus, Ohio. *317 Northridge Road, Columbus 2, Ohio*
ALEXANDER, Prof. C. K., Ph.D. (Calif. Inst. of Tech.) Head of Dept., Occidental Coll., Los Angeles 41, Calif.
ALEXANDER, Prof. J. W., Ph.D. (Princeton) Inst. for Advanced Study, Princeton, N.J. *29 Cleveland Lane*
ALFIERI, F. A., *1919 McGraw Ave., Apt. 2-G, N.Y.C. 62, N.Y.*
ALLEN, Prof. E. B., Ph.D. (Rensselaer) Head of Dept., Math. and Astr., Rensselaer Poly. Inst., Troy, N.Y. *4 Sheldon Ave.*
ALLEN, Prof. ELBERT F., Ph.D. (Missouri) Oklahoma A. and M. Coll., Stillwater, Okla. *1409 College Ave.*
ALLEN, Asst. Prof. FLORENCE E., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *219 Lathrop St.*
ALLENDOERFER, Prof. C. B., Ph.D. (Princeton) Haverford Coll., Haverford, Pa.
ALLIOT, Prof. EUGENE, Lic.es Sc. (Paris) St. Edmund's Juniorate, Swanton, Vt.
AMMERMAN, JANE, B.S. (Duke) Instr., Newark Coll., Rutgers Univ., Newark, N. J. *757 Sterling Dr., Orange, N.J.*
AMUNDSON, Asst. Prof. N. R., Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis 14, Minn. *Chemistry Bldg.*
ANDERSEN, Prof. MAE R., Ph.D. (Chicago) Head of Dept., Concordia Coll., Moorhead, Minn. *507 Tenth St. S.*
ANDERSON, A. H., M.E. (Marquette) Head of Science Dept., Whitefish Bay Schools, Whitefish Bay, Wis. *5143 N. Berkeley Blvd., Milwaukee 11, Wis.*
ANDERSON, A. T., A.M. (Michigan) Instr., Cooper Union, New York, N.Y. *P.O. Box 124, Floral Park, N.Y.*

ANDERSON, Asst. Prof. E. W., Ph.D. (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
ANDERSON, Asst. Prof. H.M., Ph.M. (Wisconsin) Gustavus Adolphus Coll., St. Peter, Minn.
ANDERSON, P. H., Ph.D. (Illinois) Economic Analyst, Marketing Div., Office of Domestic Commerce, Dept. of Commerce, Washington 25, D.C. *1228 Blair Mill Road, Silver Spring, Md.*
ANDERSON, R. D., A.B. (Minnesota) WH 19, Univ. of Texas, Austin, Tex. *Dept. of Math.*
ANDERSON, Asst. Prof. R. Lucile, Ph.D. (Bryn Mawr) Hunter Coll., New York 21, N.Y. *695 Park Ave.*
ANDERSON, Prof. W. E., Ph.D. (Pennsylvania) Miami Univ., Oxford, Ohio. *112 E. Walnut St.*
ANDERTON, ETHEL L., Ph.D. (Yale) Teacher, High School, West Haven, Conn. *215 Park Terrace Ave.*
ANDREE, R. V., Ph.M. (Wisconsin) Grad. Teaching Asst., Univ. of Wisconsin, Madison, Wis. *North Hall*
ANDREWS, J. J., A.M. (St. Louis Univ.) Lecturer, St. Louis Univ., St. Louis, Mo. *155 S. Sappington Road, Kirkwood 22, Mo.*
ANDRIS, PETER, B.S. in E.E. (Illinois) Teacher, Harrison Tech. High School, Chicago, Ill. *6649 S. Bishop St., Chicago 36, Ill.*
ANITA, Sister M., A.M. (Seton Hall Coll.) Instr., Caldwell Coll., Caldwell, N.J.
ANNEAR, Asst. Prof. P. R., M.S. (Case) Head of Dept., Math. and Astr., Baldwin-Wallace Coll., Berea, Ohio. *280 Eastland Road*
ANN ELIZABETH, Sister, Ph.D. (Wisconsin) Prof., Registrar, St. Mary Coll., Xavier, Kans.
ANNING, Asst. Prof. NORMAN, M.A. (Queen's Univ.) Univ. of Michigan, Ann Arbor, Mich. *R.R. 7*
ANTHONY, M. L., *5624 S. Maryland St., Chicago 37, Ill.*
APPUHN, Prof. W. E. F., A.M. (Columbia) Head of Dept., St. Francis Coll., Brooklyn, N.Y. *548 State St., Brooklyn 17, N.Y.*
ARASHIRO, N. N., B.S. in M.E. (North Dakota) Mech. Engineer, Suite 348, Bankers Bldg., Milwaukee, Wis. *208 E. Wisconsin Ave.*
ARCHIBALD, Prof. R. C., Ph.D. (Strasbourg), Dr. (Univ. Padua), LL.D. (Mt. Allison) Emeritus, Brown Univ., Providence, R.I.
ARCHIBALD, Asso. Prof. R. G., Ph.D. (Chicago) Queens Coll., Flushing, N.Y.
ARENA, J. F., A.M. (Michigan) *1020 N. 14th St., Fargo, N.D.*
ARENSON, D. L., Student, Illinois Inst. of Tech., Chicago, Ill. *6646 N. Glenwood Ave.*
ARGUE, W. T., A.B. (Tulsa) Chief Computer, Seismograph Service Corp., Tulsa 1, Okla. *Box 1590*
Armstrong, Asst. Prof. Buelah M., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *364 Math. Bldg.*
ARMSTRONG, J. D., B.S.E. (Florida) *1409 Mayfair Road, Jacksonville, Fla.*
ARNDT, Prof. W. F. C., Ph.D. (Gottingen) Univ. Coll. of the Orange Free State, Bloemfontein, Union of South Africa
ARNOLD, H. A., Ph.D. (Calif. Inst. of Tech.) *68 West 3 St., Apt. 14, New York, N.Y.*
ARNOLD, H. C., A.M. (Ohio State), M.S. (Carnegie Inst. of Tech.) Tech. Director, Federal Enamel and Stamping Co., McKees Rocks, Pa. *76 Standish Blvd., Pittsburgh 16, Pa.*
ARNOLD, Prof. H. E., Ph.D. (Yale) Wesleyan Univ., Middletown, Conn.
ARNOLD, Asst. Prof. K. J., Ph.D. (Mass. Inst. of Tech.) Univ. of Wisconsin, Madison 6, Wis. *North Hall*
ARNOLD, Asso. Prof. W. C., M.S. (Chicago) DePauw Univ., Greencastle, Ind. *P.O. Box 466*
ARNOLDY, Sister M. NICHOLAS, Ph.D. (Catholic Univ). Head of Dept., Marymount Coll., Salina, Kans.
AROIAN, Asst. Prof. L. A., Ph.D. (Michigan) Hunter Coll., New York 21, N.Y. *247 Wadsworth Ave., New York 33, N.Y.*
ARTIN, Prof. EMIL, Ph.D. (Leipzig) Princeton Univ., Princeton, N.J. *Fine Hall*
ASHBAUGH, LAURA M. (Mrs. F. R.), A.M. (Pennsylvania) Instr., Moravian Coll. for Women, Bethlehem, Pa. *115 Wall St.*
ASHBURN, Asso. Prof. A. W., Ph.D. (Virginia) Texas State Coll. for Women, Denton, Tex. *Box 344, Main Office*
ASHCRAFT, Prof. T. B., Ph.D. (Johns Hopkins) Colby Coll., Waterville, Me. *34 Pleasant St.*
ASPREY, Asst. Prof. WINIFRED A., Ph.D. (Iowa) Vassar Coll., Poughkeepsie, N.Y.
ASTRACHAN, Asso. Prof. MAX, Ph.D. (Brown) Chm. of Dept., Antioch Coll., Yellow Springs, Ohio
ATKIN, Asso. Prof. EDITH I., A.M. (Columbia) Emeritus, Illinois State Normal Univ., Normal, Ill. *815 S. Fell Ave.*

ATKINS, H. P., JR., M.S. (Brown) Instr., Univ. of Rochester, River Campus, Rochester 3, N.Y.

ATKINSON, ROBERT, A.M. (Haverford) Chm., Math. and Sci. Depts., The Shipley School, Bryn Mawr, Pa.

AUDE, Prof. H. T. R., M.S. (Colgate) Colgate Univ., Hamilton, N.Y.

AURORA, SILVIO, A.B. (Columbia) Grad. Student, Columbia Univ., New York, N.Y. *30–58 83rd St., Jackson Heights, N.Y.*

AYLOR, M. W., M.S. (Virginia) Instr., Univ. of Virginia, Charlottesville, Va. *Madison, Va.*

AYOUB, R. G., M.Sc. (McGill Univ.) *6848 Drolet St., Montreal, P.Q., Can.*

AYRE, Asso. Prof. H. G., Ph.D. (Peabody) Western Illinois State Coll., Macomb, Ill.

AYRES, Prof. FRANK, Jr., Ph.D. (Chicago) Dickinson Coll., Carlisle, Pa.

AYRES, Asso. Prof. H. C., Ph.D. (California) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*

AYRES, Prof. W. L., Ph.D. (Pennsylvania) Head of Dept., Purdue Univ., Lafayette, Ind. *Dept. of Math.*

BABBITT, A. E., A.M. (Illinois) Vice-Pres. and Actuary, Lamar Life Ins. Co., Box 880, Jackson 107, Miss.

BABCOCK, L. E., M.S. (Illinois) Head of Dept. West Virginia Inst. of Tech., Montgomery, W. Va.

BABCOCK, Dean R. W., Ph.D. (Wisconsin) School of Arts and Sci., Kansas State Coll., Manhattan, Kans.

BABCOCK, Asso. Prof. WEALTHY, Ph.D. (Kansas) Univ. of Kansas, Lawrence, Kans. *Route 6*

BACHILLER, T. R., Dr.Sc. (Univ. of Madrid) Catedratico de la Fac. de Ciencias de la Universidad de Madrid, Madrid, Spain

BACHMANN, J. G., B.S. (New York Univ.) Pres., Bachmann & Co., Inc., 8511 Sunset Blvd., Hollywood 46, Calif.

BACON, Asso. Prof. H. M., Ph.D. (Stanford) Stanford Univ., Stanford University, Calif. *Box 1144*

BADGER, GLADYS F., A.M. (Northwestern) Teacher, Roosevelt High School, Chicago, Ill. *4845 N. Kimball Ave., Chicago 25, Ill.*

BAEZ, A. V., A.M. (Syracuse) Instr., Physics, Stanford Univ., Stanford University, Calif.

BAGBY, L. C., A.M. (Kansas) Engr., Conklin Industrial Service, Inc., Detroit 2, Mich. *62 Glendale Ave., Highland Park 3, Mich.*

BAIDAFF, Prof. B. I., Dr. (Buenos Aires) Analisis, Univ. Nac., Buenos Aires, Argentina. *Ave. de Mayo 560*

BAILEY, Asst. Prof. A. H., Ph.D. (Ohio State) Georgia School of Tech., Atlanta, Ga. *Dept. of Math.*

BAILEY, E. A., M.S. (Emory Univ.) Dean and Registrar, LaGrange Coll., LaGrange, Ga.

BAILEY, Asso. Dean H. W., Ph.D. (Illinois) Lib. Arts and Sci., Univ. of Illinois in Chicago, Navy Pier, Chicago 11, Ill.

BAILEY, Asso. Prof. R. P., Ph.D. (Pennsylvania) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*

BAIRD, ARTHUR C., A.B. (Wooster) Retired, High School Vice-Prin. *502 Lawn Ave., Hamilton, Ohio*

BAKER, Asso. Prof. FRANCES E., Ph.D. (Chicago) Vassar Coll., Poughkeepsie, N.Y.

BAKER, Asst. Prof. G. A., Ph.D. (Illinois) Coll. of Agric., Univ. of California; Asst. Statistician, Experiment Sta., Davis, Calif. *606 C St.*

BAKER, Mrs. KATHRYN C., M.S. (Chicago) Asst., Randolph-Macon Woman's Coll., Lynchburg, Va. *R.F.D. 4*

BAKER, S. R., A.B. (Ursinus) Instr., North Carolina State Coll., Raleigh, N.C. *125 Brooks Ave.*

BAKST, AARON, Ph.D. (Columbia) *315–12 77th Ave., Flushing, N.Y.*

BALDWIN, Prof. J. W., A.M. (Michigan) Wayne Univ., Detroit, Mich. *16191 Roselawn Ave., Detroit 21, Mich.*

BALL, Asso. Prof. N. H., Ph.D. (Mass. Inst. of Tech.) U. S. Naval Acad., Annapolis, Md. *R.F.D. #1, Box 235*

BALLANTINE, Prof. J. P., Ph.D. (Chicago) Univ. of Washington, Seattle 5, Wash.

BALLARD, RUTH MASON (Mrs. F. K.), Ph.D. (Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill. *6739 Bosworth Ave., Chicago 26, Ill.*

BALLINGER, J. G., B.S. (Illinois), B.S. (Notre Dame) Teacher, High School, Chicago, Ill. *4824 Eddy St., Chicago 41, Ill.*

BALLOU, Asst. Prof. D. H., Ph.D. (Harvard) Middlebury Coll., Middlebury, Vt. *27 Weybridge St.*

BALOF, C. A., M.S. (Iowa) Business Manager, Lincoln Coll., Lincoln, Ill. *623 N. Union St.*

BANCROFT, T. A., Ph.D. (Iowa State) Dir. of Statistics Lab., Alabama Poly. Inst., Auburn, Ala.
BANKIER, J. D., Ph.D. (Rice) Lecturer, McMaster Univ., Hamilton, Ont., Can. *Dept. of Math.*
BANKS, Prof. G. B., Ph.D. (Niagara Univ.) Physical Sci., Niagara Univ., Niagara University, N.Y.
BARBOUR, Asso. Prof. J. M., Ph.D. (Cornell), Mus.D. (Toronto) Music, Michigan State Coll., East Lansing, Mich. *607 Division St.*
BARDELL, Asso. Prof. R. H., Ph.D. (Chicago) Acting Dir., Univ. of Wisconsin in Milwaukee, 623 W. State St., Milwaukee 3, Wis.
BAREIS, Asst. Prof. GRACE M., Ph.D. (Ohio State) Retired, Ohio State Univ., Columbus, Ohio. *164–13th Ave., Columbus 1, Ohio*
BARLAZ, Asst. Prof. JOSHUA, Ph.D. (Cincinnati) Rutgers Univ., New Brunswick, N.J. *Dept. of Math.*
BARNARD, Asso. Prof. R. W., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill.
BARNES, J. C., B.S. (North Georgia Coll.) Head of Dept., North Georgia Coll., Dahlonega, Ga.
BARNETT, Prof. I. A., Ph.D. (Chicago) Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math.*
BARNETT, Asst. Prof. JOSEPH, Jr., A.M. (Columbia) Oklahoma A. and M. Coll., Stillwater, Okla. *520 Maple Ave., Apt. 5*
BARR, Prof. C. F., M.S. (Chicago) Univ. of Wyoming, Laramie, Wyo.
BARRAL-SOUTO, Prof. JOSE, Sc.D. Statistics, Faculty of Econ. Sciences, Univ. of Buenos Aires, Buenos Aires, Argentina. *Cordoba 1459*
BARRICK, D. L., M.S. (Oklahoma) Instr., Engg. Math., Univ. of Colorado, Boulder, Colo. *403 Highland Ave.*
BARROW, Prof. D. F., Ph.D. (Harvard) Univ. of Georgia, Athens, Ga. *260 Cherokee Ave.*
BARTEN, H. J., B.S. in M.E. (Johns Hopkins) Armour Research Foundation, 35 W. 33rd St., Chicago, Ill.
BARTHOLOMAY, A. F., A.M. (Syracuse) Instr., Rensselaer Poly. Inst., Troy, N.Y. *21 Locust Ave.*
BARTKY, Prof. WALTER, Ph.D. (Chicago) Asso. Dean, Physical Sci., Univ. of Chicago, Chicago, Ill.
BARTON, Prof. HELEN, Ph.D. (Johns Hopkins) Head of Dept., Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *1027 Spring Garden St.*
BARTOO, Prof. G. C., A.M. (Michigan) Western Michigan Coll., Kalamazoo, Mich. *1341 Hillcrest Ave.*
BASOCO, Prof. M. A., Ph.D. (Calif. Inst. of Tech.) Univ. of Nebraska, Lincoln 8, Nebr.
BASS, T. J., Jr., A.M. (California) Instr., San Francisco Jr. Coll., San Francisco, Calif. *80 Alviso St., San Francisco 12, Calif.*
BASYE, R. E., Ph.D. (Texas) Instr., A. and M. Coll. of Texas, College Station, Tex. *Box 5664*
BATCHELDER, Asso. Prof. P. M., Ph.D. (Harvard) Univ. of Texas, Austin, Tex. *910 West 22 St., Austin 21, Tex.*
BATEMAN, P. T., Ph.D. (Pennsylvania) Instr., Yale Univ., New Haven, Conn. *208 Leet Oliver Hall*
BATEN, Asso. Prof. W. D., Ph.D. (Michigan) Brooklyn Coll., Bedford Ave., & Ave. H., Brooklyn 10, N.Y.
BATES, GRACE E., Ph.D. (Illinois) Instr., Mount Holyoke Coll., South Hadley, Mass.
BATTIG, LEON, A.M. (Wisconsin) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *2535 Elizabeth St., Sheboygan, Wis.*
BATTIN, Asst. Prof., I.L., A.M. (Swarthmore) Drew Univ., Madison, N.J.
BAUER, L. M., A.M. (New Mexico) Teacher, Menaul School, Albuquerque, N.M. *P.O. Box 1295*
BAUMANN, DOROTHEA M., B.S. (Marquette) Teacher, Rufus King High School, Milwaukee, Wis. *2206 S. 30th St., Milwaukee 7, Wis.*
BAUMGART, Asst. Prof. J. K., A.M. (Michigan) Elmhurst Coll., Elmhurst, Ill. *1441 S. 13th Ave., Maywood, Ill.*
BAUMGARTNER, R. A., A.M. (Illinois) Head of Dept., Senior High School, Freeport, Ill.
BAUSER, A. V., A.M. (New York Univ.) Engr., Elliott Co., Ridgway, Pa. *105 N. Ferguson St., Shenandoah, Pa.*
BAUSERMAN, THOMAS, B.S. (W. Va. Inst. of Tech.) Instr., W. Va. Univ., Morgantown, W. Va. *313 Raymond St.*
BAY, J. C., Librarian, The John Crerar Library, 86 E. Randolph St., Chicago, Ill.
BAYS, Dr. SEVERIN. Prof. ord. math. pures, Univ. de Fribourg, Fribourg, Switzerland. *Route de Bertiguy 41*

BEACH, J. W., M.S. (Iowa State) Instr., Iowa State Coll., Ames, Iowa. *155 Campus Ave.*
BEAL, Asst. Prof. JUNA LUTZ (Mrs. A. G.), A.M. (Chicago) Butler Univ., Indianapolis 7, Ind. *3262 Broadway, Indianapolis 5, Ind.*
BEARD, Asst. Prof. HELEN P., Ph.D. (Mass. Inst. of Tech.) Newcomb Coll., New Orleans 18, La.
BEARMAN, Asst. Prof. J. E., Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *1817–2nd Ave., S., #202, Minneapolis 4, Minn.*
BEASLEY, Asst. Prof. S. LOUISE, Ed.M. (Peabody) Head of Dept., Carleton Coll., Northfield, Minn.
BEATLEY, Asso. Prof. RALPH, A.M. (Columbia) Harvard Grad. School of Educ., Cambridge 38, Mass.
BEATTY, Asst. Prof. H. M., A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *200 Tibet Road*
BEATTY, Prof. SAMUEL, Ph.D. (Toronto) Dean, Faculty of Arts, Univ. of Toronto, Toronto 5, Ont., Can. *Dept. of Math.*
BEATY, Prof. E. B., A.M. (California) Oregon State Coll., Corvallis, Ore.
BEAUMONT, Asst. Prof. R. A., Ph.D. (Illinois) Univ. of Washington, Seattle 5, Wash. *Dept. of Math.*
BEAVER, Prof. R. A., Ph.D. (Cornell) New York State Coll. for Teachers, Albany 3, N.Y.
BEBERMAN, MAX, B.S. (C.C.N.Y.) Teacher, Math. and Sci., Nome Territorial High School, Nome, Alaska. *P.O. Box 577*
BECKENBACH, Prof. E. F., Ph.D. (Rice) Univ. of California at Los Angeles, Los Angeles 24, Calif.
BECKER, H. W., *1214 N. 34th St., Omaha, Nebr.*
BECKWITH, MABLE L., A.M. (Claremont) High School, San Rafael, Calif. *418 Mission Ave.*
BECKWITH, Asso. Prof. W. S., A.M. (Harvard) Univ. of Georgia, Athens, Ga. *731 Cobb St.*
BEDWELL, T. H., A.B. (South Dakota) Instr., Physics, Univ. of South Dakota, Vermillion, S.D.
BEEGLE, Prof. B. L., M.S. (Washington) Dean, Seattle Pacific Coll., Seattle 99, Wash *3307 Third Ave. W.*
BEELER, Asso. Prof. F. A., A.M. (Indiana) Western Michigan Coll., Kalamazoo, Mich. *1554 Academy St., Kalamazoo 50, Mich.*
BEEMAN, W. E., M.S. (N. Tex. State T.C.) Instr., A. and M. Coll. of Texas, College Station, Tex. *703A E. 31st, Bryan, Tex.*
BEENKEN, MAY M., Ph.D. (Chicago) Chm. of Dept., Immaculate Heart Coll., Los Angeles 27, Calif. *1906 Park Ave., Los Angeles 26, Calif.*
BEER, Asst. Prof. F. P., Ph.D. (Geneva) Mechanics Dept., Lehigh Univ., Bethlehem, Pa.
BEESLEY, Asso. Prof. E. M., Ph.D. (Brown) Acting Head of Dept. Math. and Mech., Univ. of Nevada, Reno, Nev.
BEGLE, Asst. Prof. E. G., Ph.D. (Princeton) Yale Univ., New Haven, Conn. *24 Wakefield St., Hamden, Conn.*
BEINERT, R. L., A.B. (Hobart) Asst., Cornell Univ., Ithaca, N.Y. *White Hall*
BELGODERE, PAUL, Agrege de Math. (Paris, E.N.S.) Secretaire General, Intermediare des Recherches Mathematiques, 55 rue de Varenne, Paris 7, France
BELL, Asso. Prof. CLIFFORD, Ph.D. (California) Univ. of California at Los Angeles, Los Angeles 24, Calif.
BELL, Prof. E. T., Ph.D. (Columbia) California Inst. of Tech., Pasadena 4, Calif.
BELL, Asst. Prof. J. H., Ph.D. (Wisconsin) Michigan State Coll., East Lansing, Mich. *625 Cherry Lane*
BELL, R. F., M.S. (Michigan) Instr., Eastern Washington Coll. of Educ., Cheney, Wash.
BELL, Prof. TALMON, A.B. (Sterling) Sterling Coll., Sterling, Kans.
BELLAMY, B. C., B.S. (Wyoming) Civil Engineer, Laramie, Wyo. *Box 37, Lamont, Wyo.*
BELLARDO, J. E., M.S. (St. Bonaventure) Instr., Aquinas Coll., East Grand Rapids, Mich. *1607 Robinson Road*
BEMMELS, Prof. W. D., Ph.D. (Colorado) Math. and Physics, Ottawa Univ., Ottawa, Kans.
BENDER, Asso. Prof. H. A., Ph.D. (Illinois) Rhode Island State Coll., Kingston, R.I.
BENEDICTA, Sister M. (Boyle), M.S. (St. Bonaventure) Prof., Good Counsel Coll., White Plains, N.Y.
BENNER, Asso. Prof. J. A., A.M. (Lafayette) Lafayette Coll., Easton, Pa. *2 East Campus*
BENNETT, Prof. A. A., Ph.D. (Princeton) Brown Univ., Providence 12, R.I.
BENNETT, Prof. THEODORE, Ph.D. (Illinois) Marietta Coll., Marietta, Ohio
BERG, W. D., Ph.D. (Iowa) Visiting Asst. Prof., Kenyon Coll., Gambier, Ohio
BERGER, EDLA G., A.M. (Minnesota) Mathematician, Equitable Life Assur. Soc., New York, N.Y. *130 W. Twelfth St., New York 11, N.Y.*

BERGLUND, WINIFRED V., M.A. (Northwestern) Instr., Univ. of Illinois, Chicago, Ill. *120 Green Bay Rd., Hubbard Woods, Ill.*
BERKOFSKY, LOUIS, A.B. (Brooklyn) *690 Sheffield Ave., Brooklyn 7, N.Y.*
BERNARD, ALFRED, Brother (Welch), A.M. (Catholic Univ.) Head of Dept., Manhattan Coll., Spuyten Duyvil Parkway, New York 63, N.Y.
BERNARDI, S. D., M.S. (Notre Dame), M.Ed. (Springfield Coll.) Instr., New York Univ., New York, N.Y. *8 Eld St., New Haven, Conn.*
BERNHART, Asst. Prof. ARTHUR, Ph.D. (Michigan) Univ. of Oklahoma, Norman, Okla. *Faculty Exchange*
BERNSTEIN, Prof. B. A., Ph.D. (California) Univ. of California, Berkeley, Calif. *2785 Shasta Road, Berkeley 8, Calif.*
BERNSTEIN, Asst. Prof. DOROTHY L., Ph.D. (Brown) Univ. of Rochester, Rochester, N.Y.
BERRY, Prof. A. C., Ph.D. (Harvard) Lawrence Coll., Appleton, Wis.
BERRY, E. M., Ph.D. (Iowa) Head of Dept., State Teachers Coll., Chadron, Nebr., *353 Chapin St.*
BERRY, T. E., B.S. (George Washington) Analyst, Social Security Board, Washington, D.C. *200 Rhode Island Ave. N.E., Washington 2, D.C.*
BERRY, Prof. W. J., A.M. (Harvard) Retired, Poly. Inst. of Brooklyn, Brooklyn, N.Y. *The Berrypatch, P.O. 298, Stony Brook, N.Y.*
BERT, Prof. O. F. H., A.M. (Geneva) Washington and Jefferson Coll., Washington, Pa. *28 N. Lincoln St.*
BERTRAND, Sister M. (Walton), Ph.D. (Fordham) Teacher, Marywood Coll., Scranton, Pa.
BESSELL, Prof. W. W., Jr., C.E. (R.P.I.) Col., U. S. Military Acad., West Point, N.Y. *Dept. of Math.*
BETTINGER, Asst. Prof. A. K., A.M. (Wisconsin) Creighton Univ., Omaha, Nebr.
BETZ, WILLIAM, A.M. (Rochester) Specialist in math., Rochester public schools; Univ. of Rochester, Rochester, N.Y. *652 Melville St.*
BEVERIDGE, Prof. H. R., Ph.D. (Illinois) Monmouth Coll., Monmouth, Ill. *1043 E. Detroit Ave.*
BEY, Asst. Prof. D. R., A.M. (Illinois) Illinois State Normal Univ., Normal, Ill. *402 W. Locust St.*
BIBB, Asso. Prof. S. F., M.S. (Chicago) Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill.
BICKERSTAFF, Prof. T. A., A.M. (Mississippi) Univ. of Mississippi, University, Miss. *Box 7*
BICKLEY, JEANETTE A. (Mrs.), A.B. (Harris T. C.) Computer, General American Life Ins. Co., St. Louis, Mo. *2632 Louis Ave., Brentwood 17, Mo.*
BIESELE, Asst. Prof. F. C., Ph.D. (Texas) Univ. of Utah, Salt Lake City, Utah. *1876 S. Woodside St., Salt Lake City 7, Utah*
BIGELOW, W. W., A.B. (Beloit), B.S.C.E. (George Washington) Analyst, A. O. Smith Corp., Milwaukee, Wis. *Rockton, Ill.*
BIGGERSTAFF, J. S., M.S. (Washington) Instr., Rensselaer Poly. Inst., Troy, N.Y. *Dept. of Math.*
BILLIG, A. L., Ed.D. (Temple) Teacher, Science, High School, Allentown, Pa. *1328 Gordon St.*
BING, R. H., Ph.D. (Texas) Univ. of Wisconsin, Madison, Wis. *Dept. of Math.*
BINGLEY, G. A., A.M. (Princeton) Tutor, St. John's Coll., Annapolis, Md.
BIRCHBY, Asst. Prof. W. N., A.M. (Colorado Coll.) California Inst. of Tech., Pasadena, Calif.
BIRCHENOUGH, Prof. HARRY, A.M. (Columbia) New York State Coll. for Teachers, Albany 3, N.Y.
BIRD, Asst. Prof. M. T., Ph.D. (Illinois) San Jose State Coll., San Jose 14, Calif. *Dept. of Math.*
BIRKHOFF, Prof. GARRETT, A.B. (Harvard) Harvard Univ., Cambridge, Mass. *45 Fayerweather St.*
BISER, ERWIN, Ph.D. (Pennsylvania) Instr., Rutgers Univ., New Brunswick, N.J.
BISSINGER, B. H., Ph.D. (Cornell) Vice-Pres., Geo. Gillis Shoe Corp., Fitchburg, Mass.
BLACK, Prof. A. H., Ph.D. (Cornell) Lebanon Valley Coll., Annville, Pa. *440 Maple St.*
BLACK, Asso. Prof. FLORENCE L., Ph.D. (Kansas) Univ. of Kansas, Lawrence, Kans. *1300 Louisiana St.*
BLACK, Prof. H. L., Ph.D. (Illinois) Michigan State Coll., East Lansing, Mich. *209 Cedar St.*
BLACK, L. G., A.M. (Colorado) Instr., Purdue Univ., West Lafayette, Ind. *220 S. Salisbury St.*
BLACK, L. T., A.M. (Michigan) Instr., Long Beach City Coll., Long Beach, Calif. *1230 Hungerford St., Long Beach 5, Calif.*
BLACKALL, C. J., Ph.D. (Cornell) 2nd Lt., CAC. *North Baltimore, Ohio*

BLACKWELL, Asso. Prof., R. C., Ph.D. (North Carolina) Furman Univ., Greenville, S.C. *322 University Ridge*
BLAIR, Prof. HAROLD, A.M. (Michigan) Western Michigan Coll., Kalamazoo, Mich. *1220 Academy St., Kalamazoo 49, Mich.*
BLAIR, Asso. Prof. R. V., Ph.D. (Peabody) Vanderbilt Univ., Nashville 4, Tenn. *2421 Kensington Pl., Nashville 5, Tenn.*
BLAKE, ARCHIE, Ph.D. (Chicago) Sr. Statistician, Office of Army Surgeon General, Washington, D.C. *419 Oneida Place, N.E., Washington 11, D.C.*
BLAKE, R. G., A.B. (Florida) Instr., Univ. of Florida, Gainesville, Fla. *Route 2, Box 25*
BLAKELEY, LAURA, M.A. (Peabody) Teacher, Dayton Sr. High School, Dayton, Ky. *122 Forest Ave., Ft. Thomas, Ky.*
BLANCHE, E. E., Ph.D. (Illinois) Prin. Admin. Analyst, Chief, Statistics Branch, Research & Development Div., War Dept. General Staff, Washington, D.C. *9409 Montgomery Ave., North Chevy Chase 15, Md.*
BLEICK, Asso. Prof. W. E., Ph.D. (Johns Hopkins) U. S. Naval Acad., Annapolis, Md. *8 Taney Ave.*
BLAU, L. W., Ph.D. (Texas) Research Consultant, Humble Oil & Refining Co., Houston 1, Tex. *2027 Colquitt Ave.*
BLINCOE, Prof. J. W., Ph.D. (Virginia) Randolph-Macon Coll., Ashland, Va.
BLISS, Prof. G. A., Ph.D. (Chicago) Emeritus, Univ. of Chicago, Chicago, Ill. *Flossmoor, Ill.*
BLOOM, G. M., A.M. (Northwestern) Instr., Northwestern Univ., Evanston, Ill. *Dept. of Math.*
BLUMBERG, Asst. Prof. J. O., Ph.D. (Pittsburgh) Univ. of Pittsburgh, Pittsburgh, Pa. *308 Foster St., Greensburg, Pa.*
BLUMENTHAL, Prof. L. M., Ph.D. (Johns Hopkins) Univ. of Missouri, Columbia, Mo. *Dept. of Math.*
BLYTH, C. R., B.A. (Queen's Univ.) Student, Univ. of North Carolina, Chapel Hill, N.C. *205 Mangum Dorm*
BOARDMAN, H. C., D.E. (S. D. St. Sch. of Mines) Dir. of Research, Chicago Bridge and Iron Co., Chicago, Ill. *10357 S. Hoyne Ave., Chicago 43, Ill.*
BOAS, R. P., Jr., Ph.D. (Harvard) Executive Editor, Mathematical Reviews. *21 Chauncy St., Cambridge 38, Mass.*
BOATMAN, Prof. A. O., A.M. (Indiana) Carthage Coll., Carthage, Ill.
BOCK, W. W., M.S. (Colorado) Student Health Serv., Univ. of Colorado, Boulder, Colo.
BOEDER, PAUL, Ph.D. (Gottingen) Dir., Bureau of Visual Science, American Optical Co., Southbridge, Mass.
BOEHM, Sister ADA MARIE, B.A. (Coll. of St. C.) Teacher, The Coll. of St. Catherine, St. Paul 1, Minn.
BOEHM, FRANK, Manager, Life Insurance, 80 Maiden Lane, New York 7, N.Y.
BOEHMER, Sister M. MIRABELLA, M.S. (Catholic Univ.) Instr., Math. and Physics, Alverno Teachers Coll., 1413 S. Layton Blvd., Milwaukee 4, Wis.
BOHNENBLUST, Prof. H. F., Ph.D. (Princeton) California Inst. of Tech., Pasadena 4, Calif. *Dept. of Math.*
BOLDYREFF, Asso. Prof. A. W., Ph.D. (Michigan) Univ. of New Mexico, Albuquerque, N.M.
BOLKS, STANLEY, M.S. (Iowa State) Instr., Purdue Univ., West Lafayette, Ind. *R.R. 10, Lafayette, Ind.*
BOLTON, GRACE L., A.M. (Columbia) Instr., Barnard Coll., Columbia Univ., New York 27, N.Y.
BORGMAN, Asso. Prof. W. M., Jr., Ph.D. (Chicago) Wayne Univ., Detroit 1, Mich.
BORNMANN, W. C., A.M. (Columbia) Grad. Student, Yale Univ., New Haven, Conn.
BOROFSKY, Asst. Prof. SAMUEL, Ph.D. (Columbia) Brooklyn Coll., Brooklyn, N.Y. *335 State St., Brooklyn 2, N.Y.*
BOSWELL, Prof. A. V., A.M. (Northwestern) Tennessee A. and I. State Coll., Nashville 8, Tenn.
BOTTS, Asst. Prof. T. A., Ph.D. (Virginia) Univ. of Delaware, Newark, Del. *Dept. of Math.*
BOURNE, S. G., A.M. (Johns Hopkins) Instr., Johns Hopkins Univ., Baltimore 18, Md. *Dept. of Math.*
BOUTELLE, Asst. Prof. H. D., B.S. in Ch.E. (Worcester Poly. Inst.) Massachusetts State Coll., Amherst, Mass. *10 Allen St.*
BOWDEN, Prof. JOSEPH, Ph.D. (Yale) Emeritus, Adelphi Coll., Garden City, N.Y. *21 Carleton Pl., Baldwin, N.Y.*
BOWER, Asso. Prof. JULIA W., Ph.D. (Chicago) Connecticut Coll., New London, Conn. *Dept. of Math.*
BOWKER, Prof. J. G., Ed.M. (Harvard) Chm. of Dept., Middlebury Coll., Middlebury, Vt. *14 Adirondack View*

BOWLING, Asso. Prof. FLOYD, M.S. (Iowa) Lincoln Memorial Univ., Harrogate, Tenn.
BOYCE, Asso. Prof. FANNIE W., Ph.D. (Chicago) Wheaton Coll., Wheaton, Ill. *311 E. Wesley St.*
BOYCE, Prof. JESSIE W., A.M. (Minnesota) Chm. of Dept., State Teachers Coll., Wayne, Nebr. *518 Lincoln St.*
BOYCE, Asso. Prof. M. G., Ph.D. (Chicago) Vanderbilt Univ., Nashville 4, Tenn. *Box 152 Univ.*
BOYD, Dean P. P., Ph.D. (Cornell) Head of Dept., Math. and Astr., Univ. of Kentucky, Lexington, Ky.
BOYER, Asst. Prof. C. B., Ph.D. (Columbia) Brooklyn Coll., Bedford Ave. and Avenue H, Brooklyn 10, N.Y.
BOYER, L. E., Ed.D. (Pennsylvania State) State Teachers Coll., Millersville, Pa. *406 N. George St.*
BOYLE, Sister M. RITA CECILE, M.S. (Michigan) Asst. Prof., Barry Coll., Miami 38, Fla.
BRACEWELL, Prof. K. H., Ph.D. (Indiana) Math. and Physics, Hamline Univ., St. Paul 4, Minn.
BRADFIELD, G. F., M.S. (Northwestern Univ.) Instr., Illinois Inst. of Tech.; Northwestern Univ., Evanston, Ill. *2125 Ridge Ave.*
BRADFORD, Asst. Prof. W. H., M.S. (Louisiana) McNeese Jr. Coll. of L.S.U., Lake Charles, La.
BRADLEY, Asst. Prof. A. D., Ph.D. (Columbia) Hunter Coll., New York, N. Y. *66 Villard Ave., Hastings-on-Hudson 6, N.Y.*
BRADLEY, R. A., B.A. (Queen's Univ.) Univ. of North Carolina, Chapel Hill, N.C. *Phillips Hall*
BRADSHAW, Prof. J. W., Ph.D. (Strassburg) Univ. of Michigan, Ann Arbor, Mich. *1304 Cambridge Road*
BRADY, R. P., M.S. (Chicago) *5810 Harper Ave., Chicago 37, Ill.*
BRADY, W. G., B.S. (Washington and Jefferson) Instr., Univ. of Wyoming, Laramie, Wy. *Dept. of Math.*
BRAMBLE, C. C., Ph.D. (Johns Hopkins) Director of Computation and Ballistics, Naval Proving Ground, Dahlgren, Va.
BRAMBLETT, Asst. Prof. INA M., A.M. (Texas) Texas Christian Univ., Fort Worth, Tex. *2828 W. Lowden St., Fort Worth 4, Tex.*
BRAND, Prof. LOUIS, Ph.D. (Harvard) Chm. of Dept., Univ. of Cincinnati, Cincinnati 21, Ohio
BRANDEBERRY, Prof. J. B., Ph.D. (Michigan) Dean, Coll. of Engg., Univ. of Toledo, Toledo 6, Ohio
BRANDNER, Asst. Prof. F. A., A.M., M.S. (Chicago) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
BRANSON, Asso. Prof. HERMAN, Ph.D. (Cincinnati) Physics, Howard Univ., Washington, D.C.
BRANSON, Dean J. W., M.S. (Purdue) Acting Pres., New Mexico Coll. of A. and M.A., State College, N.M.
BRAUER, Asso. Prof. A. T., Ph.D. (Berlin) Univ. of North Carolina, Chapel Hill, N.C. *410 Patterson Pl.*
BRAUER, Prof. RICHARD, Ph.D. (Berlin) Univ. of Toronto, Toronto, Ont., Can. *Dept. of Math.*
BRAVERMAN, BENJAMIN, A.M. (Columbia) Chm. of Dept., Seward Park High School, New York, N.Y. *1309 Avenue L, Brooklyn, N.Y.*
BRAY, Prof. H. E., Ph.D. (Rice) Rice Inst., Houston, Tex.
BRENEMAN, FRANCES M., M.S. (Kansas St. T. C.) Instr., Washburn Univ., Topeka, Kans. *R.R. 5*
BRENKE, Prof. W. C., Ph.D. (Harvard) Emeritus, Univ. of Nebraska, Lincoln, Nebr. *1250 S. 21st St.*
BREWSTER, Asst. Prof. J. P., A.M. (Duke) Clemson Agric. Coll., Clemson, S.C. *Box 14*
BRIANT, R. C., Ph.D. (Pittsburgh) Sr. Indus. Fellow, Mellon Inst., Pittsburgh, Pa. *403 Glasgow Road, Pittsburgh 21, Pa.*
BRIDGER, C. A., M.S. (Oregon State Coll.) Instr., Univ. of Utah, Salt Lake City 1, Utah. *336 Douglas St., Salt Lake City 2, Utah*
BRIGHT, H. F., M.S. (Rochester) Dir. of Guidance, San Angelo Coll., San Angelo, Tex. *1607 Shafter St.*
BRINK, Prof. R. W., Ph.D. (Harvard) Chm. of Dept., Univ. of Minnesota, Minneapolis 14, Minn. *2243 Hoyt Ave., St. Paul 8, Minn.*
BRINKMAN, Asso. Prof. H. W., Ph.D. (Harvard) Swarthmore Coll., Swarthmore, Pa.
BRITTON, Asso. Prof. J. R., Ph.D. (Colorado) Univ. of Colorado, Boulder, Colo.

BRIXEY, Asso. Prof. J. C., Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla. *927 S. Pickard St.*

BROCK, PAUL, A.B. (Brooklyn) Instr., Hunter Coll., New York, N.Y. *2338–63rd St., Brooklyn 4, N.Y.*

BRODERICK, Prof. T. S., M.A. (Dublin) Trinity Coll., Dublin, Ireland

BRONSTEIN, SAMUEL, A.M. (New York Univ.) Teacher, High School, Hartford, Conn. *56 Adams St., Hartford 5, Conn.*

BROOKE, Prof. W. E., A.M. (Nebraska) Emeritus, Math. and Mech., Univ. of Minnesota, Minneapolis 14, Minn. *3135–22nd Ave. S., Minneapolis 7, Minn.*

BROOKS, Asso. Prof. FOSTER, Ph.D. (Ohio State) Kent State Univ., Kent, Ohio. *P.O. Box 482*

BROTHERS, Prof. W. H., Jr., Ph.D. (Michigan) Talladega Coll., Talladega, Ala.

BROWN, Asst. Prof. A. B., Ph.D. (Harvard) Queens Coll., Flushing, N.Y.

BROWN, Prof. B. H., Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H. *7 Ripley Road*

BROWN, Asso. Prof. B. K., A.M. (Colorado) James Millikin Univ., Decatur 24, Ill. *R. #7*

BROWN, Asso. Prof. C. H., Ph.D. (Kansas) Central Missouri State Teachers Coll., Warrensburg, Mo. *407 S. Maguire St.*

BROWN, Asso. Prof. D. M., Ph.D. (Univ. of Ill.) Central Michigan Coll., Mt. Pleasant, Mich.

BROWN, Prof. E. C., A.M. (Maine) Worcester Poly. Inst., Worcester, Mass. *51 Grafton St., Shrewsbury, Mass.*

BROWN, Asst. Prof. F. R., M.A. (T. C. Columbia Univ.) James Millikin Univ., Decatur, Ill. *1736 N. Maple Ave.*

BROWN, H. E., A.B. (Maine) Teacher, Morse High School, Bath, Me. *19 Bedford St.*

BROWN, Asst. Prof. H. K., Ph.D. (Michigan) Northeastern Univ., Boston, Mass. *87 Harvard Ave., Brookline, Mass.*

BROWN, Prof. H. S., M.S. (Lafayette) Emeritus, Hamilton Coll., Clinton, N.Y. *Colgate Inn, Hamilton, N.Y.*

BROWN, J. W., B.S. (North Carolina State) Instr., Clemson Coll., Clemson, S.C. *Calhoun Road*

BROWN, Prof. LILLIAN O., A.M. (Columbia) Hood Coll., Frederick, Md.

BROWN, Asst. Prof. M. C., A.M. (Kentucky) Univ. of Kentucky, Lexington, Ky. *448 Clifton Ave.*

BROWN, Asso. Prof. MYRTLE C., A.M. (Texas) North Texas State Teachers Coll., Denton, Tex. *1415 W. Oak St.*

BROWN, Asst. Prof. M. D., M.A. (Geo. Peabody C. T.) Macalester Coll., St. Paul 5, Minn. *139 Amherst St.*

BROWN, Prof. O. E., Ph.D. (Chicago) Case School of Appl. Sci., Cleveland, Ohio. *1387 East Blvd.*

BROWN, R. D., Jr., Engr. in charge, Patent Dept., Philco Corp., Philadelphia, Pa. *Blue Bell, Montgomery Co., Pa.*

BROWN, Prof. R. E., M.S. (Mass. Inst. of Tech.) Mech. Engg., Rhode Island State Coll., Kingston, R.I. *Box 211*

BROWN, Prof. T. H., Ph.D. (Yale) Business Statistics, Harvard Grad. School of Bus. Admin., Boston, Mass. *25 Meadow Way, Cambridge, Mass.*

BROWN, W. B., Ph.D. (Ohio State) Instr., Fenn Coll., Cleveland, Ohio. *13983 Clifton Blvd., Lakewood 7, Ohio*

BROWNE, D. H., *Indian Orchard Pl., Buffalo 10, N. Y.*

BROWNE, Prof. E. T., Ph.D. (Chicago) Univ. of North Carolina, Chapel Hill, N.C. *730 E. Franklin St.*

BRUCE, Prof. C. W., Ph.D. (Virginia) Wesleyan Coll., Macon, Ga.

BRUCE, Prof. R. E., Ph.D. (Boston Univ.) Emeritus, Boston Univ., Boston, Mass.

BRUCK, Asso. Prof. R. H., Ph.D. (Toronto) Univ. of Wisconsin, Madison Wis. *3003 Stevens St. Madison 5, Wis.*

BRUNK, H. D., Ph.D. (Rice) Instr., Rice Inst., Houston, Tex.

BRYAN, Prof. N. R., Ph.D. (Columbia) Univ. of Georgia, Athens, Ga. *Dept. of Math.*

BRYSON, A. M., A.M. (Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh, Pa. *McCutcheon Lane, Wilkinsburg, Pa.*

BUCHANAN, Asso. Prof. ANN S., A.M. (Louisiana State) Southwestern Louisiana Inst., Lafayette, La. *1112 Lee Ave.*

BUCHANAN, Prof. DANIEL, Ph.D. (Chicago) Dean of Faculty of Arts and Sci., Univ. of British Columbia, Vancouver, B.C., Can.

BUCHANAN, Prof. H. E., Ph.D. (Chicago) Tulane Univ., New Orleans 15, La.

BUCHANAN, SCOTT, Ph.D. (Harvard) The Parsonage, Richmond, Mass.

BUCK, ELSIE M., Ph.D. (California) Hd. of Dept., Boise J. Coll., Boise, Idaho
BUCK, Asst. Prof. R. C., A.M. (Cincinnati) Brown Univ., Providence, R.I. *31 Ontario St.*
BUELL, E. L., Ph.D. (Mass. Inst. of Tech.) Instr., Northwestern Univ., Evanston, Ill. *Dept. of Math.*
BUELOW, ELSA M., A.M. (Loyola Univ.) Teacher, Chicago City Coll., Wright Branch, Chicago, Ill. *3449 Pierce Ave., Chicago 51, Ill.*
BUIKSTRA, B. H., M.S. (Kansas State Col.) Instr., U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
BUKER, W. E., A.M. (Ohio State) Teacher, Public School, Pittsburgh, Pa. *3833 Oswego St.*
BULLARD, Prof. J. A., Ph.D. (Clark) Chm. of Dept., Math. and Mech., Univ. of Vermont, Burlington, Vt. *110 Summit St.*
BULLITT, W. M., B.S. (Princeton) Bullitt and Middleton, Kentucky Home Life Bldg., Louisville 2, Ky.
BULLOCK, Asso. Prof. R. C., Ph.D. (Chicago) North Carolina State Coll., Raleigh, N.C. *Box 5548, State College Sta.*
BUMER, C. T., Ph.D. (Ohio State) Head Mathematician, Evaluation and Development Div., Bureau of Ordnance, Navy Dept., Washington, D.C. *11 S. Columbus St., Arlington, Va.*
BUNCH, W. H., A.M. (Oregon) Head of Dept., High School, Adrian, Ore. *Box 282*
BUNYAN, Asso. Prof. L. H., Ph.D. (Wisconsin) Rutgers Univ., New Brunswick, N.J.
BURCKHARDT, Prof. J. J., Univ. of Zurich, Zurich, Switzerland. *Bergheimstrasse 4. Zurich 32, Switzerland*
BURDETTE, Asst. Prof. A. C., Ph.D. (Illinois) Univ. of California, Coll. of Agric., Davis, Calif. *Div. of Math. and Physics*
BURDICK, O. Z., M.S. (Illinois) Lawyer, 104 E. Washington St., Hart, Mich.
BURGESS, R. W., Ph.D. (Cornell) Chief Statistician, Western Electric Co., 195 Broadway, New York 7, N.Y.
BURINGTON, R. S., Ph.D. (Ohio State) Head Mathematician, Bureau of Ordnance, Navy Dept., Washington, D.C. *5200 N. Carlin Spring Road, Arlington, Va.*
BURK, Asso. Prof. J. D., B.A. (Toronto) Univ. of Toronto, Toronto 5, Ont., Can.
BURKE, Rev. J. G., A.M. (Mt. St. Mary) Mount St. Mary Coll., Emmitsburg, Md.
BURKE, Sister LEONARDA, Ph.D. (Catholic Univ.) Prof., Regis Coll., Weston 93, Mass. *Wellesley St.*
BURKETT, Asso. Prof. F. J. H., Ph.D. (New York Univ.) Union Coll., Schenectady, N.Y. *1030 Park Ave., Schenectady 8, N.Y.*
BURNAM, Prof. J. E., A.M. (Texas) Hardin-Simmons Univ., Abilene, Tex. *1141 Grape St.*
BURNS, G. P., M.S. (West Virginia) Research Physicist, Naval Research Lab., Washington 20, D.C. *2408 Minnesota Ave. S.E.*
BURNS, Asst. Prof. H. E., Ph.D. (Northwestern) Dir., Indiana Univ. Exten., Calumet Center, East Chicago, Ind.
BURNS, Sister MARY ROBERDETTE, A.M. (Illinois) Asso. Prof., Clarke Coll., Dubuque, Iowa
BURNS, Rev. W. F., M.S. (Boston Coll.) Asso. Prof., Coll. of the Holy Cross, Worcester, Mass.
BURR, Asso. Prof. I. W., Ph.D. (Michigan) Purdue Univ., West Lafayette, Ind. *265 Littleton St.*
BURT, MARY M. (Mrs. L. N.), A.M. (Wisconsin) *800–24th Ave., N., St. Petersburg 4, Fla.*
BURTON, L. P., A.M. (Alabama) Instr., Univ. of Alabama, University, Ala. *Box 829*
BURWELL, W. R., Ph.D. (Oxford) Chm. of the Board, Brush Devel. Co., 3311 Perkins Ave., Cleveland 14, Ohio
BUSEMAN, Asst. Prof. HERBERT, Ph.D. (Gottingen) Univ. of So. Calif., Los Angeles 7, Calif. *Dept. of Math.*
BUSH, Asso. Prof. K. A., M.A. (Columbia) Resident Head of Math., Mohawk Coll., Utica, N.Y.
BUSH, Prof. L. E., Ph.D. (Ohio State) Chm. of Dept., Coll. of St. Thomas, St. Paul 1, Minn.
BUSHEY, Asso. Prof. J. H., Ph.D. (Michigan) Hunter Coll., 695 Park Ave., New York 21, N.Y.
BUSHEY, Asso. Prof. JEWELL H. (Mrs. J. H.), Ph.D. (Chicago) Hunter Coll., New York, N. Y. *501 West 113 St., New York 25, N.Y.*
BUSHYAGER, G. R., A.M. (Pennsylvania State) *1221 Morningside Ave., Sioux City 20, Iowa*
BUSSEY, Prof. W. H., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis 14, Minn. *1421 E. River Road*
BUTCHART, Prof. J. H., Ph.D. (Illinois) Arizona State Coll., Flagstaff, Ariz.
BUTLER, Asso. Prof. C. H., Ph.D. (Missouri) Western Michigan Coll., Kalamazoo, Mich. *1941 Stevens Ave., Kalamazoo 37, Mich.*
BUTLER, L. G., A.M. (Oregon) Instr., State Coll. of Washington, Pullman, Wash. *1904 B St.*

BUTTER, F. A., Jr., Ph.D. (Stanford) Research Physicist, Hughes Aircraft Co., Culver City, Calif. *1030 Embury St., Pacific Palisades, Calif.*
BUTTERFIELD, Prof. A. D., A.M. (Columbia) Emeritus, Dir., Veteran's Educ., Univ. of Vermont, Burlington, Vt. *479 Main St.*
BUXTON, Asso. Prof. C. L., M.S. (Case School) Physics, Clarkson Coll. of Tech., Potsdam, N.Y. *91 Market St.*
BYRD, P. F., M.S. (Chicago) 1st Lt., U. S. Army Air Corps, Weather Detachment, Lockbourne AAB, Columbus 17, Ohio
BYRNE, LEE, Ph.D. (Columbia) Instr., Purdue Univ., Lafayette, Ind. *6041 Dorchester Ave., Chicago 37, Ill.*
BYRNE, MARGARET C., A.M. (Columbia) St. Joseph's Coll. for Women, Brooklyn 5, N.Y. *431 Beach 142 St., Rockaway Park, N.Y.*
BYRNE, Prof. W. E., Ph.D. (Rensselaer) Virginia Military Inst., Lexington, Va. *Box 836*

CAIN, Asso. Prof. W. H., A.M. (Columbia) Western Michigan Coll., Kalamazoo, Mich. *1402 Hillcrest Ave., Kalamazoo 39, Mich.*
CAIRNS, Prof. S. S., Ph.D. (Harvard) Chm. of Dept., Syracuse Univ., Syracuse, N.Y. *1050 Ackerman Ave., Syracuse 10, N.Y.*
CAIRNS, Prof. W. D., Ph.D. (Gottingen) Honorary Life Member; Emeritus, Oberlin Coll., Oberlin, Ohio. *1675 Kenneth Way, Pasadena 3, Calif.*
CALCAGNO, H. E., *Patria, 715, Montevideo, Uruguay*
CALKIN, Asso. Prof. J. W., Ph.D. (Harvard) Rice Inst., Houston, Tex. *Dept. of Math.*
CALKINS, Asst. Prof. ELEANOR, A.M. (Michigan) Coll. of William and Mary, Williamsburg, Va. *608 Jamestown Road*
CALKINS, Prof. HELEN, Ph.D. (Cornell) Head of Dept., Pennsylvania Coll. for Women, Pittsburgh 6, Pa.
CALLAHAN, Prof. ETHEL B., Ph.D. (Columbia) Hartwick Coll., Oneonta, N.Y. *122 Chestnut St.*
CALLAWAY, Asso. Prof. IRIS, A.M. (Peabody) Univ. of Georgia, Athens, Ga. *317 Franklin St.*
CALVERT, Asst. Prof. R. L., A.M. (Illinois) Utah State Agric. Coll., Logan, Utah. *751 No. 2nd East*
CAMERON, Prof. E. A., Ph.D. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *121 Kenan St.*
CAMERON, Prof. R. H., Ph.D. (Cornell) Univ. of Minnesota, Minneapolis, Minn. *819 Superior St.*
CAMP, Prof. B. H., Ph.D. (Yale) Wesleyan Univ., Middletown, Conn. *110 Mt. Vernon St.*
CAMP, Prof. C. C., Ph.D. (Cornell) Univ. of Nebraska, Lincon. 8, Nebr. *Dept. of Math.*
CAMP, Prof. E. J., Ph.D. (Chicago) Macelester Coll., St. Paul 5, Minn.
CAMPAIGNE, H. H., Ph.D. (Northwestern) Mathematician, Navy Dept., Washington, D. C. *210 N. Granada St., Arlington, Va.*
CAMPBELL, D. F., Ph.D. (Harvard) Consulting Actuary, 160 N. LaSalle St., Chicago, Ill.
CAMPBELL, Asso. Prof. FRANCES L., Ph.D. (Michigan) George Pepperdine Coll., 1121 W. 79th St., Los Angeles 44, Calif.
CAMPBELL, G. A., Ph.D. (Harvard) Retired, American Tel. and Tel. Co., New York, N.Y. *129 Bellevue Ave., Upper Montclair, N. J.*
CAMPBELL, Asst. Prof. J. D., Ph.D. (Illinois) Rensselaer Poly. Inst., Troy, N.Y. *Dept. of Math.*
CAMPBELL, JESSIE R., A.B. (Syracuse) Instr., High School, Hollywood, Calif. *12 The Strand, Hermosa Beach, Calif.*
CAMPBELL, Prof. J. W., Ph.D (Chicago) Univ. of Alberta, Edmonton South, Alta., Can. *Dept. of Math.*
CAMPBELL, Asst. Prof. W. B., A.M. (Cornell) Philadelphia Textile Inst., Philadelphia, Pa. *6848 N. Seventh St., Philadelphia 26, Pa.*
CANADA, Asst. Prof. GRAÇE M., A.M. (Columbia) East Central State Coll., Ada, Okla. *905 S. Townsend St.*
CANNING, JOSEPH, A.B. (Intermountain) Roundhouse Foreman, Northern Pacific R.R., Garrison, Mont.
CANFIELD, E. L., A.M. (Northwestern) Instr., Drake Univ., Des Moines, Iowa. *4110 S.W. Fifth St.*
CAPARO, Prof. J. A., Ph.D. (Notre Dame) Emeritus, Elec. Engg., Univ. of Notre Dame, Notre Dame, Ind. *1024 Leeper Blvd., South Bend 17, Ind.*
CAPESIUS, Rev. JOHN, A.M. (Alabama) Dean, St. Bernard Coll., St. Bernard, Ala.
CARACA, Prof. B. DEJ., (Ci.Ec.e Fin.) Inst. Superior de Cien., Econ. e Finan., Univ. Tecnica de Lisboa, Lisbon, Portugal. *Rua Almedia e Sousa 63, 1 andar*

CAREY, Asso. Prof. E. F.A., M.S. (California) Emeritus, Montana State Univ., Missoula, Mont.

CARIS, Asso. Prof. P. A., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia 4, Pa. *717 Shadeland Ave., Drexel Hill. Pa.*

CARIS, Asst. Prof. V. B., A.M. (Defiance Coll.) Ohio State Univ., Columbus 2, Ohio. *99 W. Weber Road*

CARLEN, MILDRED E., M.S. (Brown) Registrar, Grad. School, Brown Univ., Providence 12, R.I.

CARLSON, Prof. C. S., M.S. (Iowa) Head of Dept., St. Olaf Coll., Northfield, Minn.

CARLSON, Asst. Prof. ELIZABETH, Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *3024–14th Ave. S., Minneapolis 7, Minn.*

CARLSON, Prof. J. A., M.S. (Washington) Head of Dept., Whitworth Coll., Spokane, Wash.

CARLTON, Asst. Prof. L. VIRGINIA, A.M. (Tulane) Centenary Coll., Shreveport, La.

CARMAN, M. G., Ph.D. (Illinois) Head of Dept., Murray State Teachers Coll., Murray, Ky.

CARMICHAEL, Prof. F. L., A.M. (Princeton) Statistics, School of Commerce, Univ. of Denver, Glenarm Place at 20th St., Denver, Colo.

CARMICHAEL, Prof. R. D., Ph.D. (Princeton) Emeritus Dean of Grad. School, Univ. of Illinois, Urbana, Ill. *207 W. Washington Blvd.*

CARNAHAN, Asst. Prof. W. H., A.M. (Indiana) Mathematics Editor, D. C. Heath & Co., 285 Columbus Ave., Boston 16, Mass.

CARPENTER, Prof. D. R., A.M. (Princeton) Roanoke Coll., Salem, Va.

CARPENTER, Asst. Prof. F. M., A.M. (Illinois) Colorado School of Mines, Golden, Colo. *Dept. of Math.*

CARPENTER, J. A., A.B. (North Carolina) Actuarial Clerk, Pilot Life Ins. Co., Box P, Greensboro, N.C.

CARPENTER, Asso. Prof. P. N., M.S. (Univ. of Washington) Grove City Coll., Grove City, Pa. *214 E. Pine St.*

CARR, Prof. F. E., Ph.D. (Chicago) Math. and Astr., Oberlin Coll., Oberlin, Ohio. *284 Forest St.*

CARR, R. E., Ph.D. (Iowa State) Instr., Michigan State Coll., East Lansing, Mich. *Dept. of Math.*

CARROLL, Asso. Prof. I. S., A.M. (Columbia) Syracuse Univ., Syracuse 10, N.Y. *511 Comstock Ave.*

CARSCALLEN, Asso. Prof. G. E., A.M. (Illinois) Wabash Coll., Crawfordsville, Ind. *112 N. Barr St.*

CARSON, Prof. A. B., Ph.D. (Chicago) Army Air Forces Inst. of Tech., Wright Field, Dayton, Ohio. *Hq. A.M.C., TSAIT*

CARSON, Prof. T. C., A.M. (Duke) East Tennessee State Coll., Johnson City, Tenn.

CARTER, Asst. Prof., H. N., B.S. (Northeastern Okla. St. C.) Univ. of Tulsa, Tulsa 4, Okla.

CARVER, Prof. W. B., Ph.D. (Johns Hopkins) Cornell Univ., Ithaca, N.Y. *White Hall*

CASE, Rev. J. E., Ph.D. (Chicago) Head of Dept., St. Louis Univ., St. Louis, Mo.

CASEY, ELLA F., (Mrs. N. E.), M.Ed. (Rochester) Instr., Sampson Coll., Sampson, N.Y.

CASKEY, Asst. Prof. R. L., A.M. (Oklahoma) Oklahoma A. and M. Coll., Stillwater, Okla. *Route 4*

CASSEL, C. W., Jr., A.B. (Wittenberg Coll.) Instr., Univ. of Dayton, Dayton, Ohio. *86½ Mary Ave., Dayton 5, Ohio*

CATHEY, ELIZABETH C., M.S. (Louisiana State) Instr., Univ. of Alabama, University, Ala. *Box 788, Tuscaloosa, Ala.*

CATON, Asst. Prof. W. B., Ph.D. (Yale) Washington State Coll., Pullman, Washington. *Dept. of Math.*

CAULUM, Prof. L. L., M.A. (Columbia) Sampson Coll., Sampson, N.Y.

CAVALLI, S. F., D.Sc. (Univ. of Milan, Italy) Consulting Engr., Philco Corp., Philadelphia, Pa. *271 S. 58th St., Philadelphia 39, Pa.*

CAVANAGH, Sister M. ROSE AGNES, M.A. (Catholic Univ.) Instr., Salve Regina Coll., Ochre Point Ave., Newport, R.I.

CAWLEY, Asso. Prof. JOHN, M.S. (Lafayette) Lafayette Coll., Easton, Pa. *621 Coleman St.*

CEBULA, Rev. RICHARD, M.S. (Michigan) Instr., St. Martin's Coll., Lacey, Wash. *On leave. Univ. of Michigan, Ann Arbor, Mich. 326 Ingalls St., Ann Arbor, Mich.*

CEDERBERG, Prof. W. E.., Ph.D. (Wisconsin) Augustana Coll., Rock Island, Ill. *2542–22½ Ave.*

CELAURO, Asst. Prof. F. L., A.M. (New York Univ.) Newark Coll of Engg., Newark 2, N.J. *Dept. of Math.*

CELL, Prof. J. W., Ph.D. (Illinois) North Carolina State Coll., Raleigh, N.C. *Box 5548, State Coll. Sta.*

CHACALOS, L. H., Student, Univ. of Michigan, Ann Arbor, Mich. *534 Hill St.*

CHANLER, Asst. Prof. JOSEPHINE H., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *364 Math. Bldg.*

CHAPMAN, Asst. Prof. D.G., A.M. (California) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*

CHARLESWORTH, G. B., M.A. (Cambridge) Instr., Hofstra Coll., Hempstead, N.Y.

CHAROSH, MANNIS, M.S. (New York Univ.) Teacher, New Utrecht High School, Brooklyn, N.Y. *8305–19th Ave., Brooklyn 14, N.Y.*

CHASE, L. R., Teacher, Rogers High School, Newport, R.I. *Boulevard Terrace*

CHATELAIN, VIRGINIA L. (Mrs.), M.S. (Kansas St. T. C.) Instr., Kansas State Teachers Coll., Manhattan, Kans. *1429 Laramie, Apt. 6*

CHELLEVOLD, Asst. Prof. J. O., A.M. (Northwestern) Lehigh Univ., Bethlehem, Pa. *419 W. 119 St., New York 27, N. Y.*

CHENEY, Prof. W. F., Jr., Ph.D. (Mass. Inst. of Tech.) Head of Dept., Univ. of Connecticut, Storrs, Conn. *P.O. Box 26*

CHERNOFSKY, M. I., M.B.A. (New York Univ.) *1419 Carroll St., Brooklyn 13, N.Y.*

CHESNA, JOHN, Physicist, Eastman Kodak Co., Rochester, N.Y. *12 Otilla St., Rochester 5, N.Y.*

CHESSIN, P. L., M.A. (Univ. of Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *316 West 93 St., New York 25, N.Y.*

CHIN, LOUISE H., A.M. (California) Asso., Univ. of California, Berkeley, Calif. *2008 Cedar St., Berkeley 7, Calif.*

CHITTENDEN, Prof. E. W., Ph.D. (Chicago) Univ. of Iowa, Iowa City, Iowa. *221 Physics Bldg.*

CHRISTIAN, H. S., Jr., *2737 Eastwood Ave., Evanston, Ill.*

CHRISTIE, Asst. Prof. D. E., Ph.D. (Princeton) Bowdoin Coll., Brunswick, Me. *36 Boody St.*

CHRISTMAN, LAURA E., A.M. (Wisconsin) Teacher, Senn High School, Chicago, Ill. *1217 Elmdale Ave.*

CHURCHILL, Prof. R. V., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *924 S. Forest Ave.*

CHURCH, Prof. RANDOLPH, Ph.D. (Yale) Lt. Comdr., U.S.N.R., Postgrad. School, U. S. Naval Acad., Annapolis, Md. *316 N. Glen Ave.*

CIVIN, Asst. Prof. PAUL, Ph.D. (Duke) Univ. of Oregon, Eugene, Ore. *Dept. of Math.*

CLARE, Asso. Prof. JOSEPH, M.Eng. (Liverpool, England) Knox Coll., Galesburg, Ill.

CLARK, Prof. A. G., A.M. (Colorado) Head of Dept., Colorado State Coll. of A. and M.A., Fort Collins, Colo.

CLARK, B. G., Ph.D. (Illinois) Vanderbilt Univ., Nashville, Tenn. *Hill Rd., Brentwood, Tenn.*

CLARK, Asst. Prof. C. L., Ph.D. (Virginia) Oregon State Coll., Corvallis, Ore. *Dept. of Math.*

CLARK, C. R., A.M. (Michigan) Teacher, McKinley High School, Washington, D.C. *2707 Adams Mill Road N.W.*

CLARK, F. E., A.M. (Duke) Grad. Student, Duke Univ., Durham, N.C. *Box 4238 Duke Sta.*

CLARK, J. J., M.A. (New York Univ.) Instr. Sewanhaka High School, Floral Park; Adelphi Coll., Garden City, N.Y. *607 Euston Rd. S., West Hempstead, N.Y.*

CLARK, K. L., Ph.D. (Wisconsin) Mathematician, A. O. Smith Corp., Milwaukee, Wis. *2331 N. 25th St., Milwaukee 6, Wis.*

CLARKE, Prof. E. H., Ph.D. (Chicago) Hiram Coll., Hiram, Ohio

CLARKE, F. MARION, A.M. (Smith) Lecturer, Univ. of Southern California, Los Angeles, Calif. *3789 Menlo Ave., Apt. 204, Los Angeles 7, Calif.*

CLARKE, Asst. Prof. L. BEATRICE (Mrs.), A.M. (Michigan) Florida A. and M. Coll., Tallahassee, Florida. *Box 111.*

CLARKE, Sister M. BORGIA, A.M. (Catholic Univ.) Prof., Webster Coll., Webster Groves, Mo.

CLARKE, W. B., Senior Partner, W. B. Clarke & Co., Nurserymen, P.O. Box 343, San Jose, 2, Calif.

CLARKSON, HELEN E., A.M. (Duke) Instr., Creighton Univ., Omaha, Nebr. *528 S. 52 St., Omaha 6, Nebr.*

CLARKSON, Asst. Prof. J. A., Ph.D. (Brown) Univ. of Pennsylvania, Philadelphia, Pa.

CLAWSON, Prof. J. W., M.A. (New Brunswick) Ursinus Coll., Collegeville, Pa. *6 Glenwood Ave.*

CLAYTOR, Prof. W. W. S., Ph.D. (Pennsylvania) Hampton Inst., Hampton, Va.

CLELAND, Prof. W. E., Ph.D. (Princeton) Geneva Coll., Beaver Falls, Pa.

CLEMENT, Asst. Prof. Mary D., Ph.D. (Chicago) Univ. of Miami, Coral Gables 34, Fla. *Dept. of Math.*

CLEMENTS, Prof. G. R., Ph.D. (Harvard) U. S. Naval Acad., Annapolis, Md. *7 Thompson St.*

CLIFFORD, Asso. Prof. A.H., Ph.D. (Calif. Inst. of Tech.) Johns Hopkins Univ., Baltimore 18, Md. *Dept. of Math.*

CLIFFORD, P. C., A.M. (Columbia) Instr., State Teachers Coll., Upper Montclair, N.J.
CLINTON, FLORENTINA M. (Mrs. J. C.), A.M. (Ohio State) Teacher, Chillicothe High School, Chillicothe, Ohio. *173 Vine St.*
CLOS, CHARLES, C. E. (New York Univ.) Member, Tech. Staff, Bell Telephone Labs.; Instr. Pratt Inst., Brooklyn, N.Y. *6434–83rd St., Middle Village, N.Y.*
COBB, Asst. Prof. R. N., A.M. (Harvard) Worcester Polytechnic Institute, 26 Einhorn Road, Worcester 2, Mass.
COBLE, Prof. A. B., Ph.D. (Johns Hopkins) Univ. of Illinois, Urbana, Ill. *702 W. Washington Blvd. Visiting Prof., Haverford Coll., Haverford, Pa.*
COBURN, Asst. Prof. NATHANIEL, Ph.D. (Mass. Inst. of Tech.) Univ. of Michigan, Ann Arbor, Mich. *1304 Washtenaw Terrace*
COE, Asst. Prof. C. J., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *2022 Hill St.*
COFFIN, Prof. L. M., A.M. (Michigan) Emeritus, Coe Coll., Cedar Rapids, Iowa. *633 S. Bixel St., Los Angeles 14, Calif.*
COHEN, Prof. ABRAHAM, Ph.D. (Johns Hopkins) Emeritus, Johns Hopkins Univ., Baltimore 18, Md.
COHEN, Asso. Prof. A. C., Jr., Ph.D. (Michigan) Univ. of Georgia, Athens, Ga. *Dept. of Math.*
COHEN, H. J., M.A. (Univ. of Wis.) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis. *North Hall*
COHEN, Prof. TERESA, Ph.D. (Johns Hopkins) Pennsylvania State Coll., State College, Pa. *315 S. Atherton St.*
COKER, R. L., A.M. (Alabama) Instr., Math. and Physics, Southern Coll. of Optometry, Memphis, Tenn. *1877 Cowden Ave.*
COLE, Asst. Prof. NANCY, Ph.D. (Radcliffe) Syracuse Univ., Syracuse 10, N.Y. *Dept. of Math.*
COLE, Asst. Prof. R. H., Ph.D. (Wisconsin) Univ. of Western Ontario, London, Ont., Can.
COLEMAN, E. P., M.S. (Iowa) Major, CAC, BRL, Room 201, Aberdeen Proving Ground, Md.
COLEMAN, J. B., Ph.D. (California) Acting Prof., Presbyterian Coll., Clinton, S.C. *35 E. Carolina Ave.*
COLEMAN, RUTH T., A.M. (Columbia) Teacher, River Edge School, River Edge, N.J. *326 River Road, Bogota, N.J.*
COLLIER, Prof. MYRTIE, Ph.D. (Strasbourg) Chm. of Dept., Immaculate Heart Coll., Los Angeles, Calif. *225 Thurston Ave., Los Angeles 24, Calif.*
COLLINS, V. L., A.M. (Alabama) Head of Dept., State Teachers Coll., Troy, Ala. *Box 298*
COLSON, H. D., A.B. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *Wadena, Minn.*
COLVIN, B. H., Ph.D. (Wisconsin) Instr., Univ. of Wisconsin, Madison 6, Wis. *North Hall*
COMBELLACK, Asst. Prof. W. J., Ph.D. (Boston Univ.) Northeastern Univ., 360 Huntington Ave., Boston 15, Mass.
COMFORT, Asst. Prof. E. G. H., Ph.D. (Brown) Illinois Inst. of Tech., Chicago, Ill. *812 Prairie Ave., Wilmette, Ill.*
COMSTOCK, Prof. C. E., A.M. (Knox) Emeritus, Bradley Univ., Peoria, Ill. *203 Fredonia Ave., Peoria 5, Ill.*
CONKWRIGHT, Asso. Prof. N. B., Ph.D. (Illinois) Univ. of Iowa, Iowa City, Iowa. *209-B Physics Bldg.*
CONNORS, J. L., *30 Broad St., Fall River, Mass.*
CONLAN, Sister MARIA LOYOLA, A.M. (Fordham) Asso. Prof., Coll. of Mount St. Vincent, New York 63, N.Y.
CONSTABLE, MARY LOUISE, A.M. (Pennsylvania) Teacher, Philadelphia High School for Girls, Philadelphia, Pa. *The Whittier, 140 N. 15th St., Philadelphia 2, Pa.*
CONWELL, G. M., Ph.D. (Princeton) Master, St. Pauls School, Concord, N.H. *307 Pleasant St.*
CONWELL, Prof. H. H., Ph.D. (Wisconsin) Dean, Beloit Coll., Beloit, Wis.
COOK, Asso. Prof. A. J., Ph.D. (Chicago) Univ. of Alberta, Edmonton, Alta., Can.
COOK, Asst. Prof. G. S., A.M. (Kansas) Colorado School of Mines, Golden, Colo. *Dept. of Math.*
COOK, I. D., B.S. (Univ. of Maryland) Sales Manager, Automotive Parts, System Brakes, 10 E. Lafayette Ave., Baltimore 2, Md. *942 Cator Ave., Baltimore 18, Md.*
COOK, Sister ROSE MARGARET, M.S. (Notre Dame) Prof., Loretto Heights Coll., Loretto, Colo.
COOKE, Asst. Prof. J. V., Ph.D. (Peabody) North Texas State Teachers Coll., Denton, Tex. *Route 1, 2300 W. Oak St.*

COOLEY, Asso. Prof. H. R., Ph.D. (New York Univ.) New York Univ., 100 Washington Sq E., New York 3, N.Y.
COOLEY, Prof. J. A., Ph.D. (Illinois) Univ. of Tennessee, Knoxville, Tenn.
COOLIDGE, Prof. J. L., Ph.D. (Bonn) Emeritus, Harvard Univ., Cambridge 38, Mass. *27 Fayerweather St.*
COOPER, ELIZABETH M., Ph.D. (Illinois) Chm. of Dept., Hunter Coll. High School, New York, N.Y. *201 E. 71 St., New York 21, N.Y.*
COPE, Asso. Prof. T. F., Ph.D. (Chicago) Chm. of Dept., Queens Coll., Flushing, N.Y. *Montrose, N.Y.*
COPELAND, Prof. A. H., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *616 Oswego St.*
COPELAND, Prof. LENNIE P., Ph.D. (Pennsylvania) Emeritus, Wellesley Col.., Wellesley, Mass. *1190 Eighth St. N., St. Petersburg, Fla.*
COPP, P. T., A.M. (Ohio State) Industrial Engr., Carnegie Illinois Steel Works, Gary, Ind. *Box 64, Porter, Ind.*
CORBIN, Prof. C. E., A.M. (Northwestern) Emeritus, Coll. of the Pacific, Stockton, Calif. *P.O. Box 115, Carmel, Calif.*
CORLISS, J. J., Ph.D. (Michigan) Chm. of Undergrad. Div., Univ. of Illinois, Navy Pier, Chicago, Ill.
CORONA, Sister MARIA, Ph.D. (Fordham) Dean, Coll. of Mount St. Joseph, Mount St. Joseph, Ohio
CORRAL Y ALEMAN, J. I., Ing. de Mines. Dir. de Montes y Mines, Republic of Cuba, Havana, Cuba. *Calle B 514, entre 21 y 23, Vedado*
COSBY, Byron, A.M. (Missouri) Owner, Educ. Service Bureau, Columbia, Mo. *One Ridgeley Road*
COSBY, Asst. Prof. BYRON, Jr., Ph.D. (Chicago) Univ. of Iowa, Iowa City, Iowa. *Dept. of Math.*
COTHRAN, F. E., B.S. (California) *709 S. Norton Ave., Los Angeles 5, Calif.*
COTHRAN, Prof. J. C., Ph.D. (Cornell) Head of Dept. of Chem., State Teachers Coll., Duluth Minn. *512 N. 19th Ave. E. Duluth 5 Minn.*
COULTER, W. H., Licentiate of instr. (Peabody) Retired, Railway mail clerk. *134 N. Summit Ave., Decatur, Ill.*
COURANT, Prof. RICHARD, Ph.D. (Gottingen) Chm. of Dept., New York Univ., New York, N.Y. *142 Calton Road, New Rochelle, N.Y.*
COURT, Prof. N. A., D.Sc. (Ghent, Belgium) Univ. of Oklahoma, Norman, Okla.
COWAN, Asst. Prof. R. W., Ph.D. (California) Univ. of Florida, Gainesville, Florida. Peabody Hall.
COWLES, Prof. W. H. H., A.M. (Columbia) Head of Dept., Pratt Inst., Brooklyn, N.Y. *132 Joralemon St., Brooklyn 2, N.Y.*
COWLING, Asst. Prof. V. F., Ph.D. (Rice) Lehigh Univ., Bethlehem, Pa. *Dept. of Math.*
COX, Asst. Prof. H. M., A.M. (Duke) Dir., Bureau of Instructional Research, Univ. of Nebraska, Room 3, Admin. Annex, 1125 R St., Lincoln 8, Nebr.
COX, P. C., A.M. (New Mexico) Grad School., Univ. of Michigan, Ann Arbor, Mich. *208 N. Division St., Apt. 2*
COXETER, Asso. Prof. H. S. M., Ph.D. (Cambridge) Univ. of Toronto, Toronto, Ont., Can. *24 Strathearn Blvd., Toronto 10, Ont., Can.*
CRAIG, Prof. A. T., Ph.D. (Iowa) Univ. of Iowa, Iowa City, Iowa. *Dept. of Math.*
CRAIG, Prof. C. C., Ph.D. (Michigan) Dir., Stat. Res. Lab., Univ. of Michigan, Ann Arbor, Mich. *3020 Angell Hall*
CRAIG, Prof. H. V., Ph.D. (Wisconsin) Math. and Astr., Univ. of Texas, Austin, Tex. *Journalism Bldg.*
CRAIN, K. W., M.S. (Iowa) Instr., Purdue Univ., West Lafayette, Ind. *238 Lincoln St.*
CRAMER, G. F., Ph.D. (Missouri) *112 Quincy St., Chevy Chase, Md.*
CRAMER, Asst. Prof. PAUL, A.M. (Illinois) Monmouth Coll., Monmouth, Ill. *412 E. First Ave.*
CRAMLET, Asso. Prof. C. M., Ph.D. (Washington) Univ. of Washington, Seattle, Wash. *Physics Hall*
CRANE, R. E., B.S. (Harvard) Amer. Tel. & Tel. Co., 195 Broadway, New York, N.Y. *120 Early St., Morristown, N.J.*
CRANE, Asso. Prof. Rufus, A.M. (Ohio State) Ohio Wesleyan Univ., Delaware, Ohio. *269 W. William St.*
CRAWFORD, Asst. Prof. W. S. H., A.M. (Minnesota) Mount Allison Univ., Sackville, N.B., Can.
CRISPIN, J. W., Jr., B.S. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *363 Richton Ave., Detroit 3, Mich.*

CROCI, FRANCO, *3992 White Plains Ave., New York 66, N.Y.*
CROFT, MARJORIE L., M.A. (Loyola) Instr., Univ. of Illinois, Chicago, Ill. *4519 West End Ave.*
CROMWELL, J. W., Jr., A.M. (Dartmouth) Certified Pub. Accountant, Washington, D.C. *1815–13th St., N.W.*
CROSS, Asst. Prof. J. H., A.M. (Texas Tech.) Texas Tech. Coll., Lubbock, Tex. *2015–15th St.*
CROW, E. L., Ph.D. (Wisconsin) Mathematician, Science Dept., Research and Development, U. S. Naval Ordnance Test Station, Inyokern, Calif.
CRULL, Prof. H. E., Ph.D. (Illinois) Head of Dept., Butler Univ., Indianapolis, Ind.
CUDE, Prof. DON, A.M. (S. W. Tex. St. T. Coll.) Acting Head of Dept., Southwest Texas State Teachers Coll., San Marcos, Tex. *Box 77.*
CULMER, Prof. ORPHA ANN, A.M. (Michigan) Head of Dept., State Teachers Coll., Florence, Ala. *College Station*
CUNNINGHAM, Asst. Prof. A. B., Ph.D. (West Virginia) West Virginia Univ., Morgantown, W. Va. *Dept. of Math.*
CUNNINGHAM, Asst. Prof. L. E., Ph.D. (Harvard) Astr., Univ. of California, Berkeley 4, Calif. *Students' Observatory*
CURFMAN, Prof. L. E., M.S. (Colorado) Kansas State Teachers Coll., Pittsburgh, Kans. *406 W. Adams St.*
CURRIE, Prof. J. C., A.M. (Mississippi) Louisiana State Univ., Baton Rouge, La. *Box 6126 Univ. Sta.*
CURRIER, Asso. Prof. A. E., Ph.D. (Harvard) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
CURRY, Prof. H. B., Ph.D. (Gottingen) Pennsylvania State Coll., State College, Pa.
CURTIS, Prof. H. B., Ph.D. (Cornell) Lake Forest Coll., Lake Forest, Ill. *11 College Campus*
CURTIS, MARJORY R., M.A. (Univ. of Mich.) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *712 Liberty St., Flint, Mich.*
CURTISS, Prof. D. R., Ph.D. (Harvard) Emeritus, Northwestern Univ., Evanston, Ill. *632 Buena Vista St., Redlands, Calif.*
CURTISS, J. H., Ph.D. (Harvard) Asst. to Director, Nat'l. Bureau of Standards, Washington 25, D.C. *4802 Bradley Blvd., Chevy Chase, Md.*
CUTLER, Asst. Prof. E. H., Ph.D. (Harvard) Lehigh Univ., Bethlehem, Pa.
CUTTING, L. H., A.M. (Missouri) Teacher, Kansas City High School, Kansas City, Mo. *406 E. 43rd St., Kansas 4, Mo.*
CYPRIAN LUKE, Brother (Roney), Ph.D. (Catholic Univ.) Head of Dept., St. Michael's Coll., Cerillos Road, Santa Fe, N.M.

DADOURIAN, Prof. H. M., Ph.D. (Yale) Math. and Nat. Phil., Trinity Coll., Hartford, Conn. *125 Vernon St.*
DALAL, R. D., Retired, Stock Broker, London, England
DALY, J. F., Ph.D. (Princeton) Statistician, Bureau of the Census, Washington, D.C. *4213–18th St. N.E., Washington 18, D.C.*
DAMSGARD, Asst. Prof. L. C., A.B. (Union Coll., Nebr.) Math. and Astr., Pasadena Jr. Coll., Pasadena, Calif. *2000 Loma Vista St., Pasadena 7, Calif.*
DANCER, Prof. WAYNE, Ph.D. (Michigan) Univ. of Toledo, Toledo 6, Ohio
DANCEY, Prof. L. S., A.M. (Illlinois) Carroll Coll., Waukesha, Wis. *125 N. Charles St.*
DANIELLS, Asst. Prof. MARIAN E., M.S. (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
DANSKIN, J. M., Jr., A.B. (U.C.L.A.) Teaching Asst., International House, Berkeley 4, Calif.
DAPPERT, J. W., C.E. (Valparaiso Univ.) Civil Eng., Taylorville, Ill. *505 Gandy Ave.*
D'ARCO, PAUL, M.S. (DePaul) Instr., De Paul Univ., Chicago, Ill. *1116 W. Polk St.*
DARKOW, Asso. Prof. MARGUERITE D., Ph.D. (Chicago) Hunter Coll., New York, N.Y. *16 East 82 St., New York 28, N.Y.*
DARLING, F. W., A.B. (Cornell) Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C. *45 Kenilworth Ave., Garrett Park, Md.*
DARRAUGH, J. E., A.M. (Brooklyn) Clerk, Consolidated Edison Co., 4 Irving Pl., New York, N.Y. *2917 Glenwood Road, Brooklyn, N.Y.*
D'ATRI, A. J., C.E. (Brooklyn Poly. Inst.) Civil Engr., Dept. of Public Works, New York, N.Y. *1821 Madison Pl., Brooklyn 29, N.Y.*
DAUGHERTY, J. D., A.M. (Pennsylvania) Head of Dept., Eastside High School, Paterson, N.J.
DAUM, Asst. Prof. J. A., Ph.D. (Nebraska) Texas A. and M. Coll., College Station, Tex., *Box 1011*
DAUS, Prof. P.H., Ph.D. (California) U.C.L.A., Los Angeles 24, Calif.

DAVIES, ROBERT, Ph.M. (Wisconsin) Grad. Student, Univ. of Wisconsin, Madison, Wis. *1615 Hoyt St., Madison 5, Wis.*
DAVIS, Rev. A. F., A.B. (Catholic Univ.) Instr., St. Johns Seminary, Graymoor, Garrison, N.Y. *St. Paul's Friary*
DAVIS, Asst. Prof. A. W., Ph.D. (Iowa State) Iowa State Coll., Ames, Iowa. *Theoret. and Appl. Mech. Dept.*
DAVIS, CONSTANCE H., (Mrs. Leslie) *238 Audley St., South Orange, N.J.*
DAVIS, Prof. D. R., Ph.D. (Chicago) State Teachers Coll., Montclair, N.J. *43 College Ave., Upper Montclair, N.J.*
DAVIS, Asso. Prof. H. A., Ph.D. (Cornell) West Virginia Univ., Morgantown, W. Va. *307 Duquesne Ave.*
DAVIS, Dean J. B., M.S. (Northwestern) Amarillo College, 2201 Washington St., Amarillo, Texas. *2012 Madison St.*
DAVIS, Asst. Prof. JAMES EDGAR, A.M. (Ohio State) Pharmacy, Univ. of Illinois, 808 S. Wood St., Chicago 12, Illl.
DAVIS, Prof. JAMES ELMER, A.M. (Wisconsin) Drexel Inst. of Tech., 32nd and Chestnut St., Philadelphia 4, Pa.
DAVIS, M. S., M.S. (Mississippi) Research Asst., Yale Univ., Observatory, Prospect and Canner Sts., New Haven, Conn.
DAVIS, Asso. Prof. MARGARET R., Ed.D. (Columbia) Southeastern Louisiana Coll., Hammond, La.
DAVIS, Asst. Prof. U. P., A.M. (Florida) Univ. of Florida, Gainesville, Fla. *1635 W. Mechanic St.*
DAVIS, Prof. W. M., Ph.D. (Chicago) Cornell Coll., Mt. Vernon, Iowa. *616–7th Ave., N.*
DAVISON, Asso. Prof. RACHEL, A.M. (Oberlin) Houghton Coll., Houghton, N.Y.
DEAN, ALICE C., A.M. (Rice) Teaching Fellow, Rice Inst., Houston 1, Tex. *P.O. Box 1892*
DEAN, Mrs. MILDRED W., Ph.D. (Johns Hopkins) Instr., Queens Coll., Flushing, N.Y. *254–09 West End Drive, Great Neck, N.Y.*
DEARBORN, Prof. D. C., Ph.D. (Duke) Registrar, Catawba Coll., Salisbury, N.C.
DEAUX, Prof. ROLAND, D.Sc. (Univ. of Gand) Faculte Polytechnique, Mons, Belgium, *47 Chaussee de Binche*
DECHERD, Asst. Prof. MARY E., A.M. (Texas) Univ. of Texas, Austin, Texas. *2313 Nueces St., Austin 21, Texas*
DE CICCO, Asso. Prof. JOHN, Ph.D. (Columbia) Illinois Inst. of Tech., Chicago, Ill. *6340 Blackstone Ave., Apt. 303, Chicago 37, Ill.*
DECK, Prof. L. J., A.M. (Pennsylvania) Muhlenberg Coll., Allentown, Pa.
DECKER, Prof. F. F., Ph.D. (Syracuse) Syracuse Univ., Syracuse, N.Y. *312 Marshall St.*
DECLEENE, Rev. L. A. V., Ph.D. (Catholic Univ.) *1015 S. Monroe Ave., Green Bay, Wis.*
DECOU, Prof. E. E., M.S. (Chicago) Univ. of Oregon, Eugene, Ore. *929 Hilyard St.*
DEDERICK, L. S., Ph.D. (Harvard) Asso. Director, Ballistic Research Lab., Aberdeen Proving Ground, Md.
DE FRANCESCO, H. F., B.E.E. (Virginia) Grad. Student, Univ. of Virginia, Charlottesville, Va. *512 Lexington Ave.*
DE LA GARZA, ELEUTERIO, *Box 304, Brownsville, Tex.*
DE LA SALLE (Seiler), Brother LOUIS, Ph.D. (Catholic Univ.) Dean of Studies, St. Mary's Coll., Winona, Minn.
DEMING, Prof. R. M., M.S. (Iowa State) Upper Iowa Univ., Fayette, Iowa
DEMOSS, M. L., M.S. (Kansas St. T. C.) Instr., General Motors Inst., Flint, Mich. *2213 Flushing Road*
DENBOW, Asso. Prof. CARL, Ph.D. (Chicago) U. S. Naval Postgrad. School, Annapolis, Md. *Dept. of Math.*
DENNIS, Asso. Prof. F. L., Ph.D. (Illinois) Ursinus Coll., Collegeville, Pa. *930 Main St.*
DENNISON, C. H., B.S. (Mass. Inst. of Tech.) Chemist, Archer Rubber Co., Milford, Mass. *4 Crescent St., Natick, Mass.*
DENNY, C. E., B.S. (U. S. Naval Acad.) Instr., Central Coll., Fayette, Mo. *105 Lucky St.*
DENTON, Asst. Prof. W. W., Ph.D. (Illinois) Univ. of Ariz., Tucson, Ariz. *643 No. 4th Ave., Tucson, Ariz.*
DE REGT, M. P., B.S. (Webb Inst.) *920 Kains Ave., Albany, Calif.*
DERFLINGER, R. B., B.S. (Geneva Coll.) Asst., Purdue Univ., Lafayette, Ind. *Iroq 3-33 W. Lafayette, Ind.*
DERNHAM, M. A., LL.B. (California) Attorney-at-Law, 405 Montgomery St., Room 1114, San Francisco 4, Calif.
DEUTSCH, J. G., A.M. (Columbia) Chm. of Dept., Andrew Jackson High School, Queens, N.Y. *66 St. Pauls Place, Brooklyn 26, N.Y.*

DIAMOND, Prof. N. H., Ph.D. (California) Head of Dept., Okla. A. and M. Coll., Stillwater, Okla.
DICKINSON, Mrs. ALICE B., M.A. (Columbia) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich. *507 S. Fifth Ave.*
DICKSON, Prof. L. E., Ph.D. (Chicago) Honorary Life Member, Emeritus, Univ. of Chicago, Chicago, Ill.
DILLINGHAM, Prof. ALEXANDER, A.M. (Harvard) U. S. Naval Acad., Annapolis, Md.
DILLON, G. M., B.A. (Long Island Univ.) Student, Columbia Univ., New York, N.Y. *51 Union St., New Rochelle, N.Y.*
DILWORTH, Asso. Prof. R. P., Ph.D. (Calif. Inst. of Tech.) Calif. Inst. of Tech., Pasadena, Calif.
DIMICK, Prof. C. E., A.M. (Pennsylvania) Retired, U. S. Coast Guard Acad., New London, Conn. *Gales Ferry, Conn.*
DIMOND, Sister AGNES THERESA, A.M. (Columbia) Prof., Coll. of Mount St. Joseph, Mount St. Joseph, Ohio
DIMSDALE, BERNARD, Ph.D. (Minnesota) Ballistic Research Laboratory, Aberdeen Proving Ground, Md.
DINES, Prof. L. L., Ph.D. (Chicago) Emeritus, Carnegie Inst. of Tech., Pittsburgh, Pa. *131 North 14th St., Quincy, Ill.*
DINKINES, Adj. Prof. FLORA, A.M. (Michigan) Univ. of South Carolina, Columbia, S.C. *On leave, Grad. Student, Univ. of Chicago. Room 201, Blackstone Hall, 5748 S. Blackstone Ave., Chicago, Ill.*
DIX, Prof. L. E., M.S. (Norwich) Norwich Univ., Northfield, Vt. *5 Spring St.*
DOBBIN, Sister MARIOLA, A.M. (Wisconsin) Prof., Rosary Coll., River Forest, Ill.
DOBBINS, J. W., Student, St. Mary's Univ., San Antonio Tex. *122 Pear Walk, San Antonio 3, Tex.*
DOBSON, W. P., M.A. (Toronto) Dir. of Research, Hydro-Electric Power Commission of Ontario, 620 University Ave., Toronto 2, Ont., Can.
DODSON, Asso. Prof. N. E., M.A. (Univ. of Ala). Lenoir Rhyne Coll., Hickory, N.C. *718–16th St., Hickory, N.C.*
DOERFLER, HILARY, A.M. (St. John's Univ., Minn.) Head of Dept., St. Gregory's Coll., Shawnee, Okla.
DOERMANN, F. W., Ph.D. (Vienna) Retired
DOLCIANI, MARY P., A.M. (Cornell) Inst. for Advanced Study, Princeton, N.J.
DOLL, W. R., A.B. (Columbia Coll.) *709 Kennedy Bldg., Tulsa 3, Okla.*
DOMINIC, Brother, A.B. (St. Mary's Coll.) Instr., St. Mary's Coll., St. Mary, Calif.
DONALDSON, J. D., A.M. (Pittsburgh) Consolidated Vultee Aircraft Co., San Diego, Calif. *2027 Drescher St., San Diego 10, Calif.*
DONCHIAN, P. S., A.B. (Yale) Pres. and Treas., Samuel Donchian Rug Co., Hartford, Conn. *85 Gillett St.*
DONER, Prof. R. D., Ph.D. (Illinois) Head of Dept., Alabama Poly. Inst., Auburn, Ala. *477 E. Samford Ave.*
DORROH, Asso. Prof. J. L., Ph.D. (Texas) Texas Coll. of Arts and Industries, Kingsville, Texas. *Dept. of Math.*
DORWART, Asso. Prof. H. L., Ph.D. (Yale) Washington and Jefferson Coll., Washington, Pa. *R.D. 6*
DOSTAL, Asst. Prof. B. F., A.M. (Indiana) Univ. of Florida, Gainesville, Fla. *106 Peabody Bldg.*
DOTTERER, Prof. J. E., A.M. (Illinois) Head of Dept., Manchester Coll., North Manchester, Ind.
DOUGHERTY, LUCY T., A.M. (Kansas) *1428 Tennessee, Lawrence, Kans.*
DOUGLAS, E. G., A.M. (Mercer Univ.) Instr., Univ. of South Carolina, Columbia, S.C. *1520 Senate St.*
DOUGLAS, NELLE C., M.Ed. (Duke) Instr., Univ. of South Carolina, Columbia, S.C. *Dept. of Math.*
DOUGLASS, Prof. R. D., Ph.D. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge, Mass. *18 Oak Ave., Belmont, Mass.*
DOWNIE, E. J., A.B. (Colgate) Instr., Colgate Univ., Hamilton, N.Y. *17 Madison St.*
DOWNING, F. A., Asst. Director of Traffic, North Carolina Utilities Com., Raleigh, N.C. *212 Taylor St.*
DOWNING, Prof. H. H., Ph.D. (Chicago) Math. and Astr., Univ. of Kentucky, Lexington, Ky. *138 State St., Lexington 44, Ky.*
DOWNING, Prof. R. H., Ph.D. (West Virginia) Army Air Forces Inst. of Tech., Wright Field, Dayton, Ohio. *405 Corona Ave., Dayton 9, Ohio*
DOYLE, Asst. Prof. T. C., Ph.D. (Princeton) Dartmouth Coll., Hanover, N.H. *Graduate Club*

DOYLE, Rev. W. C., Ph.D. (St. Louis Univ.) Asso. Prof., Rockhurst Coll., Kansas City 4, Mo.
DRAGOO, Asst. Prof. R. C., A.M. (Oklahoma) Univ. of Oklahoma, Norman, Okla. *625 S. Lahoma St.*
DRAPER, Prof. OLIVE M., A.M. (Michigan) Taylor Univ., Upland, Ind.
DRESDEN, Prof. ARNOLD, Ph.D. (Chicago) Swarthmore Coll., Swarthmore, Pa. *606 Elm Ave.*
DRESSEL, Asst. Prof. F. G., Ph.D. (Duke) Duke Univ., Durham, N.C. *309 Francis St.*
DRESSLER, B. B., A.M. (Illinois) Grad. Asst., Ohio State Univ., Columbus, Ohio. *2400 Indianola Ave., Columbus 2, Ohio*
DREW, Asst. Prof. J. W., A.M. (Cornell) Virginia Union Univ., Richmond, Va.
DRIBIN, D. M., Ph.D. (Chicago) Research Analyst, Army Security Agency, Washington, D.C. *4320 Fourth St. N., Arlington, Va.*
DRIVER, D. D., A.M. (Nebraska) Instr., Hesston Coll., Hesston, Kans.
DROUSSENT, LUCIEN, Ing. (Ecole Centrale des Artes and Manuf., Paris) Ministere de l'-Armement. *36 Rue de la Cartoucherie, a Clermont-Ferrand, Puy-de-Dome, France*
DUBE, Prof. L. H., Ph.D. (Gregorian Univ.) Ottawa Univ., Ottawa, Ont., Can.
DUBISCH, Asst. Prof. ROY, Ph.D. (Chicago) Triple Cities Coll., Syracuse Univ., 304 Lincoln St., Endicott, N.Y.
DUERKSEN, J. A., A.B. (Bethel) Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C. *3134 Monroe St. N.E., Washington 18, D.C.*
DUFFIE, Prof. J. A. H., B.S. (Univ. of New Brunswick) Chemistry, Univ. of Ottawa, Ottawa, Ont., Can.
DUFFNER, R. T., G.E. (Colo. School of Mines) *2801 E. Colfax Ave., Denver 6, Colo.*
DUNCAN, D. C., Ph.D. (California) Chm. of Dept., East Los Angeles Jr. Coll., 5023 E. Sixth St., Los Angeles 22, Calif.
DUNCAN, R. L., A.B. (Tulsa) Landman, Carter Oil Co., Tulsa, Okla. *Box 801*
DUNFORD, Prof. NELSON, Ph.D. (Brown) Yale Univ., New Haven, Conn. *374 Fountain St.*
DUNKEL, Prof. OTTO, Ph.D. (Harvard) Emeritus, Washington Univ., St. Louis 5, Mo.
DUNLAP, Asso. Prof. L. T., A.M. (Pennsylvania State) Pennsylvania State Coll., State College, Pa. *Math. Dept.*
DUNLAP, P. R., A.B. (Taylor Univ.) *Kalkaska, Mich.*
DUPASQUIER, Prof. L. G., Ph.D. (Zurich) Univ. of Neuchatel, Faubourg du Cret, 4a, Neuchatel, Switzerland
DURAND, JANET C., A.M. (Pennsylvania) *223 W. Sedgwick St., Philadelphia 19, Pa.*
DUREN, Prof. W. L., Ph.D. (Chicago) Tulane Univ., New Orleans 15, La. *3323 Joseph St.*
DURFEE, Dean WALTER H., Ph.D. (Cornell) Hobart Coll., Geneva, N.Y. *Coxe Hall*
DURFEE, WILLIAM H., Ph.D. (Cornell) Instr., Dartmouth Coll., Hanover, N.H. *13 E. Wheelock St.*
DURHAM, GENEVA E., A.M. (Northwestern) Instr., Atlantic Union Coll., South Lancaster, Mass.
DURHAM, R. L., B.S. (Trinity Coll.) Emeritus, President, Southern Sem. and Jr. Coll., Buena Vista, Va.
DURST, JEANETTE R. (Mrs. T. N.), A.M. (Tennessee) Instr., Univ. of South Carolina, Columbia, S.C. *3017 Kirkwood Road*
DWIGHT, Prof. L. A., M.A. (Oklahoma) Southeastern State Coll., Durant, Okla.
DWYER, Prof. P. S., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *640 Oxford Road*
DYE, Asso. Prof. L. A., Ph.D. (Cornell) The Citadel, Charleston, S.C.

EAGLE, E. L., A.B. (Miami) Instr., Univ. of Arkansas, Fayetteville, Ark. *Box 1011*
EARHART, F. C., A.M. (Wisconsin) Epworth High School, Epworth, Iowa. *Hopkinton, Iowa*
EARL, Prof. J. M., Ph.D. (Minnesota) Municipal Univ. of Omaha, Omaha, Nebr. *528 S. 53rd St.*
EASTHAM, Asst. Prof. J. N., Ph.D. (Catholic Univ.) Cooper Union, Cooper Square, New York, N.Y. *149-41 Hawthorne Ave., Flushing, N.Y.*
EATON, J. E., Ph.D. (Yale) Instr., Queens Coll., Flushing, N.Y. *132 N. Terrace Ave., Mount Vernon, N.Y.*
EAVES, Asso. Prof. E. D., Ph.D. (Texas) Univ. of Tennessee, Knoxville 16, Tenn. *Univ. Sta.*
EAVES, J. C., A.M. (Kentucky) Student, Univ. of North Carolina, Chapel Hill, N.C. *Dept. of Math.*
EBERHART, Prof. PAUL, Ph.D. (Brown) Head of Dept., Math. and Engg., Washburn Univ,. Topeka, Kans. *2068 Lane St.*
EBERLEIN, W. F., Ph.D. (Harvard) Institute for Advanced Study, Princeton, N.J.
EDINGTON, Prof. W. E., Ph.D. (Illinois) Head of Dept., Math. and Astr., DePauw Univ., Greencastle, Ind.

EDISON, T. M., B.S. (Mass. Inst. of Tech.) Pres., Calibron Products, Inc., 51 Lakeside Ave., West Orange, N.J. *Llewellyn Park*
EDMONDSON, Prof. NAT, Jr., Ph.D. (Rice) Applied Physics Lab., Johns Hopkins Univ., Baltimore, Md. *8621 Georgia Ave., Silver Spring, Md.*
EDMUNDSON, H. P., B.A. (U.C.L.A.) Student, U.C.L.A., Los Angeles, Calif. *539 N. Kilkea Drive, Los Angeles 36, Calif.*
EDWARD JOSEPH, Sister, M.S. (Notre Dame) Instr., St. Mary's Coll., Notre Dame, Holy Cross, Ind.
EDWARDS, Asst. Prof. L. P., M.A. (Acadia) Univ. of New Brunswick, Fredericton, N.B., Can. *Alexander Coll., Apt. 34*
EDWARDS, Prof. P. D., Ph.D. (Indiana) Ball State Teachers Coll., Muncie, Ind.
EGGERS, Asst. Prof. H. C. T., Ph.D. (Michigan) Univ. of Minnesota, Minneapolis 6, Minn. *2100–31st Ave. S.*
EGGERT, O. E., Postal Clerk, U. S. Railway Mail Service, Philadelphia, Pa. *1916 Forest Ave., Morton, Pa.*
EGGETT, HAZEL E., A.M. (California) *6457 Regent St., Oakland 9, Calif.*
EIDE, Asso. Prof. MARGARET C. (Mrs. R. B.), A.M. (Wisconsin) State Teachers Coll., River Falls, Wis. *308 S. Second St.*
EIESLAND, Prof. J. A., Ph.D. (Johns Hopkins) Emeritus, West Virginia Univ., Morgantown, W. Va.
EILENBERG, Prof. SAMUEL, Ph.D. (Warsaw) Columbia Univ., New York 27, N.Y. *Hamilton Hall*
EISELE, CAROLYN, A.M. (Columbia) Instr., Hunter Coll., New York, N. Y. *257 W. 86 St., New York 24, N.Y.*
EISEN, N. A., B.S.E. (Missouri) Chemist, MidContinent Petroleum Corp., Tulsa, Okla. *1127 S. Braden St., Tulsa 4, Okla.*
EISENHART, L. P., Ph.D. (Johns Hopkins) Executive Officer, American Philosophical Society, Independence Square, Philadelphia, Pa. *25 Alexander St., Princeton, N.J.*
EKMAN, Prof. W. E., A.M. (South Dakota) Math. and Astr., Univ. of South Dakota, Vermillion, S.D. *104 S. Yale St.*
ELLIOTT, Prof. W. W., Ph.D. (Cornell) Duke Univ., Durham, N.C. *Box 4721*
ELLIS, J. R., M.S. (New Mexico) Teacher, Booker T. Washington High School, Tulsa, Okla. *1110 E. Pine St., Tulsa 6, Okla.*
ELSTON, J. S., A.B. (Cornell) Asst. Actuary, Travelers Ins. Co., Hartford, Conn.
EMMONS, Asso. Prof. H. W., D.S. (Harvard) Mech. Engg., Grad. School of Engineering, Harvard Univ., Cambridge, Mass. *307 Pierce Hall*
ENGEL, R. W., La Pagueteria, Guatemala, Guatemala
ENGSTROM, H. T., Ph.D. (Yale) Vice Pres. for Research and Engineering, Engineering Research Assoc., Washington, D.C. *213 Prince St., Alexandria, Va.*
EPSELL, MAX, B.S. (C.C.N.Y.) *698 Ashford St., Brooklyn 7, N.Y.*
EPSTEIN, BENJAMIN, Ph.D. (Illinois) Mathematician, Research Lab., Carnegie Inst. of Tech., Pittsburgh, Pa. *633 Rivermont Drive, Pittsburgh 7, Pa.*
ERICKSEN, W. S., Ph.D. (Wisconsin) Mathematician, Forest Products Laboratory, Madison, Wis. *1924 Rowley Ave., Madison 5, Wis.*
ERICKSON, R. L., M.S. (Wisconsin) Instr., Purdue Univ., Fort Wayne, Ind.
ERICKSON, Prof. R. W., Ph.D. (Minnesota) Psychology, Mississippi State Coll. for Women, Columbus, Miss.
ERIKSON, Prof. C. M., Ph.D. (Michigan) Michigan State Normal Coll., Ypsilanti, Mich. *101 Wallace Blvd.*
ERKILETIAN, Asst. Prof. D. H., Jr., A.M. (Illinois) Univ. of Missouri, School of Mines & Met., Rolla, Mo. *Dept. of Math.*
ERNSBERGER, IVA B., A.M. (Nebraska) Instr., Fullerton Junior Coll., Fullerton, Calif.
ERNSDORFF, Rev. L. E., M.S. (Notre Dame) Instr., Loras Coll., Dubuque, Iowa
ERRERA, Prof. ALFRED, L'Univ. libre de Bruxelles, Belgium. *140 Avenue Moliere.*
ERWIN, M. C., Teacher, Industrial Arts Dept., High School, Reynolds, Ind. *Box 202, Monticello, Ind.*
ESPOSITO, J. P., M.S. (De Paul) Teacher, Crane Tech. High School, Chicago, Ill. *1640 N. Parkside Ave., Chicago 39, Ill.*
ETTINGER, W. J., B.S. in M.E. (Lewis Inst.) Designing Engr., Edison General Elec. Appliance Co., 5600 W. Taylor St., Chicago, Ill.
ETTLINGER, Prof. H. J., Ph.D. (Harvard) Univ. of Texas, Austin, Tex. *3110 Harris Park Ave.*
EULENBERG, M. D., A.M. (Loyola Univ.) Instr., Evening School, Illinois Inst. of Tech., Chicago, Ill. *3025 Logan Blvd., Chicago 47, Ill.*
EVANS, Prof. G. C., Ph.D. (Harvard) Univ. of California, Berkeley 4, Calif.

EVANS, Prof. H. B., Ph.D. (Pennsylvania) Emeritus, Univ. of Pennsylvania, Philadelphia, Pa. *88 Merbrook Lane, Merion Station, Pa.*
EVANS, Prof. H. P., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison 6, Wis. *North Hall*
EVENSEN, E. J., Actuarial Section, Metropolitan Life Ins. Co., Pacific Coast Head Office, San Francisco, Calif.
EVERETT, E. E., B.S. (Brigham Young Univ.) Instr., Dixie Jr. Coll., St. George, Utah. *159 S. Fourth West*
EVERETT, H. S., Ph.D. (Chicago) Extension Prof., Univ. of Chicago, Chicago 37, Ill.
EVERETT, Prof. J. P., Ph.D. (Columbia) Emeritus, Western Michigan Coll., Kalamazoo, Mich. *907 W. South St.*
EVERETT, Asso. Prof. J. R., A.M. (Wisconsin) Colorado School of Mines, Golden, Colo. *1700 Washington St.*
EVES, Asst. Prof. H. W., A.M. (Harvard) Oregon State Coll., Corvallis, Ore. *Dept. of Math.*
EWING, Asso. Prof. G. M., Ph.D. (Missouri) Univ. of Missouri, Columbia, Mo. *208 Engineering Bldg.*
EWY, D. J., A.B. (California) Instr., Bethel Coll., North Newton, Kans. *Box 181*

FAGERSTROM, Asst. Prof. W. H., Ph.D. (Columbia) Coll. of the City of New York, New York 31. *706 Riverside Drive, New York 31, N.Y.*
FALVEY, FRANCES E., A.M. (Southern Methodist) *302 E. Cotton St., Longview, Tex.*
FARNELL, Asst. Prof. A. B., Ph.D. (California) Univ. of Colorado, Boulder, Colo. *Dept. of Math.*
FAULKNER, F. D., B.S. (Kansas State T. C.) Univ. of Michigan, Ann Arbor, Mich. *Dept. of Engg. Research*
FAWCETT, Prof. H. P., Ph.D. (Columbia) Ohio State Univ., Columbus, Ohio. *54 W. Beechwold Blvd., Columbus 2, Ohio*
FAY, E. A., A.M. (Harvard) *415 S. 17th St., Apt. 2B, Richmond, Calif.*
FEDERICO, P. J., A.M. (George Washington) Principal Examiner, U. S. Patent Office, Washington, D.C. *3634 Jocelyn St. N.W.*
FEEMSTER, Prof. H. C., A.M. (Nebraska) Emeritus, Math. and Astr., York Coll., York, Nebr. *812 E. 7th St.*
FEHR, Asso. Prof. H. F., Ph.D. (Columbia) State Teachers Coll., Upper Montclair, N.J. *544 Highland Ave.*
FEINSTEIN, I. K., M.A. (Northwestern) Instr., Univ. of Ill., Undergraduate Div., Chicago, Ill. *3348 Wilson Ave., Chicago 25, Ill.*
FELD, Asst. Prof. J. M., Ph.D. (Columbia) Queens Coll., 65–30 Kissena Blvd., Flushing, N.Y.
FELDER, VIRGINIA I., M.S. (Tulane) Head of Dept., Odessa Jr. Coll., Odessa, Tex. *511 N. Kelley Ave.*
FELLER, Prof. WILLIAM, Ph.D. (Gottingen) Cornell Univ., Ithaca, N.Y.
FELTGES, EDNA M., A.M. (Wisconsin) Chm. of Dept., Wilson Jr. Coll., Chicago, Ill. *2552 E. 76th St., Chicago 49, Ill.*
FERNS, Prof. H. H., Ph.D. (Toronto) Univ. of Saskatchewan, Saskatoon, Sask., Can. *Dept. of Math.*
FERRY, F. C., Ph.D. (Clark) Emeritus, Pres., Hamilton Coll., Clinton, N.Y. *324 Hart St., New Britain, Conn.*
FETTERER, Rev. R. A., M.S. (Catholic Univ.) Instr., St. Francis Seminary, Milwaukee, Wis. *3257 S. Lake Drive, Milwaukee 7, Wis.*
FETTIS, H. E., A.B. (Wittenberg) Aero. Engr., Air Tech. Service Command. *931 Five Oaks Ave., Dayton 6, Ohio*
FICKEN, Prof. F. A., Ph.D. (Princeton) Univ. of Tennessee, Knoxville 16, Tenn.
FIELD, Prof. FLOYD, A.M. (Harvard) Dean of Men, Georgia School of Tech., Atlanta Ga. *2865 Tupelo St. S.E.*
FIELD, Prof. PETER, Ph.D. (Cornell) Emeritus, Univ. of Michigan, Ann Arbor, Mich.
FIELD, S. E., A.M. (Michigan) Head of Dept., Wright High School, Ironwood, Mich. *635 E. Cloverland Drive*
FIELDS, Asst. Prof. W. L., A.M. (Indiana) Louisville Municipal Coll., Louisville, Ky. *2239 W. Chestnut St.*
FINAN, Prof. E. J., Ph.D. (Ohio State) Catholic Univ. of America, Washington, D. C. *604 Girard St. N.E., Washington 17, D.C.*
FINCH, J. V., A.M. (Wisconsin) *6104 S. Woodlawn Ave., Chicago 37, Ill.*
FINDLAY, Prof. WILLIAM, Ph.D. (Chicago) Emeritus, McMaster Univ., Hamilton, Ont., Can. *32 South Oval*
FINE, Asst. Prof. N. J., Ph.D. (Pennsylvania) Univ. of Penn., Philadelphia, Pa. *Apt. D-5, Sunderland Crt., 3425 Powelton Ave.*

FINKEL, DANIEL, B.S. (C.C.N.Y.) *690 Hendrix St., Brooklyn 7, N.Y.*
FIRESTONE, C. D., B.S. (New Mexico) Instr., Rutgers Univ., New Brunswick, N.J. *Dept. of Math.*
FIRST, DOUGLAS, Physicist, Material Lab., New York Naval Shipyard, Brooklyn, N.Y. *88-06 Sutphin Blvd., Jamaica 2, N.Y.*
FISCHER, Asso. Prof. C. H., Ph.D. (Iowa) Univ. of Michigan, Ann Arbor, Mich. *3016 Angell Hall*
FISCHER, Rev. F. J., M.S. (DePaul) Instr., DePaul Univ., Chicago, Ill. *1010 Webster Ave., Chicago 14, Ill.*
FISCHER, Asst. Prof. I. C., M.S. (Marquette) Engg., Exten. Div., Univ. of Minnesota, Minneapolis, Minn. *4252 Columbus Ave., Minneapolis 7, Minn.*
FISHER, ANSELM, A.M. (New York Univ.) Instr., Central High School, Washington, D.C. *1723 Harvard St., N.W.*
FITE, Prof. W. B., Ph.D. (Cornell) Emeritus, Columbia Univ., New York, N.Y. *411 West 114 St., New York 25, N.Y.*
FITZGERALD, Rev. E. L., M.A. (Gonzaga Univ.) Manresa Hall, Port Townsend, Washington
FITZPATRICK, Asst. Prof. J. D., A.M. (Creighton) Marquette Univ., Milwaukee, Wis. *2102 W. Wisconsin Ave., Apt. 6, Milwaukee 3, Wis.*
FLAGG, Asst. Prof. ELINOR B., M.S. (Illinois) Illinois State Normal Univ., Normal, Ill. *29 Payne Place*
FLANDERS, HARLEY, B.S. (Chicago) Research Asst., Univ. of Chicago, Chicago, Ill. *6208 Ellis Ave., Chicago 37, Ill.*
FLANDERS, Prof. R. L., M.C.E. (Cornell) Civ. Engr., Oklahoma A. and M. Coll., Stillwater, Okla.
FLEISHER, Asso. Prof. EDWARD, Ph.D. (New York Univ.) Brooklyn Coll., Brooklyn, N.Y. *20 Plaza St., Brooklyn 17, N.Y.*
FLEMING, WALTER, A.M. (Minnesota) Lecturer, Univ. of Manitoba, Fort Garry Site, Winnipeg, Man., Can. *Dept. of Math.*
FLEXNER, Asso. Prof. W. W., Ph.D. (Princeton) Cornell Univ., Ithaca, N.Y.
FLOGSTAD, IDA, M.S. (Iowa State) Chm. of Dept., State Teachers Coll., Superior, Wis. *1009 N. 18th St.*
FLORIS, ATHANASIUS, Grad. Engr. (Tech. Hochschule Munich) Pacific Elec. Ry. Co., Los Angeles 14, Calif. *748 S. Kingsley Drive, Los Angeles 5, Calif.*
FLOYD, E. E., B.A. in Ed. (Alabama) Student, Univ. of Virginia, Charlottesville, Va. *15 Monroe Apts.*
FOARD, C. W., Ph.D. (Iowa) Physicist, Eastman Kodak Co., Rochester 4, N.Y. *686 Marshall Road, Rochester 11, N.Y.*
FOBES, Asst. Prof. M. P., A.M. (Harvard) Coll. of Wooster, Wooster, Ohio
FOCKE, Prof. T. M., Ph.D. (Göttingen) Emeritus, Dean, Case School of Appl. Sci., Cleveland, Ohio. *2517 Wellington Road, Cleveland 18, Ohio*
FOLLEY, Asso. Prof. K. W., Ph.D. (Toronto) Wayne Univ., Detroit, Mich. *19230 Gainsborough St., Detroit 23, Mich.*
FOLSOM, Asst. Prof. R. B., A.M. (Columbia) The Citadel, Charleston, S.C.
FOOTE, KATHERINE S., M.S. (Louisiana State) Critic Teacher, Mississippi Southern Coll., Hattiesburg, Miss. *204 Fourth Ave.*
FORAKER, Prof. F. A., M.S. (Ohio Northern) Univ. of Pittsburgh, Pittsburgh, Pa. *1313 Macon Ave.*
FORD, CLARENCE, A.M. (Kentucky) Teacher, Male High School, Louisville, Ky. *235 So. 39th St., Louisville 12, Ky.*
FORD, Prof. L. R., Ph.D. (Harvard) Chm. of Dept., Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill. *Technology Center*
FORD, Prof. W. B., Ph.D. (Harvard) Emeritus, Univ. of Michigan, Ann Arbor, Mich. *Hayt Corners, Seneca Co., N.Y.*
FORMAN, J. W., B.A. (Univ. of Omaha) Grad. Asst., Univ. of Iowa, Iowa City, Iowa. *Physics Bldg. 316A*
FORMAN, WILLIAM, A.M. (Brooklyn Coll.) Instr., Brooklyn Coll. Evening Session, Brooklyn, N.Y. *494 Powell St., Brooklyn 12, N.Y.*
FORRAY, M. J., M.S. (New York Univ.) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *729 Kelly St., New York 55, N.Y.*
FORSYTH, Prof. C. H., Ph.D. (Michigan) Dartmouth Coll., Hanover, N.H.
FORT, M. K., Jr., A.M. (Virginia) Instr., Univ. of Virginia, Charlottesville, Va. *318 Park Place*
FORT, Prof. TOMLINSON, Ph.D. (Harvard) Head of Dept., Univ. of Georgia, Athens, Ga.

FOSTER, Asso. Prof. MARY L., M.S. (Louisiana State) Henderson State Coll., Arkadelphia, Ark.
FOSTER, Prof. R. M., B.S. (Harvard) Head of Dept., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *136 Stanmore Place, Westfield, N.J.*
FOUST, Prof. J. W., Ph.D. (Michigan) Central Michigan Coll. of Educ., Mt. Pleasant, Mich.
FOX, Asso. Prof. A. H., Ph.D. (Yale) Union Coll., Schenectady, N.Y. *1101 Millington Rd.*
FOX, Sister CATHERINE DOROTHEA, Ph.D. (Boston Coll.) President Trinity Coll., Washington 17, D.C.
FRAME, Prof. J. S., Ph.D. (Harvard) Head of Dept., Michigan State Coll., East Lansing, Mich. *821 E. Grand River Ave.*
FRANCIS, S. A., A.M. (California) Chm. of Dept. Math. & Engg., San Mateo Jr. Coll., San Mateo, Calif. *1705 Cowper St., Palo Alto, Calif.*
FRANCIS XAVIER, Sister, Ph.D. (Fordham) Head of Dept. St. Joseph's Coll. for Women, Brooklyn 5, N.Y. *232 Clinton Ave.*
FRANK, Asst. Prof. EVELYN, Ph.D. (Northwestern) University of Ill., Evanston, Ill. *2226 Sherman Ave.*
FRANK, F. T., A.B. (Stanford) Asst. to Mgr., Ins. Dept., Parrott and Co., San Francisco, Calif. *708 Ashbury St., San Francisco 17, Calif.*
FRANKEL, E. T., B.S. (C.C.N.Y.) Budget Analyst Federation of Social Agencies, 519 Smithfield St., Pittsburgh 22, Pa.
FRANKEL, R. W., A.B. (Michigan) Michigan Bell Telephone Co., Ypsilanti, Mich. *1309 Sudbury, Willow Run, Mich.*
FRANKENBUSH, BERTHA E., A.M. (Tulane) Retired, High School Teacher, New Orleans, La. *1217 Jefferson Ave.*
FRANKLIN, Prof. PHILIP, Ph.D. (Princeton) Massachusetts Inst. of Tech., Cambridge, Mass. *312 Pleasant St., Belmont 78, Mass.*
FRASER, G. C., A.M. (Pennsylvania) Teacher George School, Bucks Co., Pa.
FRASER, W. A., Ph.D. (Iowa State) Research Engr., The Perkin-Elmer Corp., Glenbrook, Conn.
FRASER, W. C. G., Ph.D. (Toronto) Instr., Dartmouth Coll., Hanover, N.H. *1 S. Park St., c/o Dept. of Math., Rensselaer Polytechnic Inst., Troy, N.Y.*
FREAS, ELIZABETH, A.M. (Louisiana State) Asst., Louisiana State Univ., University State, Baton Rouge, La. *Dept. of Math.*
FRECHETTE, EMILE, Lic. es Sci. Math. (Montreal) Teacher, Seminary of Philosophy, 3880 Cote-des-Neiges Road, Montreal 25, P.Q., Can.
FREDERICKSON, ELVY L., A.B. (Willamette) *34 S.E. 70th Ave., Portland, Ore.*
FREE, N. S., M.A. (Univ. of British Columbia) Lecturer, Univ. of Birtish Columbia, Vancouver, B.C., Can. *Dept. of Math.*
FREESE, Asst. Prof. FRANCES, A.M. (Radcliffe) Dean of Women, Mount Union Coll., Alliance, Ohio
FRICK, Prof. C. H., Ph.D. (North Carolina) Mary Washington Coll., Fredericksburg, Va. *Box 1116 Coll. Sta.*
FRIERSON, Asso. Prof. JACK, M.S. (South Carolina) Coker Coll., Hartsville, S.C.
FRINK, Prof. ORRIN, Jr., Ph.D. (Columbia) Pennsylvania State Coll., State College, Pa. *706 Sunset Road*
FRY, THORNTON C., Ph.D. (Wisconsin) Dir., Switching Research, Bell Telephone Labs., 463 West St., New York 14, N.Y.
FRY, W. J., M.S. (Pennsylvania State) *907 W. Eureka St., Champaign, Ill.*
FUDGE, HELEN G., Ph.D. (Pennsylvania) Teacher, Holmes Jr. High School, Philadelphia, Pa. *Rosemont, Pa.*
FULLER, Prof. GORDON, Ph.D. (Michigan) Alabama Poly. Inst., Auburn, Ala. *346 Payne St.*
FULLER, GRACE A., A.B. (Fresno State Coll.) Head of Dept., Union High School, Madera, Calif. *901 W. Yosemite Ave.*
FULLER, Asso. Prof. K. G., A.M. (Nebraska) Teachers Coll. of Connecticut, New Britain, Conn. *316 Barbour Rd.*
FULLER, L. E., M.S. (Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis. *North Hall*
FULMER, Asso. Prof. H. K., A.M. (Columbia) Georgia School of Tech., Atlanta, Ga.
FULTON, C. M., Ph.D. (Tech. Hochschule) Instr., Univ. of California, Davis, Calif. *Dept. of Math.*
FUNKHOUSER, H. G., Ph.D. (Columbia) Instr., Phillips Exeter Acad., Exeter, N.H. *Cilley Hall*
FURMAN, Asst. Prof. ALBERT, M.S. (New Hampshire) Kansas State Coll., Manhattan, Kans. *Dept. of Math.*

GABRIELLE MARIE, Sister, M.A. (Catholic Univ.) Instr., Dunbarton Coll., Washington, D.C. *2935 Upton St., Washington 8, D.C.*

GAFFNEY, M. P., Jr., B.S. (Harvard) Student, Univ. of Chicago, Chicago, Ill. *525 Ash St., Winnetka, Ill.*

GAGE, Prof. W. H., M.A. (Univ. of B.C.) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*

GAGER, Acting Asso. Prof. W. A., Ph.D. (Peabody) Univ. of Florida, Gainesville, Fla. *Peabody Hall*

GAINES, Prof. R. E., A.M. (Furman) Univ. of Richmond, Richmond, Va.

GALBRAITH, Asst. Prof. M. G., M.S. (Rutgers) Rutgers Univ., New Brunswick, N.J. *121 Magnolia St.*

GALE, Prof. A. S., Ph.D. (Yale) Emeritus, Univ. of Rochester, Rochester, N.Y. *93 Bellevue Drive, Rochester 7, N.Y.*

GANDY, Asso. Prof. W. W., M.S. (Texas A. and M. Coll.) Northwestern State Coll., Natchitoches, La. *Box 1086 Coll.*

GARABEDIAN, Prof. C. A., Ph.D. (Harvard) Wheaton Coll., Norton, Mass.

GARCIA, Asso. Prof. MARIANO, Jr., Ph.D. (Virginia) Coll. of Agriculture, Univ. of Puerto, Rico, Mayaguez, P.R.

GARNER, Asso. Prof. L. L., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *507 North St.*

GARRETT, Prof. J. A., A.M. (Peabody) Arkansas A. and M. Coll., Monticello, Ark.

GARRETT, Prof. W. H., A.M. (Illinois Coll.) Vice President, Baker Univ., Baldwin, Kans.

GARRISON, Asso. Prof. L. M., Ed.M. (Peabody) Louisiana Poly. Inst., Ruston, La. *Box 67 Tech. Sta.*

GARVIN, Sister MARY CLEOPHAS, Ph.D. (St. Louis Univ.) Head of Dept. of Science, Notre Dame Coll., South Euclid 21, Ohio

GASAWAY, SADIE C., M.S. (Illinois) Instr., Tennessee A. and I. State Coll., Nashville, Tenn. *Box 105*

GASKELL, Asst. Prof. R. E., Ph.D. (Michigan) Iowa State Coll., Ames, Iowa. *Dept. of Math.*

GATEWOOD, Asso. Prof. B. E., Ph.D. (Wisconsin) Army Air Forces Inst. of Tech., Wright Field, Dayton, Ohio. *919 Hampshire Road, Dayton 9, Ohio*

GAULT, Prof. A. E., M.S. (Chicago) Dean, Arts and Sci., Bradley Univ., Peoria 5, Ill.

GAUTHIER, Prof. ABEL, A.M. (Columbia) Univ. of Montreal, 2900 Blvd. Mont-Royal, Montreal, P.Q., Can.

GAVER, Prof. W. H., A.B. (Randolph-Macon) Newberry Coll., Newberry, S.C. *Box 345*

GAY, Prof. H. J., A.M. (Clark) Worcester Poly. Inst., Worcester, Mass. *7 Belvidere Ave., Worcester 5, Mass.*

GAYLORD, Asst. Prof. LESLIE J., M.S. (Chicago) Agnes Scott Coll., Decatur, Ga.

GEHMAN, Prof. H. M., Ph.D. (Pennsylvania) Univ. of Buffalo, Buffalo 14, N.Y. *163 Winspear Ave., Buffalo 15, N.Y.*

GELBART, Asso. Prof. ABE, Ph.D. (Mass. Inst. of Tech.) Institute for Advanced Study, Princeton, N.J.

GENTRY, Prof. F. C., Ph.D. (Illinois) Louisiana Poly. Inst., Ruston, La. *Box 423*

GENTZLER, W. E., A.M. (Columbia) Bursar, Columbia Univ., New York 27, N.Y.

GEORGES, J. S., Ph.D. (Chicago) Chm. of Dept., Wright Jr. Coll., Chicago, Ill. *4515 N. Kildare Ave., Chicago 30, Ill.*

GEPHART, LANDIS, B.S. (Dayton) Instr., Univ. of Dayton, Dayton, Ohio. *337 Irving Ave., Dayton 9, Ohio*

GERE, Asso. Prof. B. H., Ph.D. (Mass. Inst. of Tech.) Hamilton Coll., Clinton, N.Y. *College Hill*

GERGEN, Prof. J. J., Ph.D. (Rice) Duke Univ., Durham, N.C. *Box 4771 Duke Sta.*

GERST, Rev. F. J., Ph.D. (Johns Hopkins) Prof. Chm., of Dept., Loyola Univ., 6525 Sheridan Road, Chicago 26, Ill.

GERST, IRVING, A.M. (Columbia) Tech. Consultant, Army Service Forces. *1055 Clarkson Ave., Brooklyn, N.Y.*

GETCHELL, B. C., Ph.D. (Michigan) Research Analyst, U. S. War Dept., Washington, D.C. *903 N. Wayne St., Apt. 305, Arlington, Va.*

GHANI, S. A., M.A. (Muslim Univ., India) Lecturer, Nizam Coll., Hyderabad, Deccan, India

GHENT, Asso. Prof. K. S., Ph.D. (Univ. of Chicago) Univ. of Oregon, Eugene, Ore. *2105 McMillan St.*

GIBBENS, Asso. Prof. GLADYS, Ph.D. (Chicago) Univ. of Minnesota, Minneapolis 14, Minn. *121 Folwell Hall*

GIBBONS, Sister MARY SERAPHIM, A.M. (Minnesota) Coll. of St. Catherine, St. Paul 1, Minn.

GIBNEY, ESTHER F., Ph.D. (Northwestern) Chm. of Dept. Roosevelt High School, Chicago, Ill. *2970 Sheridan Road, Chicago 14, Ill.*
GIESE, J. H., Ph.D. (Princeton) Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md. *667 Green St., Havre de Grace, Md.*
GILBERT, W. M., B.A. (Oregon) Student Univ of Oregon, Eugene, Ore. *170 Walnut Lane*
GILL, Prof. B. P., Ph.D. (Columbia) Coll. of the City of New York, New York, N.Y. *493 Warwick Ave., West Englewood, N.J.*
GILLAM, Asso. Prof. B. E., Ph.D. (Missouri) Head of Dept., Drake Univ., Des Moines, Iowa
GILLETTE, E. F., A.B. (Hamilton) Master, The Taft School, Watertown, Conn.
GILLIS, Prof. M. E., A.M. (Chicago) Blue Mountain Coll., Blue Mountain, Miss.
GILMAN, Asso. Prof. R. E., Ph.D. (Princeton) Brown Univ., Providence, R.I. *44 E. Manning St.*
GILMORE, A. L., Jr., B.S. (Miss. Southern Coll.) Grad. Fellow, Tulane Univ., New Orleans, La. *401 Dearborn St., Hattiesburg, Miss.*
GINGRICH, Prof. C. H., Ph.D. (Chicago) Math. and Astr., Carleton Coll., Northfield, Minn. *213 Maple St.*
GINNINGS, R. M., M.S. (Chicago) Emeritus, Head of Dept., Western Illinois State Teachers Coll., Macomb, Ill. *314 N. Ward St.*
GINSBURG, Prof. JEKUTHIEL, A.M. (Columbia) Head of Dept., Yeshiva Coll., New York, N.Y. *610 West 139 St., New York 31, N.Y.*
GIVENS, Prof. J. W., Ph.D. (Princeton) Univ. of Tennessee, Knoxville, Tenn. *Dept. of Math.*
GLANDER, HAROLD, B.S. (Milwaukee St. T. C.) Student, Marquette Univ. Grad. School, Milwaukee, Wis. *3750-A So. 14 St., Milwaukee 7, Wis.*
GLAZIER, Asst. Prof. HARRIET E., A.M. (Chicago) Emeritus, U.C.L.A., Los Angeles, Calif. *1307 Lucile Ave., Los Angeles 26, Calif.*
GLEASON, A. M., B.S. (Yale) Jr. Fellow, Harvard Univ., Cambridge, Mass. *A-24 Winthrop House, Cambridge 38, Mass.*
GLENN, W. H., JR., A.M. (U.C.L.A.) Head, Div. Natural Sci., Muir Jr. Coll., Pasadena, Calif. *1425 Beech St., South Pasadena, Calif.*
GLOVER, Prof. B. C., A.M. (Chicago) Otterbein Coll., Westerville, Ohio. *220 Hiawatha Ave.*
GODDERZ, H. W., B.S. (Northern St. T. C.) Coll. of St. Thomas, St. Paul 1, Minn. *824 Grand Ave., St. Paul 5, Minn.*
GODFREY, Asst. Prof. E. L., A.M. (Indiana) Head of Dept., Math. and Astr., Defiance Coll., Defiance, Ohio. *125 Session St.*
GOEN, F. MONICA, M.S. (Iowa) Instr., Mississippi State Coll., State College, Miss. *Box 1015*
GOLD, Prof. J. S., A.M. (Bucknell) Math. and Astr., Bucknell Univ., Lewisburg, Pa. *306 S. Third St.*
GOINS, Asst. Prof. MARY A., A.M. (Michigan) Marshall Coll., Huntington, W. Va. *435 Ninth Ave.*
GOLDBERG, MICHAEL, A.M. (George Washington) Principal Engr., Bureau of Ordnance, Navy Dept., Washington, D.C. *5823 Potomac Ave. N.W., Washington 16, D.C.*
GOLDSTINE, Asst. Prof. H. H., Ph.D. (Chicago) Institute for Advanced Study, Princeton, N.J.
GOLDMAN, I. L., A.B. (Columbia) Lecturer, Columbia Univ., New York, N. Y. *107 Broadway, Hicksville, N.Y.*
GOLOMB, Asso. Prof. MICHAEL, Ph.D. (Berlin) Purdue Univ., Lafayette, Ind. *R.R. 10*
GOLUB, ABRAHAM, A.B. (Brooklyn) Mathematician, Ballistics Research Lab., Aberdeen Proving Ground, Aberdeen, Md.
GOMAN, E. G., M.S. (Oregon State Coll.) Instr., College of Puget Sound, Tacoma, Wash.
GONZALEZ, Prof. M. O., D.P.M.S. (Havana) Univ. of Havana, Havana, Cuba. (*On leave*) *Univ. of Alabama, University, Ala. Dept. of Math.*
GOOD, J. M., M.S. (Brown) *928 Lassen St., Richmond, Calif.*
GOOD, Asst. Prof. R. A., Ph.D. (Wisconsin) Univ. of Maryland, College Park, Md. *Dept. of Math.*
GOODHUE, Asso. Prof. E. A., M.S. (Missouri School of Mines) Missouri School of Mines, Rolla, Mo. *Box 317, Rolla, Mo.*
GOODMAN, A. W., Ph.D. (Columbia) Instr., Rutgers Univ., New Brunswick, N.J. *Dept. of Math.*
GOODRICH, M. T., A.M. (Clark Univ.) Head of Dept., Keene Teachers Coll., Keene, N.H. *36 Wyman Way*
GOORMAGHTIGH, RENE-VICTOR, C.E. (Univ. of Ghent) General Mgr., La Brugeoise Steelworks, Bruges, Belgium. *15 Blvd. Elisabeth*
GORDON, Asso. Prof. W. O., A.M. (Pennsylvania State) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*

GORE, Prof. G. D., Ph.D. (Chicago) Chm. of Dept., Roosevelt Coll., Chicago, Ill. *430 S. Michigan Ave., 10th Floor, Chicago 5, Ill.*
GORMAN, J. R., A.M. (U.C.L.A.) Instr., U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
GORMSEN, S. T., B.S. (Ohio State) Instr., Univ. of Florida, Gainesville, Fla. *1659 W. Columbia St.*
GORRELL, Prof. G. W., A.M. (Ohio State) Emeritus, Univ. of Denver, Denver, Colo. *Box 1243, Estes Park, Colo.*
GORSLINE, K. E., A.M. (Denver) Teacher, East High School, Denver, Colo. *1451 Grape St., Denver 7, Colo.*
GOTTLIEB, Asst. Prof. M. J., Ph.D. (Washington Univ.) Washington Univ., St. Louis, Mo. *On Leave. Institute for Advanced Study, Princeton, N.J.*
GOTTSCHALK, Asst. Prof. W. H., Ph.D. (Virginia) Univ. of Pennsylvania, Philadelphia 4, Pa. *On Leave. Institute for Advanced Study, Princeton, N.J.*
GOUGH, Prof. E. St. J., B.A. (Montreal) Jacques Cartier Normal School, Montreal 24, P.Q., Can. *4415 Coronation Ave., Montreal 28, P.Q., Can.*
GOULD, ALICE B., A.B. (Bryn Mawr) *35 Congress St., Boston, Mass.*
GOULD, Asst. Prof. S. H., Ph.D. (Yale) Purdue Univ., Lafayette, Ind. *Dept. of Math.*
GOURRICH, G. E., B.S. (California) Student, Univ. of California, Los Angeles, Calif. *450 N. Rossmore Ave., Los Angeles 4, Calif.*
GOUWENS, Asso. Prof. CORNELIUS, Ph.D. (Chicago) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
GOVE, H. E., M.S. (Washington Univ.) Union Elec. Co. of Missouri, 315 N. 12th Blvd., St. Louis 1, Mo. *Engg. and Const. Dept.*
GRABBE, REV. HYACINTH, A.M. (Catholic Univ.) Prof., St. Joseph's Coll. and Milit. Acad. Hays, Kans.
GRABLE, Asst. Prof. E. S., A.M. (Washington and Jefferson) Univ. of Richmond, Richmond, Va. *Box 45*
GRAD, ARTHUR, A.M. (Columbia) Research Assoc. & Acting Instr., The Knoll, Stanford University, Calif.
GRAEBER, ALICE, A.B. (California) Teacher, Bates School, San Francisco, Calif. *210 Eleventh Ave., San Francisco 18, Calif.*
GRAESSER, Prof. R. F., Ph.D. (Illinois) Univ. of Arizona, Tucson, Ariz. *1648 E. Fifth St.*
GRAFF, Asst. Prof. F. G., A.M. (Pittsburgh) Amherst Coll., Amherst, Mass. *85 Dana St.*
GRAHAM, Asso. Prof. MARIA D., A.M. (Columbia) Retired, East Carolina Teachers Coll., Greenville, N.C. *Warrenton, N.C.*
GRAHAM, Asso. Dean P. H., A.M. (Virginia) Chm. of Dept., New York Univ., Washington Square Coll., Washington Square E., New York, N.Y.
GRAHAM, Asso. Prof. W. W., Ph.D. (Geo. Peabody Coll.) Vanderbilt Univ., Nashville 4, Tenn. *Box 18, Calhoun Bldg.*
GRANT, ALICE A., A.M. (Brown) Principal, Miss Grant's School, 215 College St., Toronto 2B, Ont., Can. *Apt. 241, Cawths Mansions*
GRANT, Asso. Prof. H. S., Ph.D. (Pennsylvania) Rutgers Univ., New Brunswick, N.J. *108 S. Third Ave., Highland Park, N. J.*
GRAS, E. C., A.M. (Harvard) Instr., U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
GRAU, A. A., PhD. (Michigan) *26 Pinehurst Drive, Tuscaloosa, Ala.*
GRAVATT, Prof. T. E., M.S. (Rutgers) Pennsylvania State Coll., State College, Pa. *344 E. College Ave.*
GRAVES, Asst. Prof. C. H., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa.
GRAVES, Asso. Prof. G. H., Ph.D. (Columbia) Purdue Univ., West Lafayette, Ind. *227 S. Grant St.*
GRAVES, Prof. L. M., Ph.D. (Chicago) Univ. of Chicago, Chicago 37, Ill. *Visiting Professor, Indiana Univ., Bloomington, Ind. 1610 E. First St.*
GRAVES, W. L., A.M. (Pennsylvania) Head of Dept., Drury Coll., Springfield, Mo. *920 S. Fremont Ave.*
GRAY, MARION C., Ph.D. (Bryn Mawr) Member, Tech. Staff, Bell Telephone Labs. 463 West St., New York 14, N. Y.
GRAY, MARY W. (Mrs. A. B.), A.M. (Connecticut Coll.) Instr., Barnard School for Girls, 554 Ft. Washington Ave., New York, N.Y. *87–40 Elmhurst Ave., Elmhurst, N.Y.*
GREEN, DOROTHY E., M.A. (Peabody) Chm. of Dept., Huntingdon Coll., Montgomery, Ala.
GREEN, Asst. Prof. J. W., Ph.D. (California) U.C.L.A., 405 Hilgard Ave., Los Angeles 24, Calif. *Dept. of Math.*
GREEN, Asst. Prof. L. J., Ph.D. (Chicago) Case School of Technology, Cleveland 6, Ohio
GREENE, Asst. Prof. LAURA Z., M.S. (Chicago) Washburn Univ., Topeka, Kans. *1516 Boswell Ave.*
GREENE, R. L., B.E.E. (Clarkson) Instr., Clarkson Coll., Potsdam, N.Y.

GREENLEAF, Prof. H. E. H., Ph.D. (Indiana) DePauw Univ., Greencastle, Ind. *1024 S. College Ave.*
GREENWOOD, Asst. Prof. R. E., Jr., Ph.D. (Princeton) Univ. of Texas, Austin, Tex. *1704 Windsor Road*
GREER, CASSIE C., A.M. (Chicago) Teacher, Englewood High School, Chicago, Ill. *6056 S. Kimbark Ave., Chicago 37, Ill.*
GREER, Asso. Prof. EDISON, Ph.D. (Kansas) Kansas State Coll., Manhattan, Kans. *1600 Pierre*
GREGORY, Brother JOSEPHUS, A.M. (Loyola Univ.) Instr., St. Mary's Coll., Winona, Minn.
GRENARD, MADELINE, A.M. (Nebraska) Instr., Univ. of Illinois, Chicago, Ill. *5137 Harper Ave.*
GREVILLE, T. N. E., Ph.D. (Michigan) Actuarial Mathematician, National Office Vital Statistics, U. S. Public Health Service, Washington 25, D.C.
GRIFFIN, Prof. F. L., Ph.D. (Chicago) Reed Coll., Portland 2, Ore.
GRIFFIN, Asst. Prof. HARRIET M., Ph.D. (New York Univ.) Brooklyn Coll., Brooklyn, N.Y. *3609 Farragut Road, Brooklyn 10, N.Y.*
GRIFFITH, Asso. Prof. W. C., A.M. (Oregon) Centenary Coll., Shreveport, La.
GRIMES, A. C., A.M. (Mississippi) Instr., Mississippi State Coll., State College, Miss. *Box 1073*
GROSSMAN, GEORGE, A.M. (Columbia) Chm. of Dept., De Witt Clinton High School, New York, N.Y.; Instr., Poly. Inst. of Brooklyn. *3425 Gates Place, New York 67, N.Y.*
GROTTS, G. W., M.S. (Illinois) Instr., General Motors Inst., Flint, Mich. *2534 Bagley St.*
GROVE, Asst. Prof. C. C., Ph.D. (Johns Hopkins) Emeritus, Coll. of the City of New York, New York 10, N.Y. *143 Milburn Ave., Baldwin, N.Y.*
GROVE, Prof. V. G., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *438 Rosewood Ave.*
GROVES, MARJORIE J., A.M. (Chicago) Teacher, Morton Jr. High School, Hammond, Ind. *6819 Rosewood St.*
GUARD, Asst. Prof. H. T., M.S. (Colorado) Colorado State Coll. of A. and M.A., Fort Collins, Colo. *Dept. of Math.*
GUNDER, Prof. D. F., Ph.D. (Wisconsin) Mechanics Dept., Coll. of Engr., Cornell Univ., Ithaca, N.Y.
GUNDERSON, N. G., A.M. (Cornell) Instr., Univ. of Rochester, Rochester 3, N.Y. *Dept. of Math.*
GUSTIN, W. S., A.B. (U.C.L.A.) Indiana University, Bloomington, Ind. *Dept. of Math.*
GUTHRIE, Prof. R. V., Jr., M.S. (Iowa) Southwestern Louisiana Inst., Lafayette, La.
GUY, W. T., Jr., B.S. (Texas A. and M. Coll.) Instr., Univ. of Texas, Austin, Tex. *305 Journalism Bldg.*
GYSLAND, Mrs. MARIAN S., A.M. (California) Instr., Univ. of Colo., Extension Div., Denver, Colo. *1158 S. Clayton St.*

HAAG, V. H., A.M. (Duke) Instr., Hershey Jr. Coll., Hershey, Pa. *Box 13*
HACKER, Asso. Prof. S. G., Ph.D. (Princeton) State Coll. of Washington, Pullman, Wash. *Dept. of Math.*
HADLEY, J. R., B.S. (Ohio State) Industrial Engineer, Buda Co., Harvey, Ill. *743 Oakdale Ave., Apt. B., Chicago 14, Ill.*
HADLEY, Lena C. (Mrs. A. J.), A.M. (Missouri) *426 E. Market St., Warrensburg, Mo.*
HADLOCK, Prof. E. H., Ph.D. (Cornell) Hastings Coll., Hastings, Nebr.
HADNOT, B. F., Student, Univ. of Alabama, University, Ala. *P.O. Box 565*
HAFNER, Ralph, M.A. (Michigan) Instr., Univ. of Dayton, Dayton, Ohio. *Box 134*
HAGEN, Asso. Prof. BEATRICE L., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
HAILPERIN, Asst. Prof. THEODORE, Ph.D. (Cornell) Lehigh Univ., Bethlehem, Pa. *Dept. of Math.*
HAIR, C. L., A.M. (Duke) Head of Dept., The Citadel, Charleston, S.C.
HALL, Prof. D. W., Ph.D. (Virginia) Univ. of Maryland, College Park, Md. *Box 162*
HALL, Prof. N. A., Ph.D. (Calif. Inst. of Tech.) Univ. of Minnesota, Minneapolis 14, Minn. *Dept. of Mech. Engg.*
HALLER, Asst. Prof. MARY E., Ph.D. (Washington) Univ. of Washington, Seattle 5, Wash. *Dept. of Math.*
HALLER, M. N., B.S. in E.E. (Louisville) *Cowen, W. Va.*
HALLERBERG, Asst. Prof. A. E., A.M. (Illinois) Illinois Coll., Jacksonville, Ill. *605 W. State St.*
HALLETT, W. N., Ph.D. (Pennsylvania) Instr., State Teachers Coll., Frostburg, Md. *324 Forest Ave., Ben Avon, Pittsburgh 2, Pa.*

HALMOS, Asst. Prof. P. R., Ph.D. (Illinois) Inst. for Advanced Study, Princeton, N.J.
HALPERIN, Asso. Prof. ISRAEL, Ph.D. (Princeton) Queen's Univ., Kingston, Ont., Can. *115 King St. West*
HAMILTON, Asso. Prof. H. J., Ph.D. (Brown) Pomona Coll., Claremont, Calif. *735 Yale St.*
HAMILTON, Prof. O. H., Ph.D. (Texas) Oklahoma A. and M. Coll., Stillwater, Okla. *503 Pine St.*
HAMILTON, W. H., A.M. (Columbia) Head of Dept., High School, Highland Park, N.J. *232 Wayne St.*
HAMILTON, W. M., A.M. (Michigan) Retired, *1512 Granger Ave., Ann Arbor, Mich.*
HAMMER, Asso. Prof. P. C., Ph.D. (Ohio State) Oregon State Coll., Corvallis, Ore.
HAMMER, Dean W. B., A.M. (Iowa) Junior Coll., Estherville, Iowa
HAMMING, R. W., Ph.D. (Illinois) Member, Tech. Staff, Bell Telephone Labs., Murray Hill, N.J.
HAMMOND, Prof. E. S., Ph.D. (Princeton) Bowdoin Coll., Brunswick, Me. *9 Thompson St.*
HAMMOND, Asso. Prof. J. R., A.M. (Harvard) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
HANCOCK, CLARA L., A.M. (Iowa) Instr., Junior Coll., Virginia, Minn. *Box 706*
HAND, Sister MIRIAM, A.B. (Immaculate Heart) Instr., Immaculate Heart Coll., 2021 N. Western Ave., Hollywood, Calif.
HANHAUSER, M. A., A.B. (St. Bonaventure) Siena Coll., Loudonville, N.Y.
HANNA, Asst. Prof. J. R., M.S. (Kans. St. T. C.) Univ. of Wichita, Wichita 6, Kans.
HANSMAN, Asst. Prof. Margaret M., Ph.D. (Illinois) Colorado Coll., Colorado Springs, Colo. *2038 Armstrong Ave.*
HANSON, E. H., Ph.D. (Ohio State) Director, North Texas State Teachers Coll., Denton, Texas. *619 W. Oak St.*
HANSON, W. R., M.S. (Chicago) Instr., San Francisco Jr. Coll., San Francisco 12, Calif. *327 Judson Ave*
HARARY, FRANK, A.M. (Brooklyn) Teaching Asst., Univ. of California, Berkeley 4, Calif. *Dept. of Math.*
HARDIN, J. A., A.M. (Chicago) Head of Dept., Centenary Coll., Shreveport 16, La.
HARDING, HOWARD, B.M.E. (Michigan) Rochester Gas and Elec. Corp., Rochester, N.Y. *29 Kingston St.*
HARDY, Prof. G. H., M.A. (Trinity, Cambridge) Trinity Coll., Cambridge Univ., Cambridge England
HARDY, H. M., B.S. (S.H.S.T.C.) Grad. Asst., Sam Houston St. Teachers Coll., Huntsville, Tex. *1323 Avenue J.*
HARMAN, H. H., M.S. (Chicago) Research Asso., National Research Council, Washington 25, D.C. *4111 Maryland Ave., Washington 16, D.C.*
HARMON, ELIZABETH H. (Mrs. J. P.), M.S. (Washington Univ.) *4800 Forbes St., Pittsburgh 13, Pa.*
HARP, E. L., Jr., A.M. (New Mexico) Head of Dept., High School, Roswell, N. M. *Box 6651*
HARPER, Prof. F. S., Ph.D. (Iowa) Head of Dept., Actuarial Science, Drake Univ., Des Moines, Iowa
HARRELL, E. G., Ph.D. (Iowa) Head of Dept., State Teachers Coll., Platteville, Wis.
HARRINGTON, C. E., M.S. (Buffalo) Engr., W. H. Smith, Inc., Springville, N.Y. *52 Winter St., Buffalo 13, N.Y.*
HARRINGTON, Asst. Prof. W. J., Ph.D. (Cornell) Penn. State College, State College, Pa. *Dept. of Math.*
HARRIS, Asso. Prof. ISABEL, A.M. (Columbia) Westhampton Coll., Univ. of Richmond, Richmond, Va.
HARRIS, L. J., B.E.E. (Rensselaer Poly. Inst.) Elec. Engr., Aluminum Co. of America, Alcoa Reduction Plant, Alcoa, Tenn. *P.O. Box 102, Maryville, Tenn.*
HARRIS, MINNIE W. C. (Mrs. D. P.), A.M. (Missouri) Instr., Physics, North Carolina State Coll., Raleigh, N.C. *1203 Filmore St.*
HARRISON, R. A., Ph.D. (Cornell) Master, St. Mark's School, Southboro, Mass.
HARRJE, H. J., Arch. (Cooper Union), Civ. Engr. (C.C.N.Y.) Harrje Associates, Arch. and Engr., P.O. Box 746, Jacksonville Beach, Florida
HARSHBARGER, Asso. Prof. FRANCES, Ph.D. (Illinois) Kent State Univ., Kent, Ohio
HART, BERTHA I., A.M. (Cornell) Chief Computer, Ballistic Research Lab., Aberdeen Proving Ground, Md.
HART, Prof. W. L., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *100 Folwell Hall*
HART, Asso. Prof. W. W., A.B. (Chicago) Retired, Univ. of Wisconsin, Madison, Wis. *139 Tudor Place, Kenilworth, Ill.*

HARTIG, Prof. H. E., Ph.D. (Minnesota) Head of E.E. Dept., Communication Engg., Univ. of Minnesota, Minneapolis 14, Minn. *3940 Lake Curve, Robbinsdale 12, Minn.*
HARTLEY, Asst. Prof. M. C., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *1203 W. Illinois St.*
HARTNELL, GEORGE, Retired, U. S. Coast and Geodetic Survey. *Wyoming, N.Y.*
HARTUNG, Asso. Prof. M. L., Ph.D. (Wisconsin) Univ. of Chicago, Chicago 37, Ill.
HARTZLER, Prof. H. H., Ph.D. (Rutgers) Goshen Coll., Goshen, Ind.
HARVEY, Asso. Prof. G. G., Ph.D. (Washington Univ.) Physics, Massachusetts Inst. of Tech., Cambridge 39, Mass.
HARWOOD, Asst. Prof. MAY N., A.M. (Syracuse) Syracuse Univ., Syracuse, N.Y. *803 Comstock Ave.*
HASKINS, ELIZABETH M., M.S. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge, Mass. *27 Vernon St., Worcester 4, Mass.*
HASSLER, Prof. J. O., Ph.D. (Chicago) Math. and Astr., Univ. of Oklahoma, Norman, Okla. *425 S. Lahoma Ave.*
HASTINGS, Cecil, Jr., B.S. (Florida) Research Analyst, Douglas Aircraft Corp., Santa Monica, Calif. *907 Fifth St.*
HATCH, Asso. Prof. D. A., A.M. (Columbia) Lafayette Coll., Easton, Pa. *705 High St.*
HATCHER, Prof. T. W., Ph.D. (Cornell) Head of Dept., Virginia Poly. Inst., Blacksburg, Va. *P.O. Box 172*
HATFIELD, Prof. CHARLES, A.M. (Tennessee) Head of Dept., Georgetown Coll., Georgetown, Ky. *301 Clayton Ave.*
HATFIELD, Asst. Prof. CHARLES, Jr., Ph.D. (Cornell) Univ. of Minnesota, Minneapolis, Minn. *4612 14th Ave. S., Minneapolis 7, Minn.*
HATTAN, CORRINE R., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *1008 W. Stoughton*
HAUSMANN, Rev. B. A., Ph.D. (Yale) West Baden Coll., West Baden Springs, Ind.
HAVILAND, Asst. Prof. E. K., Ph.D. (Harvard; Johns Hopkins) Johns Hopkins Univ., Baltimore, Md. *203 Ridgewood Road, Baltimore 10, Md.*
HAWTHORNE, FRANK, M.A. (Columbia) Instr., Hofstra Coll., Hempstead, N.Y. *40-C Broadfield Rd.*
HAY, Asso. Prof. G. E., Ph.D. (Toronto) Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*
HAYES, J. J., B.S. (Utah) Instr., Univ. of Utah, Salt Lake City, Utah
HAYNES, Asso. Prof. NOLA L. (Mrs. M. B.), Ph.D. (Missouri) Univ. of Mo., Columbia, Mo. *1408 Rosemary Lane*
HAYWARD, Asst. Prof. LEOTA C., M.S. (Colorado State Coll.) Acting Head of Dept., Colorado State Coll. of A. and M.A., Fort Collins, Colo. *Dept. of Math.*
HAZARD, Prof. C. T., A.M. (Indiana) Purdue Univ., West Lafayette, Ind. *344 N. Western Ave.*
HAZARD, KATHARINE E., Ph.D. (Chicago) Instr., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
HAZELTINE, Prof. B. A., A.M. (Columbia) Middlebury Coll., Middlebury, Vt.
HAZELTINE, L. A., A.M. (Columbia) *15 Tower Drive, Maplewood, N.J.*
HAZLETON, E. W., A.M. (Chicago) Teacher, Chicago Board of Educ.; Instr., Ill. Inst. of Tech. Chicago, Ill. *8039 Maryland Ave., Chicago 19, Ill.*
HAZLEWOOD, Prof. E. A., Ph.D. (Cornell) Texas Tech. Coll., Lubbock, Tex. *3218–20th St.*
H'DOUBLER, F. T., M.D. (Harvard), Ph.D. (Wisconsin) Medical Arts Bldg., Springfield, Mo.
HEARN, Rev. J. R., A.M. (Woodstock) Old St. Joseph's Church, 321 Willing's Alley, Philadelphia, Pa.
HEASLET, M. A., Ph.D. (Stanford) Physicist, Dept. of Theoretical Aerodynamics, N.A.C.A. Moffett Field, Calif. *Box 402, Los Altos, Calif.*
HEATH, R. V., Member, New York Stock Exchange, 40 Wall St., New York, N. Y.
HEBEL, I. L., M.S. (Colorado) Head of Dept., Colorado School of Mines, Golden, Colo.
HEDBERG, Asso. Prof. E. A., Ph.D. (Missouri) Univ. of South Carolina, Columbia, S.C. *Sloan Coll.*
HEDLUND, Prof. G. A., Ph.D. (Harvard) Univ. of Virginia, Charlottesville, Va. *1831 Wayside Place*
HEFNER, Prof. R. A., Ph.D. (Chicago) Dean, General Studies, Georgia School of Tech., Atlanta, Ga. *Faculty Exchange Box 2102*
HEILMAN, C. E., A.M. (Duke) Instr., Syracuse Univ., Syracuse, N.Y. *352 N. 10th St., Lebanon, Pa.*
HEIMANN, Prof. E. E., A.M. (Texas) East Central State Coll., Ada, Okla. *523 S. Highland St.*

HEINEMAN, Prof. E. R., A.M. (Wisconsin) Texas Tech. Coll., Lubbock, Tex. *2216 14th St.*
HEINKE, C. H., A.M. (Ohio State) Instr. Capital Univ., Columbus 9, Ohio
HEINS, Asso. Prof. M. H., Ph.D. (Harvard) Brown Univ., Providence, R.I. *Dept. of Math.*
HEINZMAN, Asso. Prof. W. P., A.M. (Illinois) Acting Head of Dept. New Mexico Coll. of A. and M.A., State College, N.M. *Box 144*
HELLINGER, Prof. E. D., Ph.D. (Göttingen) Northwestern Univ., Evanston, Ill. *2215 Maple Ave.*
HELLMICH, Prof. E. W., Ph.D. (Columbia) Northern Illinois State Teachers Coll., DeKalb, Ill.
HELME, G. C., M.S. (Washington Univ.) Instr., Pratt Inst., Brooklyn, N.Y. *161 Emerson Place*
HELMS, Asst. Prof. C. B., B.S. in Educ. (Temple) Pennsylvania Military Coll., Chester, Pa.
HELSEL, Asst. Prof. R. G., Ph.D. (Ohio State) Ohio State Univ., Columbus 10, Ohio. *Dept. of Math.*
HELTON, Asso. Prof. F. F., Ph.D. (Illinois) Head of Dept., Central Coll., Fayette, Mo. *Morrison Observatory*
HEMENWAY, Asso. Prof. L. D., A.M. (Harvard) Math. and Physics, Simmons Coll., 300 The Fenway, Boston, Mass.
HENDERSON, Prof. ARCHIBALD, Ph.D. (North Carolina) Head of Dept., Univ. of North Carolina, Chapel Hill, N.C. *721 E. Franklin St.*
HENDRIX, Asst. Prof. GERTRUDE, A.M. (Illinois) Eastern Illinois State Teachers Coll., Charleston, Ill. *1425 Fourth St.*
HENNESSEY, Rev. J. J., A.M. (Woodstock Coll.) Prof., Woodstock Coll., Woodstock, Md.
HENRIQUES, Asst. Prof. ANN S. (Mrs. D. E.), Ph.D. (Chicago) Univ. of Utah, Salt Lake City 1, Utah
HENRY, Asso. Prof. R. E., M.Sc. (New York Univ.) Chm. of Dept., Newark Coll., Rutgers Univ., Newark, N.J. *59 Fernwood Rd., Maplewood, N.J.*
HEREN, Prof. MABEL M., M.S. (Northwestern) Knox Coll., Galesburg, Ill.
HERLIHY, F. W., B.S. (Chicago) Vice-Pres. Herlihy Mid-Continent Co., Chicago, Ill. *Box 81, Comstock, Mich.*
HERPEL, Asst. Prof. Coleman, A.M. (Harvard) Pennsylvania State Coll. Altoona Undergrad. Center, Altoona, Pa. *212 Coleridge Ave.*
HERR, Asso. Prof. GERTRUDE A., M.S. (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
HERRERA, R. B., A.M. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif. *2626 S. Mansfield Ave., Los Angeles 16, Calif.*
HERTZIG, MORRIS, M.S. (C.C.N.Y.) Chm. of Dept., High School, Forest Hills, N.Y.
HERZOG, Asso. Prof. FRITZ, Ph.D. (Columbia) Michigan State Coll., East Lansing, Mich. *Dept. of Math.*
HESS, Prof. G. W., Ph.D. (Michigan) Head of Dept. Howard Coll., Birmingham 6, Ala. *8009 Fourth Ave. S.*
HESSE, EMMA V., A.M. (Columbia) Retired, Claremont Jr. High School, Oakland, Calif.
HESTENES, Prof. M. R., Ph.D. (Chicago) U.C.L.A., Los Angeles, Calif. *Dept. of Math.*
HEYDA, J. F., Ph.D. (Illinois) Mathematician, U. S. Naval Ordnance Plant, Indianapolis, Ind. *Research Dept.*
HICKERSON, Prof. T. F., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *108 Battle Lane*
HICKMAN, J. S., A.M. (Minnesota) Physicist, U. S. Navy Electronics Lab., San Diego, Calif. *3242 Lucinda St., San Diego 6, Calif.*
HICKSON, Asst. Prof. A. O., Ph.D. (Chicago) Duke Univ., Durham, N. C. *2712 Legion Ave.*
HIGGINS, Asso. Prof. T. J., Ph.D. (Purdue) Elec. Engg., Illinois Inst. of Tech., 3300 Federal St., Chicago, Ill.
HIGHTOWER, Prof. RUBY U., Ph.D. (Missouri) Emeritus, Shorter Coll., Rome, Ga. *Quitman, Ga.*
HILDEBRANDT, Asso. Prof. E. H. C., Ph.D. (Michigan) Northwestern Univ., Evanston, Ill.
HILDEBRANDT, Prof. T. H., Ph.D. (Chicago) Chm. of Dept., Univ. of Michigan, Ann Arbor, Mich. *1930 Cambridge Road*
HILDNER, Asso. Prof. R. C., Ph.D. (Ohio State) Univ. of New Mexico, Albuquerque, N.M. *Dept. of Math.*
HILL, Asso. Prof. J. D., Ph.D. (Brown) Michigan State Coll., East Lansing, Mich.
HILL, J. S., Minnesota Mutual Life Ins. Co., 156 E. 6th St., St. Paul, Minn.
HILL, Asso. Prof. L. S., Ph.D. (Yale) Hunter Coll., New York, N.Y. *22 Sagamore Road, Bronxville 8, N.Y.*
HILL, Prof. M. A., Jr., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *333 Tenney Circle*

HILL, Asso. Prof. P. R., M.S. (Georgia) Univ. of Georgia, Athens, Ga. *190 Morton Ave.*
HILLE, Prof. EINAR, Ph.D. (Stockholm) Yale Univ., New Haven, Conn. *72 Edgehill Road, New Haven, 11, Conn.*
HILLS, E. J., Ph.D. (Southern California) Instr., Los Angeles City Coll., Los Angeles 27, Calif. *1248 Holmby Ave., Los Angeles 24, Calif.*
HINRICHSEN, Asso. Prof. J. J. L., Ph.D. (Harvard) Iowa State Coll., Ames, Iowa. *Beardshear Hall*
HLAVATY, J. H., B.S. (C.C.N.Y.) Teacher, Bronx High School of Science, New York, N.Y. *250 Coligni Ave., New Rochelle, N.Y.*
HOARE, Prof. A. J., A.M. (Michigan) Emeritus, Univ. of Wichita, Wichita, Kans. *1717 N. Holyoke Ave.*
HOBBS, R. P., A.B. (Dartmouth) Acting Mgr., Rinehart and Co., Inc., 232 Madison Ave., New York 16, N.Y. *College Dept.*
HODGE, Asst. Prof. F. H., A.M. (Boston) Emeritus, Purdue Univ., West Lafayette, Ind. *820 N. Main St.*
HOEL, Asso. Prof. P. G., Ph.D. (Minnesota) U.C.L.A., Los Angeles 24, Calif. *Dept. of Math.*
HOERSCH, Asst. Prof. V. A., Ph.D. (Iowa) Univ. of Illinois, Urbana, Ill. *909 S. First St., Apt. 8, Champaign, Ill.*
HOFFMAN, HELEN P., M.S. (Illinois) Instr., Yankton Coll., Yankton, S.D.
HOFFMAN, JANICE M., B.S. in Ed. (Bowling Green State) Teacher, High School, Delray Beach, Fla. *306 N. Swinton St.*
HOFFMAN, RUTH I., A.M. (Colorado) Teacher, North High School, Denver, Colo. *338 Fox St., Denver 9, Colo.*
HOHN, Asst. Prof. F. E., Ph.D. (Illinois) Univ. of Maine, Orono, Me. *Math. Dept.*
HOLL, Prof. D. L., Ph.D. (Chicago) Head of Dept., Iowa State Coll., Ames, Iowa
HOLLAND, Sister MARY CHARLOTTE, A.M. (Catholic Univ.) Registrar, St. Xavier Coll., 4900 Cottage Grove Ave., Chicago 15, Ill.
HOLLCROFT, Prof. T. R., Ph.D. (Cornell) Chm. of Dept., Wells Coll., Aurora, N.Y.
HOLMES, Prof. C. T., Ph.D. (Harvard) Bowdoin Coll., Brunswick, Maine. *60 Spring St.*
HOLT, E. W., M.S. (Chicago) Muhlenberg Coll., Allentown, Pa. *1913 W. Livingston St.*
HOLTOM, Asst. Prof. CARL, Ph.D. (Chicago) Purdue Univ., Lafayette, Ind. *Dept. of Math.*
HOLTON, Asst. Prof. C. H., A.M. (Harvard) Georgia School of Tech., Atlanta, Ga. *167 Fourth St. N.W.*
HOLUBAR, LEROY, B.S.E.E. (Colorado) Instr., Engg. Math., Univ. of Colorado, Boulder, Colo. *1215 Grandview Ave.*
HOLZINGER, J. R., B.S. (Franklin and Marshall) Asst., Cornell Univ., Ithaca, N.Y. *740 E. Tower Road*
HOOD, R. T., M.A. (Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis., *79 Hamilton Blvd., Kenmore, New York*
HOOK, Asso. Prof. C. W., A.M. (North Carolina) Georgia School of Tech., Atlanta, Ga. *616 Williams St. N.W.*
HOOPES, M. F., A.B. (Oberlin) Teacher, Southern State Normal School, Springfield, S.D.
HOOPS, R. R., County Engr., Perry County, New Lexington, Ohio
HOOVER, Asso. Prof. B. P., Ph.D. (Illinois) Carnegie Inst. of Tech., Pittsburgh, Pa., *2030 Fairlawn St., Pittsburgh 21, Pa.*
HOPKINS, FANNIE, A.M. (Wisconsin) Teacher, High School, Waukesha, Wis. *138 S. East Ave.*
HOPKINS, Asso. Prof. L. A., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich.
HOPKINS, R. H., A.M. (Denver) Instr., Mississippi State Coll., State College, Miss., *Box 992*
HORNACEK, ROSE L., M.A. (Loyola) Instr., Univ. of Illinois, Chicago, Ill. *3932 W. Fillmore St., Chicago 24, Ill.*
HORNE, B. C., Jr., A.M. (North Carolina) Instr., Virginia Poly. Inst., Blacksburg, Va. *University Club*
HORSFALL, Prof. I. O., Ph.D. (Cornell) Director, Extension Division, Univ. of Utah, Salt Lake City 1, Utah, *82 Virginia St., Salt Lake City 3, Utah.*
HORTON, R. E., A.B. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif. *4655 Los Feliz Blvd., Los Angeles 27, Calif.*
HORVITZ, D. G., B.S. (Mass. State) Grad. Student, Iowa State Coll., Ames, Iowa
HOSKINS, R. H., A.B. (Harvard) John Hancock Mutual Life Ins. Co., Boston, Mass. *Actuarial Dept.*
HOUGHTON, D. B., A.M. (Michigan) *294 Western Way, Princeton, N.J.*
HOUSEHOULDER, A. S., Ph.D. (Chicago) Principal Physicist, Clinton Labs., Oak Ridge, Tenn.
HOVE, E. MARIE, M.S. (Iowa) Instr., Hofstra Coll., Hempstead, N.Y. *Dept. of Math.*

HOVEY, Asst. Prof. B. K., Ph.D. (Göttingen) Elec. Engg., Univ. of Pittsburgh, Pittsburgh 13, Pa. *208 Thaw Hall*
HOWARD, Prof. AUGHTOM S., (Mrs. N. J.) Ph.D. (Kentucky) Kentucky Wesleyan Coll., Winchester, Ky.
HOWARD, Prof. C. M., E. Mines (Alabama Poly. Inst.) Acting Head of Dept., North Texas Agric. Coll., Arlington, Tex., *605 S. Center St.*
HOWARD, HARRIET, A.M. (Boston) Head of Dept., The Ethel Walker School, Simsbury, Conn.
HOWE, G. K., B.S. (Worcester Poly. Inst.) Bell Aircraft Corp., Mariette, Ga. *359 Fifth St. N.W., Atlanta, Ga.*
HOWE, T. D., Jr., B.S. (Harvard) T. D. Howe Construction Co., Houston, Tex. *R.F.D. 12, Box 324*
HOWELL, J. V., A.M. (North Carolina) Chm. of Dept., Mars Hill Coll., Mars Hill, N.C.
HOWELL, S. W., A.M. (South Dakota) Instr., Yankton Coll., Yankton, S.D.
HOWIE, Prof. J. M., A.M. (Nebraska) Emeritus, Nebraska Wesleyan Univ., Lincoln, Nebr. *135 Ogden St., Denver 3, Colo.*
HOWLAND, Prof. L. A., Ph.D. (Munich) Dean, Wesleyan Univ., Middletown, Conn.
HOYLE, Prof. H. B., Jr., A.M. (North Carolina) Head of Dept., Queens Coll., Charlotte, N.C. *1907 Vail Ave., Apt. 3, Charlotte 4, N.C.*
HRATZ, Rev. J. A., B.S. (Iowa) Acting Head of Dept., St. Ambrose Coll., Davenport, Iowa
HSU, Asso. Prof. P. L., D.Sc. (London) Univ. of North Carolina, Chapel Hill, N.C. *The Carolina Inn*
HUBBARD, Asst. Prof. J. F., Ed.M. (Teachers Coll. of Boston) Univ. of Mass., Fort Devens, Mass.
HUBBS, Prof. H. N., Ph.D. (Cornell) Treas. and Secretary, The Colleges of the Seneca, Geneva, N.Y. *538 So. Main St.*
HUBERT, Asso. Prof. W. G., Sc.D. (New York Univ.) Chm. of Dept., Coll. of the City of New York, *125 Lee Ave., Yonkers 5, N.Y.*
HUCK, Prof. C. A., M.A. (George Peabody) Peru State Teachers Coll., Peru, Nebr.
HUCK, RAYMOND, M.S. (Illinois) *5705 Forest Rd., Cheverly, Maryland*
HUFF, Asso. Prof. G. B., Ph.D. (Illinois) Univ. of Georgia, Athens, Ga. *Dept. of Math.*
HUFF, Asst. Prof. W. N., A.M. (Pennsylvania) Univ. of Oklahoma, Norman, Okla. *Dept. of Math.*
HUFFER, Prof. R. C., Ph.D. (Chicago) Beloit Coll., Beloit, Wis. *729 Hobart Pl.*
HUGHES, Prof. H. K., Ph.D. (Michigan) Purdue Univ., Lafayette, Ind. *Dept. of Math.*
HUGHES, H. M., A.M. (Texas) Grad. Student and Teaching Asst., Univ. of California, Berkeley, Calif. *1454 Bancroft Way, Berkeley 2, Calif.*
HULL, RALPH, Ph.D. (Chicago) *702–23rd Ave., N., Seattle, Wash.*
HUME, ALFRED, D.Sc. (Vanderbilt) Emeritus, Head of Dept. and Chancellor, Univ. of Mississippi, University, Miss.
HUMMEL, Asso. Prof. P. M., Ph.D. (Ohio State) Univ. of Alabama, University, Ala. *Box 1251*
HUMPHREY, H. K., B.S. in E.E. (Union) Chm. of Board, Winnetka Trust and Savings Bank, Winnetka, Ill. *520 Ash St.*
HUMPHREYS, Asst. Prof. M. GWENETH, Ph.D. (Chicago) Sophie Newcomb Coll., New Orleans 18, La.
HUNDERTMARK, ELAINE, A.M. (Illinois) Instr., Florida State Univ., East Palatka, Fla., *Federal Point*
HUNEKE, H. V., A.M. (Oklahoma) Chm. of Dept., Northwestern State Coll., Alva, Okla. *909 Barnes Ave.*
HUNSAKER, Asst. Prof. N. C., A.M. (California) Utah State Agric. Coll., Logan, Utah
HUNT, Asst. Prof. G. H., C.E. (Cornell) U.C.L.A., Los Angeles, Calif. *237 Tavistock Ave.*
HUNT, Prof. MILDRED, Ph.D. (Chicago) Illinois Wesleyan Univ., Bloomington, Ill. *406 E. Walnut St.*
HUNTER, Asst. Prof. LOUISE S., (Mrs.), Ed.M. (Harvard) Virginia State Coll., Petersburg, Va.
HUNTINGTON, Prof. E. V., Ph.D. (Strassburg) Emeritus, Mech., Harvard Univ., Cambridge, Mass. *48 Highland St.*
HURD, C. C., Ph.D. (Illinois) Theoretical Research Engineer, Carbide & Carbon Chemicals Corp., Box P, Oakridge, Tenn., *113 Clarion Road*
HURRY, J. A., A.M. (California) *630 S. Canosa Court, Denver, Colo.*
HURST, Prof. J. W., Ph.D. (Illinois) Head of Dept., Montana State Coll., Bozeman, Mont., *522 S. Sixth St.*
HURT, J. M., A.M. (Texas) Lecturer, Univ. of Southern Calif., Los Angeles, Calif. *2535 Exposition Pl., Los Angeles 16, Calif.*

HURWITZ, SOLOMON, Ph.D. (Columbia) Instr., Coll. of the City of New York, New York, N.Y. *4014 Ave. 1, Brooklyn 10, N.Y.*
HURWITZ, Prof. W. A., Ph.D. (Göttingen) Cornell Univ., Ithaca, N.Y. *White Hall*
HUSKEY, Asst. Prof. H.D., Ph.D. (Ohio State) Univ. of Oklahoma, Norman, Okla. *On leave, Mathematician, National Physical Lab., Teddington, Middlesex, England*
HUTCHERSON, Prof. W. R., Ph.D. (Cornell) Head of Dept., Math. and Astr., Berea Coll., Berea, Ky.
HUTCHINSON, Prof. C. A., A.M. (Wittenberg) Head of Dept., Engg. Math., Univ. of Colorado, Boulder, Colo.
HUTCHINSON, Asst. Prof. L. C., Ph.D. (Mass. Inst. of Tech.) Poly. Inst. of Brooklyn, Brooklyn 2, N.Y. *Dept. of Math.*
HUTCHINSON, Prof. T. O., Ph.D. (Chicago) Head of Dept., Tennessee Poly. Inst., Cookeville Tenn. *511 N. Cedar St.*
HUTCHISON, Asst. Prof. L. P., Ph.D. (Kentucky) The Citadel, Charleston, S.C.
HYDE, Asso. Prof. EMMA, A.M. (Chicago) Kansas State Coll., Manhattan, Kans.
HYDEN, Prof. J. A., Ph.D. (Cornell) Vanderbilt Univ., Nashville 4, Tenn. *Box 111 Univ.*
HYERS, Asso. Prof. D. H., Ph.D. (Calif. Inst. of Tech.) Univ. of Southern California, Los Angeles 7, Calif. *5127 Parkglen Ave., Los Angeles 43, Calif.*
HYMAN, M. A., M.A. (Maryland) Mathematician, Naval Ordnance Lab., Washington, D.C. *4000 Kansas Ave. N.W., Washington 11, D.C.*

IKENBERRY, Prof. J. E., Ph.D. (Cornell) Head of Dept., Madison Coll., Harrisonburg, Va.
INGALLS, Prof. E. E., Ph.D. (Michigan) Albion Coll., Albion, Mich. *1111 Michigan Ave.*
INGRAHAM, Prof. M. H., Ph.D. (Chicago) Dean, Coll. of Letters and Science, Univ. of Wisconsin, Madison, Wis.
INNIS, MARY E., A.M. (Smith) Head of Dept. The Baldwin School, Bryn Mawr, Pa.
IRWIN, Asso. Prof. H. H., A.M. (State Coll. of Washington) State Coll. of Washington, Pullman, Wash.
ISAACS, RUFUS, Ph.D. (Columbia) Research Engr., Aero Physics Lab., North American Aviation, Inc., Municipal Airport, Los Angeles, Calif.
IVANOFF, V. F., A.M. (California) *2070 Fell St., Apt. 6, San Francisco 17, Calif.*

JABLONOWER, JOSEPH, Pd.M. (New York Univ.) Memb. Board of Examiners of the City of New York, New York, N.Y. *110 Livingston St., Brooklyn 2, N.Y.*
JACKSON, Prof. J. B., A.M. (Columbia) Univ. of South Carolina, Columbia, S.C. *227 S. Waccamaw Ave., Columbia 48, S.C.*
JACKSON, Prof. ROSA L., Ph.D. (Chicago) Head of Dept. Alabama Coll., Montevallo, Ala.
JACKSON, T. W., A.M. (Missouri) Head of Dept., Jamestown Coll., Jamestown, N.D. *729 Fifth St., N.E.*
JACOBSEN, N. L., B.S. in Ed. (Oregon) Instr., Graceland Coll., Lamoni, Iowa
JAEGER, Prof. C. G., Ph.D. (Missouri) Hd. of Dept., Pomona Coll., Claremont, Calif.
JAFFE, W. J., M.S. in I.E. (Columbia) Instr. Newark Coll. of Engineering, Newark 2, N. J. *1030 Anderson Ave., Palisade, N.J.*
JAMES, Asso. Prof. GLENN, Ph.D. (Columbia) U.C.L.A., Los Angeles, Calif.
JAMES, Prof. R. D., Ph.D. (Chicago) Univ. of British Columbia, Vancouver, B.C., Can., *Dept. of Math.*
JAMISON, Asst. Prof. FREE, Ph.D. (Pittsburgh) San Jose State Coll., San Jose, Calif. *310 S. 15th St.*
JANES, Asso. Prof. W. C., A.M. (Nebraska) Kansas State Coll., Manhattan, Kans.
JARMAIN, W. R., B.A. (Western Ontario) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis. *26 Champion Ave., Fond du Lac, Wis.*
JASON, Prof. W. B., A.M. (Pennsylvania) Dean, Lincoln Univ., Jefferson City, Mo.
JASPER, S. J., M.A. (Ohio S. Univ.) Instr., Univ. of Kentucky, Lexington, Ky. *Dept. of Math.*
JAUTZ, Rev. M. L., M.S. (Marquette) Curate, St. Mary Church, Racine, Wis. *3600 So. Kinnickinnic Ave., Milwaukee 7, Wis.*
JEFFERY, Prof. R. L., Ph.D. (Cornell) Queen's Univ., Kingston, Ont., Can.
JEFFRIES, Prof. J. B., M.S. (Chicago) Agric. and Tech. Coll., Greensboro, N.C. *923 Servier St.*
JEHN, L. A., B.M.E. (Dayton) Tchg. Fellow, Univ. of Michigan, Ann Arbor, Mich. *1313 Oakham Court, Willow Run Village, Mich.*
JENKE, C. W., B.S. (St. Mary's Univ.) Chemist, San Antonio Brewing Association, San Antonio, Tex. *1723 N. Olive St., San Antonio 8, Tex.*
JENKINS, Asso. Prof. E. D., Ph.D. (Ohio State) Kent State Univ., Kent, Ohio *Dept. of Math.*
JENNINGS, Asso. Prof. S. A., Ph.D. (Toronto) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*

JENNINGS, Asst. Prof. Walter, A.M. (Ohio State) Post Graduate School, Annapolis, Md.
JENSEN, C. M., Ph.D. (Minnesota) Lecturer, Univ. of Minnesota, Minneapolis, Minn. *3315–17th Ave. S., Minneapolis 7, Minn.*
JERBERT, Asso. Prof. A. R., Ph.D. (Washington) Univ. of Washington, Seattle, Wash.
JOFFE, S. A., M.S. (New York Univ.) Retired, Actuary, Mutual Life Ins. Co., New York, N.Y. *515 West 110 St., New York 25, N.Y.*
JOHANSON, Asst. Prof. R. N., Ph.D. (Chicago) Boston Univ., Boston, Mass., *688 Boylston St.*
JOHN, Prof. F. W., M.A. (Columbia) New York Univ., 100 Washington Sq. East, New York, N.Y.
JOHN, Asso. Prof. FRITZ, Ph.D. (Göttingen) Appl. Math. Group, New York Univ., 53 Washington Sq. S., New York, N.Y. *30 Summit Ave., New Rochelle, N.Y.*
JOHNSON, A. L., Jr., A.B. (Nebraska Wesleyan Univ.) Secretary, The Crete Mills, Crete, Nebr.
JOHNSON, C. A., A.M. (Northwestern Univ.) Instr., Univ. of Missouri, School of Mines and Met., Rolla, Mo.
JOHNSON, D. A., A.M. (Minnesota) Instr., Coll. of Educ., Univ. of Minnesota, Minneapolis 14, Minn. *205 Univ. High School*
JOHNSON, D. F., M.S. (Middlebury) Instr., Jr. Coll., Worcester, Mass. *13 Franconia St., Worcester 2, Mass.*
JOHNSON, E. A., Civil Engr., *P.O. Box 909, Yuma, Ariz.*
JOHNSON, Asso. Prof. EVAN, Jr., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa. *345 S. Buckhout St.*
JOHNSON, Asso. Prof. E. H., Ph.D. (Michigan) Statistics, George Washington Univ., Washington, D.C. *3939 Pennsylvania Ave. S.E., Washington 20, D.C.*
JOHNSON, Mrs. FAY H., A.B. (Howard Payne Coll.) Instr., Howard Payne Coll., Brownwood, Tex. *P.O. Box 29*
JOHNSON, L. G., A.M. (Michigan) Mathematician, Chevrolet Gear and Axle, Detroit, Mich. *1557 Glendale Ave., Detroit 6, Mich.*
JOHNSON, Asst. Prof. M. L., A.M. (Western Reserve) Kent State Univ., Kent, Ohio. *425* E. *Summit St.*
JOHNSON, O. B., Jr., B.S. (Chicago Tech. Coll.) Student, Roosevelt Coll., Chicago, Ill. *5852 S. Michigan Ave., Chicago 37, Ill.*
JOHNSON, Prof. R. A., Ph.D. (Harvard) Brooklyn Coll., Bedford Ave. and Ave. H, Brooklyn, N.Y.
JOHNSON, Asso. Prof. R. E., Ph.D. (Wisconsin) Smith Coll., Northampton, Mass. *211 Elm St.*
JOHNSON, Asso. Prof. ROBERTA F., Ph.D. (Cornell) Wilson Coll., Chambersburg, Pa.
JOHNSON, Asso. Prof. R. P., Ph.D. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *Dept. of Math.*
JOHNSON, Prof. W. L., A.M. (Univ. of Tennessee) Head of Dept. Mississippi Southern Coll., Hattiesburg, Miss. *200 N. 25th Ave.*
JOHNSON, W. W., Instr., Huntington Poly. Inst., Cleveland, Ohio. *10722 Orville Ave., Cleveland 6, Ohio*
JOHNSTON, Asst. Prof. ERNEST, M.S. (Illinois) Univ. of Minn., Minneapolis Minn. *2024 Commonwealth Ave., St. Paul, Minn.*
JOHNSTON, Prof. F. E., Ph.D. (Illinois) George Washington Univ., Washington 6, D.C.
JOHNSTON, Prof. L. S., A.M. (Missouri) Univ. of Detroit, Detroit 21, Mich. *227 Engineering Bldg.*
JOLIAT, Prof. J. S., M.S. (St. Louis Univ.) John Carroll Univ., Cleveland Heights, Ohio
JONAH, F. C., Ph.D. (Brown) Staff Project Engr., Chance Vought Aircraft, Stratford, Conn. *381 Laughlin Road*
JONAH, Asso. Prof. H. F. S., Ph.D. (Purdue) Purdue Univ., West Lafayette, Ind. *Dept. of Math.*
JONES, A. W., Rensselaer Poly. Inst., Troy, N.Y. *Carnegie Bldg.*
JONES, Prof. B. W., Ph.D. (Chicago) Cornell Univ., Ithaca, N.Y. *On leave 1747 S. Villa St., Pasadena 4, Calif.*
JONES, Prof. E. F. W., C.E. (Manhattan Coll.) Rollins Coll., Winter Park, Fla.
JONES, Brig. General HARRIS, B.S. (Mass. Inst. of Tech.) Dean, U. S. Military Acad., West Point, N.Y. *Headquarters.*
JONES, K. R., M.S. (Michigan) Asso. Physicist, NEPA Project, Oak Ridge, Tenn.
JONES, Prof. L. O., A.M. (Peabody) Chm. of Dept., William Jewell Coll., Liberty, Mo. *478 E. Mill St.*
JONES, Asst. Prof. MARGARET E., A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *164–13th Ave., Columbus 1, Ohio*

JONES, P. C., B.S. (Mass. Inst. of Tech.) Science Editor, Bell Telephone Labs., 463 West St., New York 14, N.Y.
JONES, P. S., A.M. (Michigan) Instr., Univ. of Michigan, Ann Arbor, Mich. *406 South Wing*
JONES, Asst. Prof. R. W., A.M. (Pennsylvania) Univ. of Delaware, Newark, Del. *121 Townsend Road*
JORDAN, Asso. Prof. H. E., Ph.D. (Chicago) Univ. of Kansas, Lawrence, Kans. *1600 Kentucky St.*
JORDAN, H. J., *147–36 97th Ave., Jamaica 4, N.Y.*
JURDAK, Prof. M. H., M.A. (Amer. Univ. of Beirut) American Univ. of Beirut, Beirut, Lebanon
JUSTICE, Prof. H. K., Ph.D. (Cincinnati) Ass't Dean, Coll. of Engg., Univ. of Cincinnati, Cincinnati 21, Ohio
JUSTIN, E. M., M.S. (Case School) Instr., Engg., U.C.L.A., Los Angeles, Calif. *17138 Ventura Blvd., Encino, Calif.*

KAC, Prof. MARK, Ph.D. (John Casimir Univ.) Cornell Univ., Ithaca, N.Y. *Dept. of Math.*
KAELIN, G. R., A.M. (California) Instr., Los Angeles City Coll., 855 N. Vermont Ave., Los Angeles 27, Calif.
KALISCH, Asst. Prof. G. K., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *126 Folwell Hall*
KALISH, AIDA, A.M. (Columbia) *420 Crown St., Brooklyn 25, N.Y.*
KALTENBORN, Prof. H. S., Ph.D. (Michigan) Head of Dept., Memphis State Coll., Memphis 11, Tenn. *Box 252*
KAPLAN, E. L., B.S. (Carnegie Inst. of Tech.) Asst. Physicist, Naval Ordnance Lab., Navy Yard, Washington 25, D.C. *1427 N. St., N.W., Washington, D.C.*
KAPLAN, SIDNEY, A.M. (Brooklyn Coll.) *2821–28th St., S.E., Washington 20, D.C.*
KAPLAN, Asst. Prof. WILFRED, Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *1308 Olivia Ave.*
KAPLANSKY, IRVING, Ph.D. (Harvard) Instr., Univ. of Chicago, Chicago 37, Ill. *Dept. of Math.*
KARAPETOFF, Prof. VLADIMIR, M.M.E. (Leningrad) Emeritus, Elec. Engr., Cornell Univ., Ithaca, N.Y. *39 Claremont Ave., Apt. 84, New York, N.Y.*
KARL, Sister MARY CORDIA, Ph.D. (Johns Hopkins) Head of Dept., Coll. of Notre Dame of Maryland, N. Charles Street, Baltimore 10, Md.
KARLIN, MEYER, Ph.D. (Columbia) Instr., Brooklyn Coll. Evening Session, Brooklyn, N.Y. *1804 East 4th St., Brooklyn 23, N.Y.*
KARLIN, SAMUEL, B.S. (Ill. Inst. of Tech.) Calif. Inst. of Tech., Pasadena, Calif.
KARNES, Asso. Prof. H. T., Ph.D. (Peabody) Dean of Men, Louisiana State Univ., University Sta., Baton Rouge 3, La.
KARNOW, HERMAN, A.M. (Colorado) Teacher, Brooklyn Tech. High School, Brooklyn, N.Y. *210 Avenue of Americas, New York 14, N.Y.*
KARPINSKI, Prof. L. C., Ph.D. (Strassburg) Univ. of Michigan, Ann Arbor, Mich. *1315 Cambridge Road*
KARR, Asso. Prof. LOIS, A.M. (Wisconsin) Lindenwood Coll., St. Charles, Mo.
KASNER, Prof. EDWARD, Ph.D. (Columbia) Columbia Univ., New York, N.Y. *430 West 116 St., New York 27, N.Y.*
KATZ, MAX, Accountant, Katz, Zuckerman and Co., 50 Broad St., New York, N.Y.
KAUFMAN, HYMAN, M.S. (McGill Univ.) Grad. Student, Brown Univ., Providence, R.I. *830 St. Louis St., Lachine, Montreal 32, P.Q., Can.*
KAUFMAN, NORBERT, S 2/c, U. S. Navy. *6930 Sheridan Road, Chicago 26, Ill.*
KEARNEY, DORA E., A.M. (Minnesota) Instr., Itasca Jr. Coll., Coleraine, Minn. *Box 763*
KECK, WINFIELD, A.M. (Pennsylvania) Instr., Physics, Brown Univ., Providence, R.I. *74 Benefit St., Providence 3, R.I.*
KEELER, Prof. B. C., A.M. (Columbia) Webb Inst., of Naval Architecture, New York, N.Y. *48 Sagamore Road, Bronxville, N.Y.*
KEFFER, RALPH, A.M. (Wisconsin) Actuary, Aetna Life Ins. Co., Hartford 15, Conn.
KEHL, W. B., A.M. (Harvard) *3 Comeau St., Wellesley Hills 82, Mass.*
KELLER, E. R., A.M. (Tennessee) Teacher, Central High School, Fountain City, Tenn. *Route 8, Maryville, Tenn.*
KELLER, Asso. Prof. M. W., Ph.D. (Indiana) Purdue Univ., West Lafayette, Ind. *138 Sheetz St.*
KELLEY, Asso. Prof. J. L., Ph.D. (Virginia) Univ. of Calif., Berkeley 4, Calif. *Dept. of Math.*
KELLS, Asso. Prof. L. M., Ph.D. (Columbia) U. S. Naval Acad., Annapolis, Md. *23 Thompson St.*

KELLY, Asso. Prof. K. D., M.S. (Chicago) Fenn Coll., Cleveland 15, Ohio. *14719 Clifton Blvd., Lakewood 7, Ohio*
KELLY, L. M., A.M. (Boston Univ.) Instr., Univ. of Missouri, Columbia, Mo. *Dept. of Math.*
KELLY, MAY B., A.M. (Yale) Teacher, Bulkeley High School, Hartford, Conn. *93 Newport Ave., West Hartford 7, Conn.*
KELLY, P. J., Ph.D. (Wisconsin) Instr. Univ. of Southern California, Los Angeles, Calif. *11773 Montana Ave., West Los Angeles, Calif.*
KEMPNER, Prof. A. J., Ph.D. (Göttingen) Head of Dept., Univ. of Colorado, Boulder, Colo.
KENDALL, Prof. CLARIBEL, Ph.D. (Chicago) Univ. of Colorado, Boulder, Colo. *1305 Euclid Ave.*
KENNA, Sister ESTHER MARIA, A.M. (St. Elizabeth) Prof., Coll. of St. Elizabeth, Convent Station, N.J.
KENNEDY, E. C., Ph.D. (Rice) Research Engr., Consolidated Vultee Aircraft Corp., Daingerfield, Tex. *Box 295, Hughes Springs, Texas*
KENNEDY, EVELYN M., A.M. (Cincinnati) Mathematician, Applied Physics Lab., The Johns Hopkins Univ., Silver Spring, Md. *5911–16th St. N.W., Washington 11, D.C.*
KENNEDY, Adj. Prof., E. S., Ph.D. (Lehigh) American Univ. of Beirut, Beirut, Lebanon
KENNEY, Asst. Prof. J. F., A.M. (Michigan) Univ. of Wisconsin at Milwaukee, 623 W. State St., Milwaukee 3, Wis.
KENNISON, Asst. Prof. L. S., Ph.D. (Calif. Inst. of Tech.) Brooklyn Coll., Brooklyn, N.Y. *110-11 215 St., Queens Village 9, N.Y.*
KENT, Asst. Prof. J. R. F., M.A. (Queen's Univ.) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*
KEPPLER, KATHARINE B., A.M. (Bryn Mawr) Head of Dept., Foxcroft School, Middleburg, Va.
KERALLA, J. A., B.S. (William & Mary) Research Div., Elec. Storage Co. (Exide), Philadelphia, Pa. *1154 Sanger St.*
KERR, F. J., *Merrick St., Pleasantville, Pa.*
KETCHUM, Asso. Prof. P. W., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *803 W. Illinois St.*
KEYES, Lt. (j.g.) G. A., M.S. (Syracuse) Asso. Instr., U. S. Merchant Marine Cadet School, Pass Christian, Miss.
KEYFITZ, NATHAN, B.S. (McGill Univ.) Mathematical Advisor, Dominion Bureau of Statistics, Ottawa, Ont., Can. *5 Bristol Ave.*
KIEFER, Prof. E. C., M.S. (Michigan) James Millikin Univ., Decatur 24, Ill.
KIELY, Prof. E. R., Ph.D. (Columbia) Iona Coll., New Rochelle, N.Y.
KIEVAL, H. S., Ph.D. (Cincinnati) Instr., Brooklyn Coll., Brooklyn 10, N.Y. *Dept. of Math.*
KILLEN, Prof. C. G., M.S. (Louisiana State) Northwestern State Coll., Natchitoches, La. *429 Henry Ave.*
KIMBALL, Prof. S. H., Ph.D. (Harvard) Head of Dept., Univ. of Maine, Orono, Maine. *135 Stevens Hall*
KIMBALL, T. C., A.M. (Princeton) Master, Lawrenceville School, Lawrenceville, N.J. *Box 40*
KIMME, E. G., Student, Calif. Inst. of Tech., Pasadena, Calif. *3766 Gaviota Ave., Long Beach 7, Calif.*
KINGSLEY, E. H., B.S. (Central Y.M.C.A. Coll.) Lecturer Roosevelt Coll., 430 S. Michigan Ave., Chicago, Ill.
KINGSTON, Prof. H. R., Ph.D. (Chicago) Head of Dept., Math. and Astr., Univ. of Western Ontario, London, Ont., Can.
KINGSTON, Asst. Prof. J. M., Ph.D. (Toronto) Univ. of Washington, Seattle 5, Wash. *Dept. of Math.*
KINNEY, J. M., Ph.D. (Chicago) Emeritus, Wilson Jr. Coll., Chicago, Ill. *8058 Bennett Ave., Chicago 17, Ill.*
KINSMAN, BLAIR, B.S. (Chicago) Tutor, St. John's College, Annapolis, Md. *1802 Kenyon St., N.W., Washington 10, D.C.*
KIRBY, Asst. Prof. A. R., A.M. (Columbia) Math. and Commerce, School of Educ., Fordham Univ., New York 7, N.Y. *2921 Briggs Ave., New York 58, N.Y.*
KIRCHNER, Prof. W. H., B.S. (Worcester Poly. Inst.) Emeritus, Univ. of Minnesota, Minneapolis, Minn. *722 Tenth Ave., S.E., Minneapolis 14, Minn.*
KIRKHAM, Asst. Prof. W. J., Ph.D. (Indiana) Oregon State Coll., Corvallis, Ore. *Dept. of Math.*
KLAUBER, L. M., A.B. (Stanford) President, San Diego Gas and Elec. Co., San Diego, Calif. *233 W. Juniper St., San Diego 1, Calif.*
KLEE, V. L., Jr., A.B. (Pomona) Instr., Univ. of Virginia, Charlottesville, Va. *21 Montibello Circle*

KLEENE, Asso. Prof. S. C., Ph.D. (Princeton) Univ. of Wisconsin, Madison, Wis. *Dept. of Math.*
KLEVAN, JULIUS, A.B. (Brooklyn) *135 Kingston Ave., Brooklyn 13, N.Y.*
KLIMCZAK, W. J., A.M. (Yale) Instr., Univ. of Rochester, Rochester, N.Y.
KLINE, Prof. J. R., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *529 Riverview Rd., Swarthmore, Pa.*
KLINE, Asso. Prof. MORRIS, Ph.D. (New York Univ.) New York Univ., New York, N.Y. *1188 E. Eighth St., Brooklyn 30, N.Y.*
KLINGER, E. L., A.M. (Illinois) Instr., Purdue Univ., West Lafayette, Ind. *126 North St.*
KLIPHARDT, R. A., B.S. (Armour Inst. of Tech.) Instr., Civil Engg., Northwestern Univ., Evanston, Ill. *5116 N. Glenwood St., Chicago 40, Ill.*
KLIPPLE, Asso. Prof. E. C., Ph.D. (Texas) Texas A. and M. Coll., College Station, Texas, *P.O. Box 4852*
KLOS, Sister CONSTANTIA, B.S. (St. Joseph's Coll.) St. Joseph's Coll., Emittsburg, Md.
KLOTZ, CELIA E., M.A. (Nebraska State) Instr., Washington State Coll., Pullman, Wash. *1201 Kamiaken St.*
KLOYDA, Sister M. THOMAS A KEMPIS, Ph.D. (Michigan) Head of Dept., Coll. of St. Teresa, Winona, Minn.
KNEALE, S. G., A.B. (Kansas) Instr., Univ. of Kansas, Lawrence, Kans. *Dept. of Math.*
KNEBELMAN, Prof. M. S., Ph.D. (Princeton) Head of Dept., State Coll. of Washington, Pullman, Wash. *2008 Indiana St.*
KNEDLER, Prof. P. A., A.M. (Pennsylvania) State Teachers Coll., Kutztown, Pa. *East Texas, Pa.*
KNIGHT, Asst. Prof. L. C., A.M. (Columbia) Emeritus, Coll. of Wooster, Wooster, Ohio. *1028 N. Bever St.*
KNIGHT, Asst. Prof. L. C., JR., A.M. (Kent) Muskingum Coll., New Concord, Ohio. *156 Lakeside Drive*
KNOWLER, Asso. Prof. L. A., Ph.D. (Iowa) Chm. of Dept., Univ. of Iowa, Iowa City, Iowa
KNOX, Prof. J. J., M.S. (Chicago) Dakota Wesleyan Univ., Mitchell, S.D. *1219 W. University Blvd.*
KNOX, Asso. Prof. R. H., JR., A.M. (Michigan) Virginia Milit. Inst., Lexington, Va. *313 Letcher Ave.*
KOCH, E. H., JR., B.S. in E.E. (Pennsylvania) Retired. *Ogontz Manor, Apt. F61, Ogontz and Olney Aves., Philadelphia 41, Pa.*
KOCH, R. J., *1126 Pratt Blvd., Chicago 26, Ill.*
KOEHLER, Asst. Prof. FULTON, Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *Dept. of Math. and Mech.*
KOEHLER, Asst. Prof. T. L., A.M. (Pennsylvania) Muhlenberg Coll., Allentown, Pa. *625 N. 24th St.*
KOKEN, J. C., M.A. (Missouri) Instr., Univ. of Missouri, Columbia, Mo. *1318 Ross St.*
KOKOMOOR, Prof. F. W., Ph.D. (Michigan) Univ. of Florida, Gainesville, Fla. *Dept. of Math.*
KOLEY, Sister VINCENT DE PAUL, M.S. (Notre Dame) Head of Dept., Mary Manse Coll., Toledo, Ohio
KOPP, P. J., A.M. (Duke) Asst., Gulf Oil Corp., Munsey Bldg., Washington, D.C. *1305 N. Adams St., Arlington, Va.*
KORGEN, Asso. Prof. R. L., Ph.D. (Harvard) Bowdoin Coll., Brunswick, Me. *Prince's Point Road*
KOTLER, Asst. Prof. DAVID, M.A. (Columbia) Mohawk Coll., Utica, N.Y.
KOVARIK, Prof. A. F., Ph.D. (Minnesota) Physics, Yale Univ., New Haven, Conn. *Sloane Lab.*
KRABILL, Asso. Prof. D. M., Ph.D. (Ohio State) Bowling Green State Univ., Bowling Green, Ohio. *437 N. Prospect St.*
KRALL, Asso. Prof. H. L., Ph.D. (Brown) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
KRAMER-LASSAR, EDNA E., Ph.D. (Columbia) Chm. of Dept., Thomas Jefferson High School, Brooklyn, N.Y. *Consultant, N.D.R.C., 32 Lenox Road, Brooklyn 26, N.Y.*
KRATHWOHL, Prof. W. C., Ph.D. (Chicago) Inst. for Psychological Services, Ill. Inst. of Tech., 18 S. Michigan Ave., Chicago 3, Ill.
KRATTIGER, J. T., A.M. (Southern Methodist) Instr., Univ. of Oklahoma, Norman, Okla. *Faculty Exchange*
KRAUS, G. R., M.S. (Carnegie Inst. of Tech.) Head of Dept., Math. and Physics, Gannon Coll. of Villa Maria Coll., Erie, Pa. *619 W. Ninth St.*

KRAVETZ, SAUL, Student, Harvard Univ., Cambridge, Mass. *3072 Brighton 13 St., Brooklyn 24, N.Y.*
KREIDER, Asst. Prof. O. C., M.S. (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
KRONSBEIN, JOHN, Ph.D. (Leipzig) Research Engr., Hoosier-Cardinal Corp., Evansville, Ind.; Consultant, Illinois Tool Works, Chicago, Ill. *Box 455, Evansville, Ind*
KRUEGER, Prof. R. L., Ph.D. (Marquette) Head of Dept., Wittenberg Coll., Springfield, Ohio
KRUGER, Prof. OBERT, B.A. (Nebraska) St. John's College, Winfield, Kans.
KRUSE, W. E., A.M. (Columbia) Instr., St. Peter's Coll., Jersey City, N.J. *320 Webster Ave., Jersey City 7, N.J.*
KRUSKAL, W. H., A.M. (Harvard) Vice-President, Kruskal and Kruskal, Inc., New York, N.Y. *235 East 22 St.*
KRYLOFF, Prof. NICOLAS, Dr. of Math. honoris cause (Kieff) Acad. of Sciences, Book Dept., Ul. Kropotkina 16, Moscow 34, USSR
KUBIS, Asst. Prof. J. F., Ph.D. (Fordham) Grad. School, Fordham Univ., New York, N.Y.
KUEBLER, Asst. Prof. R. R., JR., M.A. (Pennsylvania) Dickinson Coll., Carlisle, Pa.
KUHN, Rev. B. J., A.M. (St. Bonaventure) Prof. Head of Dept., Siena Coll., Loudonville, N.Y.
KUHN, Prof. H. W., Ph.D. (Cornell) Emeritus, Ohio State Univ., Columbus, Ohio. *1179 Fairview Ave.*
KULLBACH, SOLOMON, Ph.D. (George Washington) Lecturer in Statistics, George Washington Univ., Washington, D.C. *1259 Van Buren St., N.W., Washington 12, D.C.*
KUNKEL, Prof. P. V., Ph.D. (Columbia) Head of Dept., Cedar Crest Coll., Allentown, Pa. *Box 104, Trexlertown, Pa.*
KURZIN, W. H., M.S. (Chicago) Instr., Chicago City Coll., Herzl Branch, 3711 Douglas Blvd., Chicago, Ill.
KUTMAN, Mrs. HELEN L., A.M. (Columbia) Instr., Hunter Coll., New York, N.Y. *2825 Grand Concourse*

LACKAY, Sister M. PACHOMIA, A.M. (Missouri) Coll. of St. Teresa, Kansas City, Mo.
LACY, Prof. E. PAULINE, Ed.M. (Harvard) William Woods Coll., Fulton, Mo.
LADNER, Asst. Prof. A. C., A.M. (Brown) Math. and Engg., Denison Univ., Granville, Ohio. *P.O. Box 253*
LADY, C. H., A.M. (Southern California) State Teachers Coll., Slippery Rock, Pa.
LAFFERTY, W. A., A.M. (Ohio State) Instr., Univ. of Ill., Galesburg Branch, Galesburg, Ill.
LAFON, Asst. Prof. J. E., A.M. (Oklahoma) Univ. of Oklahoma, Norman, Okla. *806 Monnett St.*
LAGERSTROM, Asst. Prof. P. A., Ph.D. (Princeton) Aeronautics, California Inst. of Tech., Pasadena 4, Calif. *G.A.L.C.I.T.*
LAGRONE, Asso. Prof. J. W., A.M. (Vanderbilt) Clemson Agric. Coll., Clemson, S.C.
LAHTI, ELIZABETH L., M.A. (Michigan) Statistician, Bureau of Measurement and Guidance, Carnegie Inst. of Tech., Pittsburgh 13, Pa. *358 Lehigh Ave., Pittsburgh 6, Pa.*
LAIRD, L. E., B.S. in Ed. (Kansas St. T.C.) Grad. Fellow, State Teachers Coll., Emporia, Kans. *304½ E. 14th St.*
LALONDE, R. P., Lab. Asst., R.C.A. Victor Co., Montreal, P.Q., Can. *3910 Mentana St.*
LAMB, Prof. R. C., A.M. (Virginia) U. S. Naval Acad., Annapolis, Md. *149 Monticello Ave.*
LAMBEK, JOACHIM, B.S. (McGill Univ.) Address unknown
LAMBERT, R. J., A.B. (Drake) Instr., Iowa State Coll., Ames, Iowa. *1901–55th St., Des Moines, Iowa*
LAMBERT, W. D., A.M. (Harvard) Retired, U. S. Coast & Geodetic Survey, Washington, D.C. *Lime Rock, Conn.*
LAMPEN, Prof. A. E., A.M. (Michigan) Hope Coll., Holland, Mich. *552 College Ave.*
LAMPLAND, C. O., A.M. (Indiana) Lowell Observatory, Flagstaff, Ariz. *Bin 1640*
LANCASTER, Asst. Prof. O. E., Ph.D. (Harvard) Univ. of Maryland, College Park, Md., *4607–27th St., Mt. Rainer, Md.*
LANCKTON, A. L., A.M. (Duke) Socony Vacuum Oil Co., *P.O.B. 163, Athens, Greece*
LANCZOS, CORNELIUS, Ph.D. (Szeged) *1916 Franklin Place, Seattle 2, Wash.*
LANDERS, Asst. Prof. A. W., Ph.D. (Chicago) Brooklyn Coll., Brooklyn, N.Y. *108-48 67th Drive, Forest Hills, N.Y.*
LANDERS, Asst. Prof. MARY K. (Mrs. A. W.), Ph.D. (Chicago) Hunter Coll., New York, N.Y. *108-48 67th Drive, Forest Hills, N.Y.*
LANDIN, JOSEPH, Ph.D. (Notre Dame) Instr., Univ. of Illinois, Urbana, Ill. *Dept. of Math.*
LANDRY, Prof. A. E., Ph.D. (Johns Hopkins) Catholic Univ. of America, Washington, D.C. *3624–13th St. N.E., Brookland, D.C.*
LANE, Prof. E. P., Ph.D. (Chicago) Chm. of Dept., Univ. of Chicago, Chicago 37, Ill.
LANE, Asso. Prof. H. I., Ph.D. (Cornell) Coll. of Mines and Met., El Paso, Tex.

LANE, N.D., B.A. (Queen's Univ.) Head of Dept., St. Andrew's Coll., Aurora Ont., Can.
LANE, Asso. Prof. RUTH O., Ph.D. (Iowa) Sam Houston State Teachers Coll., Huntsville, Tex.
LANG, Asst. Prof. G. B., Ph.D. (Illinois) Univ. of Florida, Gainesville, Fla. *231 Ray St.*
LANGE, LUISE, Ph.D. (Göttingen) Wilson Jr. Coll., Chicago, Ill. *5851 Blackstone Ave.*
LANGER, Prof. R. E., Ph.D. (Harvard) Univ. of Wisconsin, Madison, Wis. *822 Miami Pass, Madison 5, Wis.*
LAPAZ, Prof. LINCOLN, Ph.D. (Chicago) Head of Dept., Univ. of New Mexico, Albuquerque, N.M. *Box 190 Univ.*
LAPIDUS, LEO, A.M. (Boston Univ.) Instr., Univ. of Maine, Orono, Me. *Dept. of Math.*
LAREW, Prof. GILLIE A., Ph.D. (Chicago) Head of Dept., Randolph-Macon Woman's Coll., Lynchburg, Va.
LARIVIERE, ROSE, M.A. (McGill) Instr., Univ. of Illinois, Chicago, Ill. *5647 Dorchester Ave., Chicago 37, Ill.*
LAROE, RACHAEL A., A.M. (Tennessee) Instr., Physics, Duke Univ., Durham, N.C. *1204 Markham Ave.*
LARRIVEE, Asst. Prof. J. A., Ph.D. (Catholic Univ.) Univ. of Vermont, Burlington, Vt. *Dept. of Math.*
LARSEN, Prof. H. D., Ph.D. (Wisconsin) Albion Coll., Albion, Mich. *310 Burr Oak St.*
LARSON, Asso. Prof. OLGA, A.M. (Missouri) State Coll. for Women, Tallahassee, Fla.
LASALLE, Asst. Prof. J. P., Ph.D. (Calif. Inst. of Tech.) Univ. of Notre Dame, Notre Dame, Ind. *1317 E. Monroe St., South Bend, Ind. On leave Princeton Univ., Princeton, N.J.*
LASHIER, Asso. Prof. H. M., A.M. (Southern California) Physics, Emmanuel Missionary Coll., Berrien Springs, Mich.
LASLEY, Prof. J. W., JR., Ph.D. (Chicago) Univ. of North Carolina, Chapel Hill, N.C. *523 E. Rosemary Lane*
LATIMER, Prof. C. G., Ph.D. (Chicago) Emory Univ., Emory University, Ga.
LATHAM, FLORENCE O., M.A. (Illinois) Instr., Univ. of Illinois, Chicago, Ill. *536 Penn Ave., Aurora, Ill.*
LATSHAW, ELMER. Tech. Engr., ACF-Brill Motors Co., Philadelphia, Pa. *1839 N. 60th St.*
LATSHAW, Asst. Prof. V. V., Ph.D. (Indiana) Lehigh Univ., Bethlehem, Pa. *708 Eighth Ave.*
LAURENTINE MARIE, Sister, A.M. (Trinity Coll., Wash., D.C.) Head of Dept., Emmanuel Coll., 400 The Fenway, Boston 15, Mass.
LAUSH, GEORGE, B.S. (Pittsburgh) Grad. Asst., Cornell Univ., Ithaca, N.Y. *Dept. of Math.*
LAUSMAN, ELLA E., A.M. (Michigan) Instr., Glendale Coll., 1500 N. Verdugo Road, Glendale 8, Calif. *1514 Opechee Way.*
LAVOIE, E. S., A.B. (Brooklyn Coll.) Patrolman, New York, N.Y. *135 Sheridan Ave., Brooklyn, N.Y.*
LAWRENCE, B. E., B.S. (Cal. Inst. of Tech.) Supervisor, Analysis Group, Electronics Section, RD & T, USNOTS, Inyokern, Cal.
LAWRENCE, H. R., A.M. (Michigan) Engr., Aerodynamics, Northorp Aircraft Co., Los Angeles, Calif. *8325 Westlawn Ave., Los Angeles 45, Calif.*
LAY, L. C., A.M. (Southern California) John Muir Jr. Coll., Pasadena, Calif. *593 W. Montana, Pasadena 3, Calif.*
LAYTON, Prof. W. I., M.S. (South Carolina) Head of Dept., Austin Peay State Coll., Clarksville, Tenn.
LAZAR, NATHAN, Ph.D. (Columbia) Teacher, Midwood High School, Brooklyn, N.Y. *1055 Carroll St., Brooklyn 25, N.Y.*
LEAVENS, D. H., A.M. (Yale) Managing Editor, *Econometrica, 1632 Wood Ave., Colorado Springs, Colorado*
LEE, Asst. Prof. H. L., Ph.D. (Duke) Univ. of Tennessee, Knoxville, Tenn. *Box 4052, Univ. Sta.*
LEE, Asst. Prof. MARY A., A.M. (Wisconsin) Cornell University, Ithaca, N.Y., *White Hall*
LEECH, Asst. Prof. J. S., Ph.D. (Yale) Chicago Univ. (The Coll.). Chicago, Ill. *5600 Blackstone St.*
LEEDS, B. R., A.B. (Brooklyn) Teacher, Alexander Hamilton High School Annex, Brooklyn, N.Y. *967 Putnam Ave.*
LEFSCHETZ, Prof. SOLOMON, Ph.D. (Clark) Chm. of Dept., Princeton Univ., Princeton, N.J. *Fine Hall*
LEHMAN, MARGARET B. (Mrs. R. M.), A.M. (U.C.L.A.) *2315–24th St., Santa Monica, Calif.*
LEHMAN, C. H., A.M. (Columbia) Instr., Cooper Union, New York 3, N.Y. *144-17 29th Ave., Flushing, N.Y.*
LEHMER, Prof. D. H., Ph.D. (Brown) Univ. of California, Berkeley, Calif. *942 Hilldale Ave., Berkeley 8, Calif.*

LEHNER, JOSEPH, Ph.D. (Pennsylvania) Mathematician, Hydrocarbon Research, 115 Broadway, New York, N.Y.
LEHR, Asso. Prof. MARGUERITE, Ph.D. (Bryn Mawr) Bryn Mawr Coll., Bryn Mawr, Pa. *Cartref*
LEIFER, H. R., A.B. (Pittsburgh) Asst. Chief, Vocational Rehabilitation & Education Div. 5500 Hays St., Pittsburgh 6, Pa.
LEIGHTON, Prof. WALTER, Ph.D. (Harvard) Washington Univ., St. Louis 5, Mo. *Box 146*
LEITHOLD, L. C., JR., M.A. (California) Instr., Phoenix Jr. Coll., Phoenix, Ariz. *Dept. of Math.*
LELEIKO, MAX, B.S. (New York Univ.) Instr., Rutgers Univ., New Brunswick, N.J. *1212 Ocean Ave., Brooklyn 30, N.Y.*
LENNAHAN, C. M., M.S. (Mass. Inst. of Tech.) Supervising Analyst, U. S. Weather Bureau, Washington 25, D.C. *24th and M St. N.W.*
LEONARD, Prof. H. B., Ph.D. (Colorado) Univ. of Arizona, Tuscon, Ariz. *Box 4024 Univ. Sta.*
LEONE, F. C., M.S. (Georgetown Univ.) Instr., Purdue Univ., Lafayette, Ind. *Dept. of Math.*
LEPORI, A. A., M.A. (Columbia) Instr. and Lecturer, Rutgers Univ., New Brunswick, N.J.; Teacher, High School, East Orange, N.J. *915 Rahway Ave., Westfield, N.J.*
LESTER, Asst. Prof. CAROLINE A., Ph.D. (Wisconsin) New York State Coll. for Teachers, Albany 3, N.Y.
LESTOURGEON, Asso. Prof. F. ELIZABETH, Ph.D. (Chicago) *Box 1452, Delray Beach, Fla.*
LEVENSON, M. E., M.S. (New York Univ.) Instr., Cooper Union, Cooper Square, New York 3, N.Y.
LEVEQUE, W. J., Ph.D. (Cornell) Instr., Harvard Univ., Cambridge 38, Mass.
LEVIN, H. S., B.S. (Chicago) Instr., Univ. of Illinois, Chicago, Ill. *532 Addison St., Chicago 13, Ill.*
LEVINE, EMANUEL, Ed.M. (Rutgers) Instr., Rider Coll., Trenton 9, N.J. *66 Beacon Ave., Jersey City 6, N.J.*
LEVINE, L. D., A.B. (Brooklyn) Instr., Inst. of Optics, New York, N.Y. *454 Central Ave., Brooklyn 21, N.Y.*
LEVIT, Asst. Prof. R. J., Ph.D. (California) Univ. of Georgia, Athens, Ga. *Dept. of Math.*
LEVITT, JOSEPH, B.M.E. (Cooper Union) Instr., Physics and Math., Univ. of Ill., Navy Pier, Chicago 11, Ill.
LEVY, GENE, M.A. (Oklahoma) Chm. of Dept., Coll. of the Ozarks, Clarksville, Ark.
LEVY, Asst. Prof. HARRY, Ph.D. (Princeton) Univ. of Illinois, Urbana, Ill. *358 Math. Bldg.*
LEVY, HERMAN, A.B. (New York Univ.) Physicist, Naval Air Exp. Station, Philadelphia, Pa. *1139 N. 41st St., Philadelphia 4, Pa.*
LEVY, S. L., B.S.E.E. (Ill. Inst. of Tech.) Instr., Illinois Inst. of Tech., Chicago 16, Ill. *3458 W. Polk St., Chicago 24, Ill.*
LEWIS, Prof. A. J., Ph.D. (Colorado) Univ. of Denver, Denver 10, Colo. *2136 S. Josephine St.*
LEWIS, Asst. Prof. Anne L., Ph.D. (Chicago) Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *Dept. of Math.*
LEWIS, Asso. Prof. C. F., M.S. (Kansas State) Kansas State Coll., Manhattan, Kans.
LEWIS, Rev. C. J., M.S. (Georgetown) Student, Weston Coll., Weston 93, Mass.
LEWIS, Prof. D. C., JR., Ph.D. (Harvard) Univ. of Maryland, College Park, Md.
LEWIS, EUNICE, A.M. (Oklahoma) Teacher, Central High School, Tulsa, Okla.
LEWIS, Prof. F. A., Ph.D. (Ohio State) Head of Dept., Univ. of Alabama, University, Ala. *Box 1444*
LEWIS, Prof. FLORENCE P., Ph.D. (Johns Hopkins) Emeritus, Goucher Coll., Baltimore, Md. *2435 N. Charles St., Baltimore 18, Md.*
LEWIS, J. H., A.M. (Washington & Jefferson) Research Mathematician, N.D.R.C., Columbia Univ., New York, N.Y. *400 West 118 St., New York 27, N.Y.*
LEWIS, MARY TERESINE, Sister, Ph.D. (Catholic Univ.) Instr., Fontbonne Coll., Wydown & Big Bend Blvd., St. Louis 5, Mo.
LEWIS, Asso. Prof. P. E., Ph.D. (Illinois) Oklahoma A. and M. Coll., Stillwater, Okla. *Dept. of Math.*
LI, J. C. R., Ph.D. (Iowa State) Instr., Oregon State Coll., Corvallis, Ore. *Dept. of Math.*
LIEBER, Prof. LILLIAN R. (Mrs. H. G.), Ph.D. (Clark) Long Island Univ., 375 Pearl St., Brooklyn 1, N.Y.
LIECHTY, MARGARET, B.A. (Oberlin) Instr., Penn. State Undergrad. Center, Hazleton, Pa.
LIGHT, Asst. Prof. F. W., JR., M.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore 18, Md. *719 E. Cold Spring Lane, Baltimore 12, Md.*
LIN, Asso. Prof. C. C., Ph.D. (Calif. Inst. of Tech.) Brown Univ., Providence, R.I.

LINARES, ENRIQUE, JR., B.S. in E.E. (Univ. of Santa Clara) Gen. Agt., Cerveceria Nacional, David, R.P. *P.O. Box 540, Panama City, R.P.*
LINDQUIST, Prof. THEODORE, Ph.D. (Chicago) Retired, State Normal Coll., Ypsilanti, Mich. *103 Elm St.*
LINDSTRUM, A. O., JR., Ph.D. (Illinois) Northwestern Univ., Evanston, Ill.
LINEHAN, Prof. P.H., Ph.D. (Columbia) Coll. of the City of New York, New York, N.Y. *924 West End Ave., New York 25, N.Y.*
LINFIELD, Asso. Prof. B. Z., Ph.D. (Harvard) Univ. of Virginia, University, Va. *1324 Hilltop Road, Charlottesville, Va.*
LINSCHEID, Asso. Prof. H. W., A.M. (Oklahoma) *420 North Caddo St., Weatherford, Oklahoma*
LIOLIOS, LEO. *4500 Malden Ave., #219 Chicago 40, Ill.*
LIPSICH, H. D., A.M. (Cincinnati) Instr., Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math.*
LITTAUER, Asso. Prof. S. B., Ph.D. (Mass. Inst. of Tech.) Indus. Engr., Columbia Univ., New York 27. N.Y. *58 Floyd Ave., Bloomfield, N.J.*
LITTERICK, W. S., Ed.D. (Rutgers) Chm. of Dept.; Dir. of Studies, The Peddie School, Highstown, N.J. *340 S. Main St.*
LITZINGER, Prof. MARIE, Ph.D. (Chicago) Mount Holyoke Coll., South Hadley, Mass.
LIVINGSTON, Prof. G. R., A.M. (California) Head of Dept., San Diego State Coll., San Diego, Calif. *4641–55th St., San Diego 5, Calif.*
LLOYD, OLWEN (Mrs. George), M.A. (Cambridge, England) Headmistress, Mount Vernon Seminary, Washington 7, D.C. *2100 Foxhall Rd.*
LOCH, JOSEPH. Grad. Student, Univ. of Chicago, Chicago 37, Ill. *11155 S. Washtenaw Ave., Chicago 43, Ill.*
LOCKE, Prof. J. F., Ph.D. (Illinois) Head of Dept., Birmingham-Southern Coll., Birmingham 4, Ala.
LOCKWOOD, E. C., A.M. (Brown) Head of Dept., Jaffna College, Vaddukoddai, Ceylon
LOCKWOOD, NEIL, A.B. (Baker Univ.) Head of Dept., Physics, State Teachers Coll., Duluth, Minn. *4317 E. Superior*
LOEWNER, Prof. CHARLES, Ph.D. (Prague) Syracuse Univ., Syracuse 10, N.Y. *406 University Place*
LOFLIN, Prof. Z. L., M.S. (Louisiana) Acting Head of Dept., Southwestern Louisiana Inst., Lafayette, La. *Box 353 S.L.I.*
LOGSDON, Prof. MAYME I., Ph.D. (Chicago) Emeritus, Univ. of Chicago, Chicago, Ill. *Visiting Prof., Univ. of Miami, Coral Gables 34, Fla.*
LONDON, LIONEL, B.S.C.E. (Illinois) Stress Analyst, Barnes and Reinecke, Chicago, Ill. *4640 W. Jackson Blvd., Chicago 44, Ill.*
LONG, Asso. Prof. FLORENCE, M.S. (Illinois) Earlham Coll., Earlham, Ind.
LONG, Asst. Prof. T. R., A.M. (Rochester) Univ. of Rochester, Rochester, N.Y.
LONGENECKER, J. V., M.S. (Iowa) Actuary, Farmers and Bankers Life Ins. Co., Wichita, Kans. *Box 580*
LONGLEY, Prof. W. R., Ph.D. (Chicago) Yale Univ., New Haven 11, Conn. *305 Lawrence St.*
LONNER, MAX, B.S. (C.C.N.Y.) Admin. Asst.; Teacher, Torah Vodaath High School, Brooklyn, N.Y. *151 S. Third St., Brooklyn 11, N.Y.*
LONSETH, Asst. Prof. A.T., Ph.D. (California) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
LORING, S. J., B.S. (Mass. Inst. of Tech.) Independent Engg. Consultant, Stratford, Conn. *29 Stiles St.*
DE LOSADA Y PUGA, Prof. CRISTOBAL, D.Sc. (Lima) Catholic Univ. of Peru, Lima, Peru. *Apartado 2708*
LOTKIN, MARK M., Ph.D. (Kiel, Germany) Mathematician, Ballistics Research Lab., Aberdeen Proving Ground, Md.
LOUD, Asst. Prof. W. S., Ph.D. (Mass. Inst. of Tech.) Univ. of Minnesota, Minneapolis 14, Minn. *Folwell Hall*
LOVE, Prof. C. E., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1915 Scotwood Ave.*
LOVETT, E. O., Ph.D. (Virginia; Leipzig) Pres., Retired, Rice Inst., Houston 1, Tex. *P.O. Box 1892*
LOWE, R. B., B.S. (Poly. Inst.) Instr., Poly. Inst. of Brooklyn, 85 Livingston St., Brooklyn 2, N.Y.
LOWEKE, Asst. Prof. G. P., Ph.D. (Berlin) Engg. Mech., Wayne Univ., Detroit 3, Mich. *Engg. College*

LOWENSTEIN, Prof. L. L., Ph.D. (Cornell) *410 Stow St., Kent, Ohio*

LUBBEN, Asso. Prof. R. G., Ph.D. (Texas) Univ. of Texas, Austin, Tex. *W.H. 16, Univ. Station*

LUBIN, Asso. Prof. C. I., Ph.D. (Harvard) Univ. of Cincinnati, Cincinnati, Ohio. *3612 Washington Ave.*

LUCAS, HELEN M. (Mrs. J. W.), M.S. (Iowa) Statistician, Bureau of the Census, Washington, D.C. *4716 B. Hudson Ave. S.E., Suitlad Manor, Washington 20, D.C.*

LUCKEY, Asso. Prof. R. R. R., Ph.D. (Cornell) Math. and Physics, Houghton Coll., Houghton, N.Y.

LUIPPOLD, R. C., A.M. (Buffalo) Tchg. Fellow, Cornell University, Ithaca, N.Y. *704 E. Buffalo St.*

LUKACS, ELIZABETH C. (Mrs. Eugene), Teacher's Certif. (Vienna) Head, Sci. Dept., Hillsdale School, Cincinnati, Ohio. *337 McGregor Ave., Cincinnati 19, Ohio*

LUNDBERG, G. H., A.M. (Vanderbilt) Instr., Vanderbilt Univ., Nashville, Tenn. *2310 Highland Ave.*

LUNN, Prof. A. C., Ph.D. (Chicago) Emeritus, Univ. of Chicago, Chicago, Ill. *5211 Kenwood Ave.*

LUTHER, Prof. C. F., Ph.D. (Stanford) Dean, Coll. of Liberal Arts, Willamette Univ., Salem, Ore.

LUTHER, Prof. H. A., Ph.D. (Iowa) Texas A. and M. Coll., College Station, Texas. *Box 1875*

LYCHE, WALTER, M.A. (Miami) Head of Dept., Wartburg Coll., Waverly, Iowa. *511-12th St., N.W.*

LYLE, Asso. Prof. G. A., M.S. (Lehigh) U. S. Naval Acad., Annapolis, Md. *227 Wardour Drive*

LYNCH, R. V., B.S. (Harvard) Instr., Phillips Exeter Acad., Exeter, N.H. *Dutch House*

LYTLE, Adj. Prof. R. A., A.M. (Virginia) Univ. of South Carolina, Columbia, S.C. *2849 Sheffield Road*

MACCOLL, L. A., Ph.D. (Columbia) Member of Tech. Staff, Bell Telephone Labs., 463 West St., New York 14, N.Y.

MACCREADIE, Asso. Prof. W. T., Ph.D. (Cornell) Bucknell Univ., Lewisburg, Pa. *104–13 St.*

MACCULLOUGH, Prof. R. H., M.S. (Lafayette) Defiance Coll., Defiance, Ohio

MACDONALD, Prof. S. L., A.M. (Columbia) Emeritus, Math. and Astr., Colorado State Coll. of A. and M.A., Fort Collins, Colo. *122 N. Thompson St., Springdale, Ark.*

MACDOUGAL, Prof. H. B., M.S. (Iowa) Acting Head of Dept., South Dakota State Coll., Brookings, S.D.

MACDUFFEE, Prof. C. C., Ph.D. (Chicago) Univ. of Wisconsin, Madison, Wis. *1815 Summit Ave.*

MACEWEN, Asst. Prof. D. M., Ph.D. (New York Univ.) Coll. of the City of New York, New York, N.Y. *55 East 65 St.*

MACKIE, Prof. E. L., Ph.D. (Chicago) Dean of Men, Univ. of North Carolina, Chapel Hill, N.C. *702 Gimghoul Road*

MACLANE, Prof. SAUNDERS, Ph.D. (Göttingen) Univ. of Chicago, Chicago, Ill. *Eckhart Hall*

MACNEILLE, Prof. H. M., Ph.D. (Harvard) Kenyon Coll., Gambier, Ohio

MACNEISH, Prof. H. F., Ph.D. (Chicago) Visiting Prof., U. of Miami, Miami, Fla. *Hotel Dallas Park*

MACPHAIL, Prof. M. S., Ph.D. (Oxford) Acadia Univ., Wolfville, N.S., Can.

MADDAUS, INGO, Jr., Ph.D. (Michigan) *81 Columbia Heights, Brooklyn 2, N.Y.*

MADDEN, Sister TERESA MARIE, A.M. (Boston Coll.) Prof., Coll. of Our Lady of the Elms, Chicopee, Mass.

MADDOX, Prof. A. C., A.M. (Columbia) Northwestern State Coll., Natchitoches, La. *405 New Second St.*

MADDRILL, J. D., Ph.D. (California) Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.

MADISON, Asst. Prof. M. L., M.S. (Colorado) Colorado State Coll. of A. and M.A., Fort Collins, Colo. *Dept. of Math.*

MADOR, R. L., A.M. (Trinity Coll.) Mathematician, U. S. Navy Electronics Laboratory, San Digeo 52, Calif. *3330–45th St., San Digeo 5, Calif.*

MAGUIRE, J. M., Supervisor in Designer Tractor Engg. Dept., Ford Motor Co., Dearborn, Mich. *1510 Hudson Ave., Detroit 8, Mich.*

MAKAROV, A. G., A.M. (Pennsylvania) Instr., Univ. of Pa., Philadelphia, Pa. *4113 Pine St., Philadelphia 4, Pa.*

MALIN, Sister M. MECHTILDIS, M.S. (Marquette) Head of Dept., Viterbo Coll., LaCrosse, Wis.

MALLORY, Prof. A. E., Ph.D. (George Peabody Coll.) Colorado State Coll. of Educ., Greeley, Colo.
MALLORY, Prof. V. S., Ph.D. (Columbia) Head of Dept., State Teachers Coll., Montclair, N.J.
MALONEY, C. J., A.M. (Minnesota) Chief, Statistics Branch, Camp Detrick, Frederick, Md.
MALONEY, Sister MARY MICHAEL, A.M. (Catholic Univ.) Head of Dept., D'Youville Coll., 320 Porter Ave., Buffalo 1, N.Y.
MANCHESTER, Prof. R. E., A.M. (Michigan) Head of Dept., Dean of Men, Kent State Univ., Kent, Ohio. *208 N. Lincoln St.*
MANCILL, Prof. J. D., Ph.D. (Chicago) Univ. of Alabama, University, Ala.
MANDEL, JOHN, M.S. (Brussels) Research Chemist, The BG Corp., New York, City, N.Y. *45 Kew Gardens Road, Kew Gardens, N.Y.*
MANNING, Prof. F. L., Ph.D. (Cornell) Ursinus Coll., Collegeville, Pa. *68 Sixth Ave.*
MANNING, Asso. Prof. H. P., Ph.D. (Johns Hopkins) Emeritus, Brown Univ., Providence, R.I. *148 Governor St., Providence 6, R.I.*
MANSFIELD, RALPH, M.S. (Chicago) Chief Engr., Jos. Weidenhoff Inc., Chicago, Ill. *8049 S. Maryland Ave., Chicago 19, Ill.*
MANSON, E. S., M.S. (Mass. Inst. of Tech.) *122 Fourth St. S.E., Washington 3, D.C.*
MANY, ANNA E., A.M. (Tulane) Counselor to Women, Sophie Newcomb Coll., New Orleans 18, La.
MAPLE, C. G., A.M. (Cincinnati) Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. *720 S. Negley Ave., Apt. c-2, Pittsburgh 6, Pa.*
MARBURGER, CLIFFORD, A.M. (Franklin and Marshall) *Denver, Pa.*
MARCH, Prof. H. W., Ph.D. (Munich) Univ. of Wisconsin, Madison, Wis.; Chief Mathematician, U. S. Forest Products Lab. *1825 Summit Ave., Madison 5, Wis.*
MARCHAND, E. W., M.S. (Wash. Univ.) Instr., Univ. of Rochester, Rochester 3, N.Y.
MARCOU, R. J., Ph.D. (Mass. Inst. of Tech.) Boston Coll., Chestnut Hill, Mass. *930 Beacon St., Newton Centre 59, Mass.*
MARDEN, Prof. MORRIS, Ph.D. (Harvard) Univ. of Wisconsin in Milwaukee, Wis. *403 E. Carlisle Ave., Milwaukee 11, Wis.*
MARIA, D. MAY HICKEY (Mrs. A. J.), Ph.D. (Rice) Brooklyn Coll., Brooklyn, N.Y. *Dept. of Math.*
MARK, A. M., Ph.D. (Cornell) Instr., Univ. of Wisconsin, Madison 6, Wis. *Dept. of Math.*
MARKLE, Asst. Prof. G. E., M.A. (Michigan) Coll. of Engg., Univ. of Detroit, Detroit, Mich.
MARM, Prof. ANNA, A.M. (Kansas) Bethany Coll., Lindsborg, Kansas. *741 N. Second St.*
MARQUIS, Prof. R. H., Ph.D. (Chicago) Chm. Dept. of Math. and Astronomy, Ohio Univ., Athens, Ohio. *Box 216*
MARRIOTT, Prof. R. W., Ph.D. (Pennsylvania) Swarthmore Coll., Swarthmore, Pa. *213 Lafayette Ave.*
MARTH, Asst. Prof. ELLA, Ph.D. (St. Louis Univ.) Harris Teachers Coll., St. Louis, Mo. *4521a Clarence Ave., St. Louis 15, Mo.*
MARTIN, JEROME, Ph.D. (California) Research Dir., Commercial Solvents Corp., Terre Haute, Ind. *1334 S. Center St.*
MARTIN, Prof. M. H., Ph.D. (Johns Hopkins) Chm. of Dept., Univ. of Maryland, College Park, Md.
MARTIN, Asst. Prof. MARGARET P., Ph.D. (Minnesota) School of Medicine, Vanderbilt Univ., Nashville 4, Tenn. *Dept. of Preventive Medicine and Public Health*
MARTIN, Rev. NORBERT, T.O.R., B.A. (Catholic Univ.) Instr., St. Francis Preparatory School, Spring Grove, Pa.
MARTIN, Prof. P. E., Ph.D. (Michigan) Chm. of Dept., Wheaton Coll., Wheaton, Ill. *922 Webster Ave.*
MARTIN, Asst. Prof. W. A., A.M. (Alabama) Georgia School of Tech., Atlanta, Ga. *Math. Dept.*
MARTIN, Prof. W. T., Ph.D. (Illinois) Massachusetts Inst. of Tech., Cambridge 39, Mass. *Dept. of Math.*
MARTINSON, MARGARET E., A.B. (Washburn Municipal) Instr., Washburn Univ., Topeka, Kans. *Route 3.*
MARTYN, W. J., M.A. (New Zealand) Master, Otago Boy's High School, Dunedin, New Zealand. *16 Sheen St., Roslyn, Dunedin, N.W. 1*
MARY AMATOR, Sister, A.M. (Pennsylvania) Asst. Prof., Immaculata Coll., Immaculata, Pa.
MARY BENEDICTA, Sister, A.B. (Good Counsel Coll.) Good Counsel Coll., White Plains, N.Y.

MARY CANISIA, Sister, B.S. (De Paul Univ.) Teacher, Holy Family Acad., Chicago, Ill. *144 W. Division St.*
MARY DANIEL, Sister, Ph.D. (Fordham) Prof., Marywood Coll., Scranton, Pa.
MARY ESTHER, Sister, A.M. (California) Instr., Mundelein Coll., 6363 Sheridan Road, Chicago, Ill.
MARY FELICE, Sister, Ph.D. (Catholic Univ.) Prof., Mount Mary Coll., Milwaukee 13, Wis.
MARY GERTRUDE, Sister, M.S. (Marquette) Instr., St. Clare Coll., 3221 S. Lake Drive, Milwaukee 7, Wis.
MARY OF MERCY, Sister, A.M. (Catholic Univ.) Instr., Incarnate Word Coll., San Antonio 2, Tex.
MARY PAULA, Sister, M.S. (Notre Dame) Prof., Marygrove Coll., 8425 W. McNichols Road, Detroit 21, Mich.
MASON, HAZEL L., A.M. (New Mexico) Head of Dept., Math. and Physics, Mary Hardin-Baylor Coll., Belton, Tex. *Box 594 Baylor Sta.*
MASON, R. H., A.M. (Syracuse) Instr., Univ. of Florida, Gainesville, Fla. *Peabody 106*
MASON, Asso. Prof. S. L., A.M. (Michigan) Univ. of North Dakota, Grand Forks, N.D. *2211 Fourth Ave. N.*
MASON, Asso. Prof. W. E., M.E. (Michigan) Engg., U.C.L.A., 405 Hilgard Ave., Los Angeles 24, Calif.
MASSEY, Asso. Prof. W. L., A.M. (Duke) Univ. of Chattanooga, Chattanooga, Tenn.
MATHANY, H. V., A.M. (State Coll. of Washington) *502 N. Aliso St., Albuquerque, N.M.*
MATHEWS, C. W., Jr., Ph.D. (Univ. of Ill.) Instr., Washington Univ., St. Louis 5, Mo. *Dept. of Math.*
MATHEWSON, Prof. L. C., Ph.D. (Illinois) Dartmouth Coll., Hanover, N.H.
MATHIAS, Asso. Prof. H. R., A.M. (Indiana) Bowling Green State Univ., Bowling Green, Ohio. *123 E. Evers Ave.*
MAUCH, Asst. Prof. MARGARET, Ph.D. (Chicago) Univ. of Akron, Akron 4, Ohio
MAY, Asst. Prof. KENNETH, Ph.D. (California) Carleton Coll., Northfield, Minn.
MAY, LIDA B., A.M. (Texas) Instr., Texas Tech. Coll., Lubbock, Texas. *2409–14th St.*
MAYER, Prof. E. S., U. S. Naval Acad., Annapolis, Md. *115 Spa View Ave.*
MAYERSON, A. L., M.A. (Michigan) Inst. of Life Insurance, N.Y.C. *96 Sterling St., Brooklyn 25, N.Y.*
MAYOR, Asso. Prof. J. R., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *441 Virginia Terrace*
MAYS, W. J., A.M. (Vanderbilt) Asst. Actuary, Liberty Life Ins. Co., Greenville, S.C.
MCBRIDE, Asst. Prof. ELNA B. (Mrs. J. S.), M.S. (Tennessee) Memphis State Coll., Memphis, Tenn.
MCBRIDE, W. H., M.S. (North Dakota) Instr., Univ. of North Dakota, Grand Forks, N.D. *1115 Plum Ave.*
MCBRIEN, Asst. Prof. V. O., Ph.D. (Catholic Univ.) Coll. of the Holy Cross, Worcester 3, Mass.
MCCAMMAN, CAROL V., A.M. (California) Teacher, Veterans H. S. Center, Central H. S. Bldg., Washington 9, D.C. *3500 14th St. N.W., Apt. 106, Washington 10, D.C.*
MCCARTHY, ANNE S., A.M. (Boston Coll.) *32 Auburn St., Brookline, Mass.*
MCCARTHY, Asst. Prof. E. D., A.M. (Pennsylvania State) Engg. Coll., Univ. of Detroit, Detroit, Mich.
MCCARTHY, J. J., M.S. (New York Univ.) Instr., Newark Colleges of Arts and Sci., Rutgers Univ., Newark 2, N.J. *128 Marine Ave., Brooklyn 9, N.Y.*
MCCARTY, A. L., A.M. (California) Retired, Instr., Jr. Coll., San Francisco, Calif. *Route 1, Box 481, Merced, Calif.*
MCCLELLAN, ADA A., B.S. (Chicago) Teacher, High School, Long Beach, Calif. *313 N. New Hampshire Ave., Los Angeles, Calif.*
MCCLELLAND, J. N., A.M. (Drake) Grad. Student, Calif. Coll. of Arts & Crafts, Oakland, Calif. *6070 Skyline Drive, Richmond, Calif.*
MCCLENON, Prof. R. B., Ph.D. (Yale) Grinnell Coll., Grinnell, Iowa
MCCLIMANS, Prof. J. W., Ph.D. (Peabody) Head of Dept., Southeastern Louisiana Coll., Hammond, La. *Box 125 College Station*
MCCOLGIN, GLADYS B. (Mrs. O. R.), A.M. (Radcliffe) *1556 Brookside Ave., Indianapolis, Ind.*
MCCONNELL, H. J., Engr., U. S. Engineer Office, Los Angeles, Calif. *3328 Madison Ave., San Diego 4, Calif.*
MCCORMICK, Asso. Prof. C. T., Ph.D. (Indiana) Illinois State Normal Univ., Normal, Ill.
MCCOY, Prof. DOROTHY, Ph.D. (Iowa) Head of Dept., Belhaven Coll., Jackson 4, Miss.
MCCOY, Prof. N. H., Ph.D. (Iowa) Smith Coll., Northampton, Mass. *53 Ridgewood Terrace*

McCrea, Prof. W. H., Ph.D. (Cambridge) Univ. of London, Royal Holloway Coll. Englefield Green, Surrey, England
McCulley, Asst. Prof. W. S., M.S. (Texas A. and M. Coll.) Texas A. and M. Coll. College Station, Tex. *Box 2795*
McCuskey, Prof. S. W., Ph.D. (Harvard) Math. and Astr., Case School of Tech., 10900 Euclid Ave., Cleveland 6, Ohio
McCutcheon, G. J., B.S. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *90 Malcolm Ave., S.E., Minneapolis 14, Minn.*
McDaniel, Asso. Prof. W. C., Ph.D. (Wisconsin) Southern Illinois Univ., Carbondale, Ill.
McDill, Prof. R. M., A.M. (Indiana) Head of Dept., Hastings Coll., Hastings, Nebr. *129 E. Seventh St.*
McDole, R. D., M.S. (Okla. A. and M. Coll.) Oklahoma A. and M. Coll., Stillwater, Okla., *8A College Courts*
McDonald, Asst. Prof. Janet, Ph.D. (Chicago) Vassar Coll., Poughkeepsie, N.Y.
McDonald, Asso. Prof. Sophia Levy (Mrs. J. H.), Ph.D. (California) Univ. of California, Berkeley, Calif. *2818 Webster St., Berkeley 5, Calif.*
McDonough, D. L., Ph.D. (Pennsylvania) Head of Dept., South Philadelphia High School for Boys, Philadelphia, Pa. *1205 Belfield Ave., Drexel Hill, Pa.*
McDougle, Edith A., A.B. (Delaware) Instr., Women's Coll., Univ. of Delaware, Newark, Del.
McDowell, E. L., Student, Illinois Inst. of Tech., Chicago, Ill. *819 Wenonah St., Oak Park, Ill.*
McEwen, Prof. G. F., Ph.D. (Stanford) Scripps Institution of Oceanography, La Jolla, Calif. *P.O. Box 109*
McEwen, Prof. W. H., Ph.D. (Minnesota) Head of Dept., Univ. of Manitoba, Winnipeg, Man., Can. *29 Rothesay Apts., Preston Ave.*
McEwen, Asst. Prof. W. R., Ph.D. (Minnesota) Univ. of Minnesota, Duluth Branch, Duluth, Minn. *Dept. of Math. and Mech.*
McFadden, Asso. Prof. Leonard, Ph.D. (Brown) Virginia Poly. Inst., Blacksburg, Va. *Box 161*
McFarlan, Prof. L. H., Ph.D. (Missouri) Univ. of Washington, Seattle 5, Wash.
McFarland, Asso. Prof. Dora, Ph.D. (Chicago) Univ. of Okla., Norman, Okla. *Faculty Exchange*
McGaughey, Prof. A. W., Ph.D. (Cincinnati) Westminster Coll., New Wilmington, Pa. *Dept. of Math.*
McGavock, Prof. W. G., Ph.D. (Duke) Davidson Coll., Davidson, N.C.
McGaw, Prof. F. M., B.S. (Wesleyan Univ.) Emeritus, Math. and Engg. Drawing, Cornell Coll., Mt. Vernon, Iowa
McGrail, J. A., B.S. (Mich. Coll. of M. & T.) Instr. Univ. of Detroit, Detroit, Mich. *631 Sheldon*
McGrath, Rev. P. H., A.M. (Woodstock Coll.) Church of St. Ignatius Loyola, 980 Park Ave., New York 28, N.Y.
McInnis, Asst. Prof. S. W., A.M. (Florida) Univ. of Florida, Gainesville, Fla. *1417 W. McCormick St.*
McKelvey, Prof. J. V., Ph.D. (Cornell) Iowa State Coll., Ames, Iowa. *2117 Graeber St.*
McKelvey, Martha M. (Mrs. J. V.), M.S. (Iowa) *2117 Graeber St., Ames, Iowa*
McKenna, Mary, A.M. (Columbia) Teacher, Washington Irving High School, New York, N.Y. *17 East 11 St., New York 3, N.Y.*
McKenney, Rev. J. L., A.M. (Manhattan) Prof. Providence Coll., Providence, R.I.
McKenzie, W. H., M.A. (California) Instr., San Francisco Jr. Coll., 726 31st Ave., San Francisco 21, Calif.
McKnelly, R. D., M.A. (Univ. of Okla) Instr., Oklahoma Univ., Norman, Okla. *809 Chautauqua*
McLaughlin, J. J., M.A. (Michigan) Chm. of Dept., State Teachers' Coll., River Falls, Wis. *136 E. Cascade*
McLaughlin, J. J. Chm. of Dept., State Teachers' Coll., River Falls, Wis. *136 E. Cascade*
McLaughlin, K. F., A.M. (Yale) Instr., U. S. Naval Acad., Annapolis, Md. *83 Market St.*
McMahon, Asst. Prof. F. A., A.M. (Brooklyn Coll.) Manhattan College, New York 63, N.Y. *445 E. Fulton St., Long Beach, N.Y.*
McMurtrie, K. A., B.S. (Brown) *246 Ohio Ave., Providence, R.I.*
McNabb, W. K., A.M. (Michigan) Instr., The Hockaday School, Dallas 6, Tex. *Belmont and Greenville*
McNair, J. S., M.S. (Chicago) Instr., Math. and Engg., Canal Zone Jr. Coll., Balboa Heights, Canal Zone. *On leave. Univ. Extension Center, Racine, Wis.*

McNeal, R. L., B.S. in E.E. (Purdue) Head of Tech. Data Section, General Motors Proving Ground, Milford, Mich. *P.O. Box 338*

McNeary, S. S., A.M. (Pennsylvania) Instr., Drexel Inst. of Tech., Philadelphia 4, Pa.

McNeil, Sister Marie Gertrude, M.S. (Notre Dame) Prof., Seton Hill Coll., Greensburg, Pa.

McNelly, Sister Presentation, A.M. (Catholic Univ) Instr., Our Lady of the Lake Coll., San Antonio, Tex.

McShane, Prof. E. J., Ph.D. (Chicago) Univ. of Virginia, Charlottesville, Va. *209 Maury Ave.*

McSweeney, Prof. A. A., A.M. (Montana) Head of Dept., John Tarleton Agric. Coll., Stephenville, Tex. *Box 368, Tarleton Station, Tex.*

Mead, Mrs. Sallie P., A.M. (Columbia) Member Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y.

Meade, Mary E., A.M. (Virginia) *3704 Bunker Hill Road, Brentwood, Md.*

Meador, Prof. G. E., A.M. (Oklahoma) Oklahoma City Univ., Oklahoma City, Okla. *1701 N. W. 30th St.*

Meadows, Asst. Prof. P. E., A.B. (Carleton Coll.) Washington and Lee Univ., Lexington, Va. *P.O. Box 902*

Mears, Prof. Florence M., Ph.D. (Cornell) George Washington Univ., Washington 6, D.C.

Mebane, Prof. W. N., Jr., A.M. (Cornell) Davidson Coll., Davidson, N.C. *Box 274*

Meder, A. E., Jr., A.M. (Columbia) Dean of Admin., Rutgers Univ., New Brunswick, N.J.

Mehlenbacher, Asso. Prof. L. E., Ph.D. (Michigan) Acting Head of Dept., Univ. of Detroit, Detroit 21, Mich. *1207 Longfellow Ave., Royal Oak, Mich.*

Mehr, Emanuel, A.M. (Brooklyn) Instr., Johns Hopkins Univ., Baltimore, Md. *Math. Dept.*

Mela, D. F., A.B. (Michigan) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich. *509 South Division St.*

Melville, Prof. C. E., A.B. (Northwestern) Clark Univ., Worcester, Mass. *950 Main St., Worcester 3, Mass.*

Mendelsohn, Asst. Prof. N. S., Ph.D. (Toronto) Univ. of Manitoba, Winnipeg, Manitoba, Can.

Menger, Prof. Karl, Ph.D. (Vienna) Illinois Inst. of Tech., Chicago, Ill. *5506 N. Wayne Ave., Chicago 40, Ill.*

Menke, Asst. Prof. H. E., A.M. (Ohio State) Heidelberg Coll., Tiffin, Ohio. *267 E. Market St., Tiffin 3, Ohio*

Mercedes, Sister M., A.B. (Coll. of St. Scholastica) Registrar, Coll. of St. Scholastica, Duluth 2, Minn.

Mercedes, Sister M., M.S. (Notre Dame) Instr., Mary Manse Coll., Toledo, Ohio. *2413 Collingwood Ave., Toledo 10, Ohio*

Mergendahl, Prof. T. E., M.S. (Tufts) Tufts Coll., Medford, Mass. *128 Professors Row, Medford 55, Mass.*

Merrill, Prof. A. S., Ph.D. (Chicago) Montana State Univ., Missoula, Mont.

Merrill, Prof. Helen A., Ph.D. (Yale) Emeritus, Wellesley Coll., Wellesley, Mass. *6 Roanoke Road, Wellesley 81, Mass.*

Merrill, L. L., Ph.D. (Rensselaer) Mathematician, Stromberg-Carlson Co., Rochester, N.Y. *35 Elmcroft Road, Rochester 9, N.Y.*

Merriman, Prof. G. M., Ph.D. (Cincinnati) Univ. of Cincinnati, Cincinnati 21, Ohio

Merriss, A. A., *3803 N.E. 24th Ave., Portland 12, Ore.*

Mertie, J. B., Jr., Ph.D. (Johns Hopkins) Senior Geologist, U. S. Geological Survey, Washington, D.C. *Room 6203, Federal Works Agency Bldg.*

Meserve, B. E., Ph.D. (Duke) Instr., Univ. of Illinois, Urbana, Ill. *356 Math. Bldg.*

Messick, C. A., Ph.D. (Chicago) *1100 Tyler St., Topeka, Kans.*

Messick, Prof. J. F., Ph.D. (Johns Hopkins) Emory Univ., Emory University, Ga.

Mewborn, Asso. Prof. A. B., Ph.D. (Calif. Inst. of Tech.) Postgrad. School, U. S. Naval Acad., Annapolis, Md. *P.O. Box 107*

Meyer, Elaine, B.A. (Huron) Head of Dept., Vermillion Public High School, Vermillion, S.D. *224 Forest Ave.*

Meyer, Asso. Prof. H. A., Ph.D. (Iowa) Univ. of Florida, Gainesville, Fla. *2015 West Columbia St.*

Meyer, H. L., Jr., M.S. (Chicago) Univ. of Chicago, Chicago 37, Ill. *Eckhart Hall*

Michael, Asst. Prof. W. B., Ph.D. (Southern Calif.) Lecturer, Psychology Dept., Princeton Univ., Princeton, N.J.

MICHAL, Prof. A. D., Ph.D. (Rice) California Inst. of Tech., Pasadena 4, Calif.
MICHALUP, ERIC, Actuary, Compania Nac. Anonima de Seguros, La Previsora, Apartado 848, Caracas, Venezuela
MICHEL, R. J., Ph.D. (Missouri) Head of Dept., Southeast Missouri State Teachers Coll., Cape Girardeau, Mo. *225 N. Ellis St.*
MICHIE, Prof. J. N., A.M. (Michigan) Head of Dept., Texas Tech. Coll., Lubbock, Tex. *Box 91, Tech. Sta.*
MICKLE, Asst. Prof. E. J., Ph.D. (Ohio State) Ohio State Univ., Columbus 10, Ohio. *Dept. of Math.*
MIDDLEMISS, Prof. R. R., M.S. (Colorado) Washington Univ., St. Louis 5, Mo.
MIKESH, J. S., A.B. (Minnesota) Head of Dept., Lawrenceville School, Lawrenceville, N.J. *Box 268*
MIKSA, F. L., Switchman, Illinois Bell Telephone Co., Aurora, Ill. *613 Spring St.*
MILES, Asso. Prof. E. J., Ph.D. (Chicago) Yale Univ., New Haven, Conn. *87 Marvel Road*
MILES, E. P., Jr., A.M. (Duke) Duke Univ., Durham, N.C. *Box 4244 Duke Sta.*
MILES, Asso. Prof. H. J., Ph.D. (California) Univ. of Illinois, Urbana, Ill. *253 Math. Bldg.*
MILKMAN, JOSEPH, A.M. (Brooklyn) Instr., U. S. Naval Acad., Annapolis, Md. *71 Shipwright St.*
MILLER, A. L., Ph.D. (Harvard) Treas. and Gen. Mgr., C. N. Miller Co., Boston, Mass. *100 Massachusetts Ave.*
MILLER, Prof. C. E., Ph.D. (Toronto) Univ. of Saskatchewan, Saskatoon, Sask., Can. *Dept. of Math.*
MILLER, Asso. Prof. D. D., Ph.D. (Michigan) Univ. of Tennessee, Knoxville, Tenn. *Box 4265 Univ. Sta.*
MILLER, Prof. E. B., A.M. (Chicago) Illinois Coll., Jacksonville, Ill. *1252 W. College Ave.*
MILLER, E. D., A.B. (Stanford) Instr., Yuba Junior Coll., Marysville, Calif.
MILLER, Prof. F. H., Ph.D. (Columbia) Head of Dept., Cooper Union, Cooper Square, New York 3, N.Y.
MILLER, Prof. G. A., Ph.D. (Cumberland) Emeritus, Univ. of Illinois, Urbana, Ill. *1203 W Illinois St.*
MILLER, Asst. Prof. G. T., Ph.D. (Purdue) Purdue Univ., Lafayette, Ind. *601 Dodge St., West Lafayette, Ind.*
MILLER, Asso. Prof. HARLAN C., Ph.D. (Texas) Texas State Coll. for Women, Denton Tex., *Box 3464 T.S.C.W. Sta.*
MILLER, HELEN T. (Mrs. H. W.), Ph.D. (Illinois) *5638 Elgin Ave., Pittsburgh 6, Pa.*
MILLER, K. W., B.S. in E.E. (Union Coll.) Chm., Elec. Engg., Armour Research Foundation, Illinois Inst. of Tech., Chicago 16, Ill. *10431 S. Oakley Ave., Chicago 43, Ill.*
MILLER, LEONARD, A.M. (Brooklyn) *1726 Union St., Brooklyn 13, N.Y.*
MILLER, M. H. A., *717 E. Main St., Vermillion, S.D.*
MILLER, Prof. NORMAN, Ph.D. (Harvard) Queen's Univ., Kingston, Ont., Can.
MILLER, Asso. Prof. W. I., Ph.D. (Pittsburgh) Bucknell Univ., Lewisburg, Pa. *220 S. Third St.*
MILLER, W. M., Ph.D. (Illinois) Head of Dept., Robert Coll., Istanbul, Turkey.
MILLINGTON, Asst. Prof. H. G., C.E. (Rensselaer) Univ. of Vermont, Burlington, Vt. *Room 339, Waterman Bldg.*
MILLS, Prof. C. N., Ph.D. (Wisconsin) Illinois State Normal Univ., Normal, Ill. *304 Virginia Ave.*
MILLS, W. H., A.B. (Swarthmore) Grad. Student, Princeton Univ., Princeton, N.J. *460 Riverside Dr., New York 27, N.Y.*
MILLSAPS, Asso. Prof. KNOX, Ph.D. (Calif. Inst. of Tech.) Aeronautical Engg., Ohio State Univ., Columbus 10, Ohio
MILNE, Prof. W. E., Ph.D. (Harvard) Hd. of Dept., Oregon State Coll., Corvallis, Ore.
MILOS, Asst. Prof. J. F., A.M. (Columbia) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
MINO, Rev. P. M., M.A. (Columbia) Grad. Student, Columbia Univ., New York, N.Y. *212 Lafayette St., Newark, N.J.*
MINOIS, Asst. Prof. SERGE, Agregation des Sci. Math. (Paris) Lycee Lakanol e Sceaux. 7 rue Candelot, Bourg-la-Reine (Seine), France
MIRICK, G. R., A.M. (Columbia) Asst. Dir., Div. of Teachers College Schools; Horace Mann-Lincoln School, 433 West 123 St., New York, N.Y.
MISER, Asst. Prof. H. J., Ph.D. (Ohio State) Williams Coll., Williamstown, Mass.
MISER, NELLIE P. (Mrs. W. L.), A.B. (Huron Coll.) *2139 Abbott Martin Rd., Nashville 9, Tenn.*
MISER, Prof. W. L., Ph.D. (Chicago) Vanderbilt Univ., Nasvhille, Tenn. *2139 Abbott Martin Rd., Nashville 9, Tenn.*

MITCHELL, Asso. Prof. A. K., Ph.D. (Johns Hopkins) Univ. of Maryland, College Park, Md. *16 Keswick St., Garrett Park, Md.*
MITCHELL, Prof. B. E., Ph.D. (Columbia) Millsaps Coll., Jackson, Miss.
MITCHELL, Asso. Prof. JOSEPHINE M., Ph.D. (Bryn Mawr) Oklahoma A. & M. Coll., Stillwater, Okla.
MITCHELL, SOLOMON, M.A. (Toronto) *1368 50th St. Brooklyn 19, N.Y.*
MODE, Prof. E. B., A.M. (Harvard) Boston Univ., Boston, Mass. *9 Longmeadow Rd., Wellesley 81, Mass.*
MOELLER, A. C., B.S. (Western Reserve) Instr., Marquette Univ., Milwaukee 8, Wis.
MOLINA, E. C., Retired, Special lecturer, Newark Coll. of Engg., Newark, N. J. *141 Dodd St., East Orange, N.J.*
MOLIVER, MARTIN, Ed.M. (Temple) Teacher, Benj. Franklin H.S., Philadelphia, Pa. *M. and Bristol Sts., Philadelphia 24, Pa.*
MONELL, I. H., A.M. (Loyola Univ.) Teacher, Austin High School, Chicago, Ill. *538 N. Cuyler Ave., Oak Park, Ill.*
MONTAGUE, Prof. HARRIET F., Ph.D. (Cornell) Univ. of Buffalo, Buffalo 14, N.Y.
MONTGOMERY, Asst. Prof. A. G., M.A. (Minnesota) Coll. of St. Thomas, St. Paul, Minn. *75 North Cleveland Ave., St. Paul 5, Minn., Apt. 3-A*
MONTGOMERY, Asso. Prof. DEANE, Ph.D. (Iowa) Yale Univ., New Haven, Conn. *Dept. of Math.*
MONTGOMERY, MABEL D., B.A. (Houghton Coll.) Instr., Univ. of Buffalo, Buffalo, N.Y. *81 Princeton Blvd., Kenmore 17, N.Y.*
MOON, Asso. Prof. PARRY, M.S. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge, Mass.
MOORE, A. H., B.M.E. (Pratt Inst.) Instr., Pratt Inst., Brooklyn, N.Y. *1585 Unionport Road, New York 62, N.Y.*
MOORE, Asst. Prof. B. C., A.M. (Princeton) Texas A. and M. Coll., College Station, Tex. *F.E. 163*
MOORE, Prof. C. N., Ph.D. (Harvard) Dir., Grad. Studies in Math., Univ. of Cincinnati, Cincinnati, Ohio. *219 Woolper Ave.*
MOORE, Asst. Prof. D. C., A.M. (Emory) Emory Jr. Coll., Valdosta, Ga.
MOORE, DEWEY, A.B. (Berea Coll.) Grad. Student, Brown Univ., Providence, R.I. *148D Sessions St., Browntown*
MOORE, Dean E. F., A.M. (Texas) Head of Dept., Hannibal-LaGrange Coll., Hannibal, Mo. *College Heights*
MOORE, Asso. Prof. G. E., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *55 E. Chalmers St., Champaign, Ill.*
MOORE, LILLIAN, Ed.D. (New York Univ.) Teacher, Far Rockaway High School, Far Rockaway, N.Y. *885 N. 28th St., Philadelphia 30, Pa.*
MOORE, Asso. Prof. L. T., Ph.D. (Johns Hopkins) Brooklyn Coll., Brooklyn, N.Y. *205 Hicks St.*
MOORE, Asso. Prof. M. G., Ph.D. (Illinois) Bradley Univ., Peoria 5, Ill. *503 W. Melbourne St.*
MOORE, Asso. Prof. T. W., Ph.D. (Yale) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
MOORE, Prof. W. A., A.M. (Chicago) Emeritus, Birmingham-Southern Coll., Birmingham 4, Ala.
MOORE, Asso. Prof. W. L., Ph.D. (Illinois) Univ. of Louisville, Louisville, Ky. *R.F.D. 1, Box 428, Coral Ridge, Ky.*
MOORMAN, Asst. Prof. R. H., Ph.D. (Peabody) Tennessee Poly. Inst., Cookeville, Tenn. *Box 549*
MOOTS, Prof. E. E., Ph.D. (Iowa) Cornell Coll., Mt. Vernon, Iowa. *710 Eighth Ave. N.*
MORAN, C. W., Ph.D. (Illinois) Instr., Wright Jr. Coll., Chicago, Ill., *2750 Jarlath St., Chicago 45, Ill.*
MORENO, Prof. JENARO, Grad. (Inst. Pedag.) Inst. Pedagogico, Santiago, Chile
MORENUS, Prof. EUGENIE M., Ph.D. (Columbia) Retired, Sweet Briar Coll., Sweet Briar, Va. *Cleveland. N.Y.*
MORGAN, AGNES L., A.M. (Columbia) Teacher, Brackenridge High School, San Antonio, Tex. *1010 A. S. St. Mary's St., San Antonio 4, Tex.*
MORGAN, EDITH L. (Mrs. Joseph), A. B. (Texas Christian) Instr., Texas Christian Univ., Fort Worth 9, Tex. *Box 398*
MORGAN, F. M., Ph.D. (Cornell) Headmaster, Clark School, Hanover, N.H.; Instr., Dartmouth Coll., Hanover, N.H.
MORGAN, W. D., *1764 St. Anthony Ave., St. Paul 4, Minn.*
MORIN, Sister M. Prudentia, A.M. (Catholic Univ.) Coll. of St. Scholastica, Duluth 2, Minn. *Dept. of Math.*

MORLEY, Prof. R. K., Ph.D. (Clark) Worcester Poly. Inst., Worcester, Mass. *43 Laconia Road, Worcester 5, Mass.*
MORREL, J. S., Ph.D. (Illinois) Principal Research Eng., Bendix Radio, Baltimore 4, Md. *Stevenson, Md.*
MORRILL, Asst. Prof. W. K., Ph.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore 18, Md.
MORRIS, Prof. C. C., A.M. (Harvard) Emeritus, Ohio State Univ., Columbus, Ohio. *1100 Urlin Ave., Columbus 12, Ohio*
MORRIS, Prof. F. R., Ph.D. (California) Head of Dept., Math. and Engg., Fresno State Coll., Fresno 4, Calif.
MORRIS, Asso. Prof. MAX, Ph.D. (Chicago) Case School of Appl. Sci., Cleveland, Ohio
MORRIS, Prof. RICHARD, Ph.D. (Cornell) Emeritus, Rutgers Univ., New Brunswick, N.J. *12 Johnson St., Highland Park, N.J.*
MORRISON, Sister CHARLES MARY, Ph.D. (Catholic Univ.) Dean, Nazareth Coll., Louisville, Ky. *851 S. Fourth St., Louisville 3, Ky.*
MORROW, Asso. Prof. D. C., Ph.D. (Chicago) Wayne Univ., Detroit 1, Mich.
MORROW, Asst. Prof. DOROTHY J., M.S. (Washington) Univ. of North Carolina, Chapel Hill, N.C. *Dept. of Mathematical Statistics*
MORROW, H. W., Jr., B.M.E. (Minnesota) Instr., Math. and Physics, Univ. of South Dakota, Vermillion, S.D. *427 Lewis St.*
MORROW, K. W., A.B. (South Dakota) Instr. Physics, Univ. of South Dakota, Vermillion, S.D. *427 Lewis St.*
MORSE, Prof. D. S., Ph.D. (Cornell) Union Coll., Schenectady 8, N.Y.
MORSE, Prof. MARSTON, Ph.D. (Harvard) Inst. for Advanced Study, Princeton, N.J.
MORTOLA, A. J., M.S. (C.C.N.Y.) Tutor, Coll. of the City of New York, New York, N.Y. *884 Riverside Drive, New York 32, N.Y.*
MOSER, LEO, M.A. (Univ. of Toronto) Lecturer, Univ. of Manitoba, Winnipeg, Manitoba, Canada. *277 Burrows Ave.*
MOSES, IRMA R., A.B. (Cornell) Instr., Temple Univ., Philadelphia 22, Pa.
MOSESSON, Z. I., Ph.D. (Harvard) Asst. Mathematician, Prudential Ins. Co. of America, Newark, N.J. *104 N. Munn Ave., East Orange, N.J.*
MOSKOVITZ, Asso. Prof. DAVID, Ph.D. (Brown) Carnegie Inst. of Tech., Pittsburgh 13, Pa.
MOSSMAN, Asso. Prof. THIRZA A., A.M. (Chicago) Kansas State Coll., Manhattan, Kans.
MOSTELLER, F. C., Ph.D. (Princeton) Lecturer, Dept. of Social Relations, Harvard Univ., Cambridge, Mass. *52 Dunster St.*
MOSTON, Prof. L. T., Ph.D. (Harvard) Dean, Waynesburg Coll., Waynebsurg, Pa. *498 Second Ave.*
MOULTON, Prof. E. J., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *1114 Colfax St.*
MOULTON, F. R., Ph.D. (Chicago) Administrative Sec'y, A.A.A.S. 1515 Massachusetts Ave., N.W., Washington 5, D.C.
MOUREAU, R. C., B.S. (U. S. Naval Acad.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.
MOURSUND, Prof. A. F., Ph.D. (Brown) Head of Dept., Univ. of Oregon, Eugene, Ore.
MOUZON, Prof. E. D., Jr., Ph.D. (Illinois) Hd. of Dept., Southern Methodist Univ., Dallas, Tex.
MOYE, Asst. Prof. W. B., M.S. (Mercer) Georgia Teachers Coll., Collegeboro, Ga.
MUEHLMANN, Rev. PAUL, A.M. (St. Louis) Prof., West Baden Coll., West Baden, Ind.
MUGGLI, Sister M. JOANNE, Ph.D. (Washington) St. Benedict's Coll., St. Joseph, Minn.
MUGUERCIA, R. V., B.S.L. (Inst. de Stgo. de Cuba) Teacher, Academia XY, Colegio Moncada, Santiago de Cuba, Cuba. *Ave. de Beljica 210*
MULLAN, C. E., A.M. (Duquesne) Mathematical Analyst, Duquesne Light Co., Planning & Development Dept. 435 Sixth Ave., Pittsburgh 19, Pa.
MULLER, G. M., A.B. (Bowdoin) *484 Gordon Road, Ridgewood, N.J.*
MULLINGS, Prof. M. E., Ph.D. (Cincinnati) Head of Dept., Abilene Christian Coll., Abilene, Tex.
MULLINS, Prof. G. W., Ph.D. (Columbia) Barnard Coll., Columbia Univ., New York, N.Y.
MUMMERY, C. R., B.S. (Toronto) Head, Mechanical and Cleaner Research Sec., The Hoover Co., North Canton, Ohio. *Engg. Dept.*
MUNDHJELD, SIGURD, A.B. (Concordia) Treas., Concordia Coll., Moorhead, Minn.
MUNRO, Asst. Prof. W. D., Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *56 Clarence Ave. S.E., Minneapolis 14, Minn.*
MUNROE, FLORENCE L., A.B. (Wellesley) Head of Dept., High School, Northampton, Mass. *51 Henshaw Ave.*
MUNSHOWER, Prof. C. W., Ph.D. (New York Univ.) Chm. of Dept., Colgate Univ., Hamilton, N.Y.

MURDOCH, Asso. Prof. D. C., Ph.D. (Toronto) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*
MURNAGHAN, Prof. F. D., Ph.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore 18, Md.
MURRAY, Prof. C. A., A.M. (Texas) West Texas State Coll., Canyon Tex. *Box 81*
MURRAY, Asst. Prof. S. B., M.S. (Chicago) Mississippi State Coll., State College, Miss., *Box 95*
MURRAY, V. F., B.S. (St. Andrew's, Scotland) *524 Hudson St., Hoboken, N.J.*
MURRAY, Prof. W. R., M.S. (Cornell) Chm. of Dept., Franklin and Marshall Coll., Lancaster, Pa.
MUSSELMAN, Prof. J. R., Ph.D. (Johns Hopkins) Chm. of Dept., Western Reserve Univ., Cleveland 6, Ohio
MYATT, D. J., B.M.E. (Louisville) Instr., Clarkson Coll. of Tech., Potsdam, N.Y.
MYERS, Asso. Prof. S. B., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *3020 Angell Hall*
MYERS, S. S., M.Ed. (Cincinnati) Teacher, Western State High School, Kalamazoo, Mich. *Box 194 Western Mich. Coll.*
MYERS, Prof. W. H., Ph.D. (Stanford) San Jose State Coll., San Jose, Calif. *Math. Dept.*

NAGLE, J. L., Div. Engr., Public Bldgs. Admin., Chicago, Ill. *1254 W. Columbia Ave., Apt. 1, Chicago 26, Ill.*
NAHIKIAN, Asso. Prof. H. M., Ph.D. (North Carolina) North Carolina State Coll., Raleigh, N.C. *Box 5548*
NAPOLES, G. A., Ph.D. (National Univ., Mexico) Dir., Instituto de Matematicas, Tacuba 5, Mexico. *Mar Negro 204, Mexico, D.F.*
NASH, Asst. Prof. E. E., M.S. (Rensselaer) Rensselaer Poly. Inst., Troy, N.Y. *Averill Park, N.Y.*
NASH, F. P., A.M. (Columbia) Teacher, Groton School, Groton, Mass.
NASSAU, J. J., Ph.D. (Syracuse) Dir. of Observatory, Case School of Appl. Sci., Cleveland 6, Ohio
NASTUCOFF, P. M., Diploma (Univ. of Moscow) Instr., Univ. of Notre Dame, Notre Dame, Ind. *202 S. Scott St., South Bend 2, Ind.*
NATHAN, D. S., Ph.D. (Cincinnati) Instr., Coll. of the City of N.Y., New York, N.Y. *3525 Perry Ave., New York 67, N.Y.*
NEALE, A. B., B.S. in M.E. (Detroit Inst. of Tech.) Research and Develop. Engr., Watson Laboratories, Red Bank, N.J. *1733 Wallace St., Phila. 30, Pa.*
NEELLEY, Prof. J. H., Ph.D. (Yale) Carnegie Inst. of Tech., Pittsburgh, Pa. *300 Broadmoor Ave., Pittsburgh 16, Pa.*
NEFF, Prof. I. F., M.S. (Chicago) Emeritus, Drake Univ., Des Moines, Iowa. *2801 Brattleboro Ave., Des Moines 11, Iowa*
NEHRBAS, C. J., A.B. (C.C.N.Y.) Moses, Nehrbas and Tyler, 20 Pine St., New York 5, N.Y.
NEISIUS, W. V., B.S. in C.E. (Georgia Tech.) Engr., Firestone Tire and Rubber Co., Atlanta, Ga. *597 St. Charles Ave. N.E.*
NELSON, Prof. A. L., Ph.D. (Chicago) Chm. of Dept., Wayne Univ., Detroit 1, Mich.
NELSON, Prof. C. A., Ph.D. (Chicago) New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
NELSON, Asso. Prof. H. E., Ph.M. (Wisconsin) Head of Dept. Gustavus Adolphus Coll., St. Peter, Minn.
NELSON, Prof. SARA L., Ph.D. (Cornell) Georgia State Coll. for Women, Milledgeville, Ga. *512 N. Columbia St.*
NELSON, T. A., M.Ed. (Maryland) Instr., Muhlenberg Coll., Allentown, Pa. *Dept. of Math.*
NELSON, Asso. Prof. W. K., M.S. in E.E. (Colorado) Engg. Math., Univ. of Colorado, Boulder Colo. *925 Grandview Ave.*
NEMECEK, VIVIAN, M.S. (Oklahoma Univ.) Instr., Oklahoma Military Academy, Claremore, Oklahoma
NEMEREVER, W. J., M.Sc. (Iowa State) Aerodynamicist, Dept. of Aeronautical Engg., University of Michigan, Ann Arbor, Mich.
NEMMERS, F. E., M.S. (Iowa) Instr., Univ. of Wisconsin in Milwaukee, Wis. *2936 N. Hackett Ave., Milwaukee 11, Wis.*
NESS, MARIE M., A.M. (Minnesota) Research Asso., Univ. of Minnesota, Minneapolis, Minn. *2530 Dupont Ave. S.*
NEUBAUER, Asst. Prof. GRETA, A.M. (Wyoming) Univ. of Wyoming, Laramie, Wyo. *Engg. Bldg.*
NEUSTADTER, S. F., A.M. (California) Teaching Asst., Univ. of Calif., Berkeley 4, Calif. *Dept. of Math.*
NEVINS, Asst. Prof. W. V., III, A.M. (Columbia) Alfred Univ., Alfred, N.Y. *Box 33*

NEWHOUSE, Asst. Prof. ALBERT, Ph.D. (Chicago) Univ. of Houston, Houston, Tex. *1562 Danville St., Houston 6, Tex.*

NEWMAN, J. J., Student, Harvard Univ., Cambridge, Mass. *220 West 107 St., New York 25, N.Y.*

NEWSOM, Prof. C. V., Ph.D. (Michigan) Chm. of Dept., Oberlin Coll., Oberlin, Ohio. *Physics Bldg.*

NEWSOM, Prof. MARY W., Ph.D. (Göttingen) Emeritus, Eureka Coll., Eureka, Ill. *Lake Dalecarlia, Lowell, Ind.*

NEWTON, Asst. Prof. ABBA V., Ph.D. (Chicago) Vassar Coll., Poughkeepsie, N.Y. *12 Collegeview Ave.*

NEWTON, Prof. G. A., A.M. (Trinity Univ.) Trinity Univ., San Antonio, Texas. *Chapman Ranch, Texas*

NICHOLLS, R. S., B.S. (London) Miles Laboratories Ltd., Western Ave., Brigdend Trading Estate, Glamorganshire, South Wales

NICHOLS, Prof. I. C., Ph.D. (Michigan) Louisiana State Univ., University Sta., Baton Rouge, La. *Nicholson Hall*

NICKERSON, HELEN K. (Mrs. W. J.), A.M. (Radcliffe) Instr., Math. and Physics, Wheaton Coll., Norton, Mass. *c/o Kelsall, 70 Preston St., Belleville, 9, N.J.*

NICKL, A. F., A.B. (Queens Coll.) Lt. (j.g.), Instr., U. S. Merchant Marine Academy, Kings Point, N.Y.

NICKOL, Prof. J. P., Ph.D. (Fribourg) Physics, Rensselaer Poly. Inst., Troy, N.Y. *Box 837, Maxwell Road, Newtonville, N.Y.*

NICOLET, JUSTIN, Structural Engr., Dept. of Subways and Highways, Chicago, Ill. *4148 N. Paulina St., Chicago 13, Ill.*

NIERSBACH, P. M., A.M. (Southern California) Chm. of Dept., High School, Bell, Calif. *6925 Passaic St., Huntington Park, Calif.*

NIVEN, Asso. Prof. IVAN, Ph.D. (Chicago) Univ. of Oregon, Eugene, Oregon. *Math. Dept.*

NOBLE, Asso. Prof. ANDREWA R., Ph.D. (California) Instr., Pacific Univ., Forest Grove, Ore.

NOBLE, Prof. C. A., Ph.D. (Göttingen) Emeritus, Univ. of California, Berkeley, Calif. *2224 Piedmont Ave.*

NOBLE, Asst. Prof. K. L., A.M. (Colo. State) Univ. of Denver, Denver, Colo. *2925 W. Douglas Place, Denver 11, Colo.*

NOEL MARIE, Sister, A.M. (Catholic Univ.) Head of Dept., Coll. of Saint Rose, Albany 3, N.Y.

NOLAN, GRACE M., M.A. (Loyola) Instr., Univ. of Ill., Chicago, Ill. *2308 W. 35th Place, Chicago 9, Ill.*

NOLAN, Asst. Prof. IRENE, M.S. (Louisiana) Southwestern Louisiana Inst., Lafayette, La. *508 E. Vermillion St.*

NOLLER, Mrs. RUTH B., Ed.M. (Buffalo) Instr., Univ. of Buffalo, Buffalo, N.Y., *145 E. Morris Ave., Buffalo 14, N.Y.*

NORDEN, M. L., B.S. (Mass. Inst. of Tech.) Mathematical Statistical Research, Project Rand, Douglas Aircraft Co., 3000 Ocean Park Blvd., Santa Monica, Calif.

NORDGAARD, Prof. M. A., Ph.D. (Columbia) Head of Dept., Upsala Coll., East Orange, N.J.

NORDHAUS, Asst. Prof. E. A., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *Dept. of Math.*

NORDSTROM, C. H., A.M. (Lehigh) Instr., Dartmouth Coll., Hanover, N.H. *9 College St.*

NORIEGA, P. L., address unknown.

NORMAN, Asst. Prof. P. B., A.M. (California) Engg., Pratt Inst., Brooklyn 5, N.Y. *2175 Cedar Ave., New York 53, N.Y.*

NORRIS, Asst. Prof. M. J., Ph.D. (Harvard) Coll. of St. Thomas, St. Paul, Minn. *75 N. Cleveland Ave., Apt. 2B, St. Paul 5, Minn.*

NORRIS, R. E., A.M. (Illinois) Dean of Instruction, State Teachers Coll., Milwaukee 11, Wis.

NORRIS, W. H., Jr., A.M. (Pennsylvania) *817 Manteo St., Apt. 6, Norfolk 7, Va.*

NORTHROP, Asso. Prof. E. P., Ph.D. (Yale) Asso. Dean of Coll., Univ. of Chicago, Chicago, Ill. *5464 Cornell Ave., Chicago 15, Ill.*

NORWOOD, L. R., A.B. (Stanford) Instr., Yale Univ., New Haven, Conn. *Dept. of Math.*

NOTLEY, LLEWELLYN, A.M. (Texas) Supt. of Schools, Teague, Tex. *P.O. Box 830*

NOVAK, Asso. Prof. J. D., M.S. (Chicago) Univ. of South Carolina, Columbia, S.C. *Dept. of Math.*

NOVOA, Asso. Prof. L. G., D.P.M.S. (Havana) Havana Univ., Havana, Cuba. *Escuela de Ciencias*

NOWLAN, Prof. F. S., Ph.D. (Chicago) Univ. of British Columbia, Vancouver, B.C., Can.

NOWLAN, MABEL I., M.S. (Michigan) Clerk, War Dept., Washington, D.C. *5810 N.W. First Place, Miami 38, Fla.*

NYSWANDER, Asso. Prof. J. A., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich.

OAKLEY, Prof. C. O., Ph.D. (Illinois) Chm. of Dept., Haverford Coll., Haverford, Pa.

O'BEIRNE, T. H., M.A. (Glasgow) Scientific Adviser, Ordnance Survey Office, Leatherhead Road, Chessington, Surrey, England

OBERG, Asso. Prof. E. N., Ph.D. (Minnesota) Univ. of Iowa, Iowa City, Iowa. *Physics Bldg.*

O'BRIEN, Prof. G. G., A.M. (Boston) Washington Missionary Coll., Takoma Park, Washington 12, D.C.

OBRIST, Sister JEANETTE, Ph.D. (Catholic) Instr., Mount St. Scholastica Coll., Atchison, Kans.

O'DONNELL, Rev. G. A., Ph.D. (St. Louis) Dean of Grad. School, Boston Coll., Chestnut Hill 67, Mass.

OEHLER, Prof. CHRISTIAN, A.M. (Columbia) Accounting, Fordham Univ.; Oehler and Sanford, Cert. Pub. Accountants, 92 Liberty St., New York 6, N.Y.

OERGEL, C. T., B.S. in M.E. (Pennsylvania State) Engr., General Electric Co., Bloomfield, N.J. *1129 Howard St., Union, N.J.*

OESCH, Asst. Prof. R. W., M.A. (Illinois) Trinity Univ., San Antonio, Tex.

OGAWA, GEORGE, A.M. (State Coll. of Washington) Student, Univ. of Chicago, Chicago, Ill. *1509 S. Millard Ave., Chicago 23, Ill.*

OGDEN, Prof. E. B., Ph.D. (Boston) Head of Dept., Union Coll., Lincoln, Nebr. *4626 Bancroft St.*

OGILVY, C. S., A.M. (Columbia) Instr., Trinity Coll., Hartford 6, Conn. *Dept. of Math.*

OGLESBY, Prof. E. J., A.M. (Virginia) Engg. Math., Univ. of Virginia, University, Va. *Box 1887*

O'HARA, Rev. C. W., Prof., Math. and Physics, Heythrop Coll., Chipping Norton, Oxon, England

OHNSORG, F. R., B.A. (St. Thomas) Student, Univ. of Minn., Minneapolis, Minn. *1230 Edmund Ave., St. Paul 4, Minn.*

OLDENBURGER, Prof. RUFUS, Ph.D. (Chicago) Illinois Inst. of Tech., Chicago 16, Ill. *1617 E. Hyde Park Blvd., Chicago 15, Ill.*

OLDS, Asso. Prof. C. D., Ph.D. (Stanford) San Jose State Coll., San Jose, Calif. *P.O. Box 462, Los Altos, Calif.*

OLDS, Asso. Prof. E. G., Ph.D. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *222 Gladstone Road, Pittsburgh 17, Pa.*

O'LEARY, Prof. A. J., M.S. (Catholic) St. Anselm's Coll., Manchester, N.H.

OLLIVIER, Asst. Prof. ARTHUR, Ph.D. (Iowa) Mississippi State Coll., State College, Miss. *Dept. of Math.*

OLLMANN, Prof. L. F., Ph.D. (Michigan) Head of Dept., Hofstra Coll., Hempstead, N.Y.

OLMSTED, Asso. Prof. J. M. H., Ph.D. (Princeton) Univ. of Minnesota, Minneapolis, 14, Minn. *100 Malcolm Ave. S.E.*

OLMSTED, Asso. Prof. MARGARET, A.M. (Illinois) Latin and Math., Augustana Coll., Rock Island, Ill. *330–19th St.*

OLPIN, J. L., M.S. (Colorado) Instr., Chm., Sci. Committee, Gila Jr. Coll., Thatcher, Ariz.

OLSEN, C. E., B.S. (U. S. Naval Acad.) Instr., Univ. of Ill., Chicago, Ill. *238 Ridgeland Ave., Waukegan, Ill.*

OLSON, C. L., Student, George Pepperdine Coll., Los Angeles, Calif. *550 W. 90 St., Los Angeles 44, Calif.*

OLSON, Asso. Prof. EMMA J., Ph.D. (Chicago) Kent State Univ., Kent, Ohio. *537 S. Lincoln St.*

OLSON, Prof. H. L., Ph.D. (Chicago) Indiana Tech. Coll., Fort Wayne, Ind.

ONDRAK, T. B., B.A. (St. Procopius) Instr., Univ. of Ill., Chicago, Ill. *2422 So. 61st Ave., Cicero 50, Ill.*

OPATOWSKI, Asst. Prof. ISAAC, D.Eng. (R. Inst. of Engg., Turin) D. Math. (R. Univ., Turin) Univ. of Michigan, Ann Arbor, Mich. *320 W. Engineering Bldg.*

O'QUINN, Asso. Prof. R. L., Ph.D. (Peabody) Louisiana State Univ., Univ. Sta., Baton Rouge, La.

ORE, Prof. OYSTEIN, Ph.D. (Oslo) Yale Univ., New Haven, Conn.

ORLIN, HYMAN, Jr., B.B.A. (C.C.N.Y.) Federal Dept. of Commerce, Coast and Geodetic Survey, Washington, D.C. *4021 Ninth St., N.W., Washington 11, D.C.*

OSBORN, R. C., A.M. (Texas) Instr., Math. and Astr., Univ. of Texas, Austin, Tex. *3003 Fountain Place, Austin 21, Tex.*

O'SHAUGHNESSY, LOUIS, Ph.D. (Pennsylvania) Dir. of Grad. Studies, Head of Dept., Applied Mech., Virginia Poly. Inst., Blacksburg, Va. *Box 93*

OSNER, H. J., M.A. (California) Instr., Modesto Junior Coll., Modesto, Calif. *209 H St.*

OSTROWSKI, Prof. A. M., Ph.D. (Göttingen) Univ. of Basle, Basle, Switzerland. *Mathem. Seminary, 21 Rheinsprung*
OTIS, F. F., B.Ed. (St. T. C., River Falls, Wis.) Grad. Asst., Univ. of Wis., Madison, Wis. *1202½ Chandler St., Madison 5, Wis.*
OTT, Asso. Prof. E. R., Ph.D. (Illinois) Rutgers Univ., New Brunswick, N. J. *199 Sterling Drive, Orange, N.J.*
OTTESON, ELLI, A.M. (Wisconsin) Chm. of Dept., High School, Eau Claire, Wis. *705 Whipple St.*
OURSLER, C. C., M.S. (Chicago) Instr., Gary College, Gary, Ind. *4832 Van Buren St.*
OVERMAN, Prof. J. R., Ph.D. (Michigan) Dean, Coll. of Lib. Arts, Bowling Green State Univ., Bowling Green, Ohio
OWENS, A. J., M.S. (Iowa) Instr., Univ. of Fla., Gainesville, Fla. *Dept. of Math.*
OWENS, Prof. F. W., Ph.D. (Chicago) Head of Dept., Pennsylvania State Coll., State College, Pa. *462 E. Foster Ave.*
OWENS, Asst. Prof. HELEN B. (Mrs. F. W.), Ph.D. (Cornell) Pennsylvania State Coll. State College, Pa. *462 E. Foster Ave.*
OXTOBY, Asso. Prof. J. C., A.M. (California) Bryn Mawr Coll., Bryn Mawr, Pa.

PALL, Prof. GORDON, Ph.D. (Chicago) Illinois Inst. of Tech., Chicago 16, Ill. *3034 W. Eastwood Ave., Chicago 25, Ill.*
PALLADINO, JAMES, B.S. (Long Island) Instr., Physics, Long Island Univ., Brooklyn, N.Y. *498 Graham Ave., Brooklyn 22, N.Y.*
PALMER, H. A., B.S. (Coll. of Idaho) Instr., Univ. of Oklahoma, Norman, Okla.
PANCOE, ARTHUR, B.S. (Wisconsin) Vice-Pres., Standard Stationery Co., Wilmette, Ill. *929 Michigan Ave.*
PARK, Asso. Prof. BART, M.S. (Mich. Coll. of Mining and Tech.) Michigan Coll. of Mining and Tech., Houghton, Mich.
PARK, EUGENE, A.M. (Lehigh) Instr., Univ. of Georgia, Athens, Ga. *R.D. 1, Andersonville, Ga.*
PARK, Prof. R. S., Ph.D. (Kentucky) Eastern Kentucky State Teachers Coll., Richmond, Ky. *213 Burnam Court*
PARKER, BOB, A.M. (Texas Tech.) Teacher, High School, Lubbock, Tex. *2606–28th St.*
PARKER, Asst. Prof. J. E., A.M. (Fisk) Knoxville Coll., Knoxville 16, Tenn.
PARKER, Asst. Prof. S. T., Ph.D. (Cincinnati) Kansas State Coll., Manhattan, Kans.
PARKER, Prof. W. V., Ph.D. (Brown) Head of Dept., Univ. of Georgia, Athens, Ga.
PARKINSON, Asso. Prof. G. A , Ph.D. (Wisconsin) Chm. of Dept., Univ. of Wisconsin, 623 W. State St., Milwaukee, Wis.
PARRISH, H. C., M.S. (North Texas St. T. C.) Ohio State Univ., Columbus, Ohio. *72 E. Blake Ave., Columbus 2, Ohio*
PATE, Prof. R. S., Ph.D. (Illinois) Michigan State Normal Coll., Ypsilanti, Mich. *224 N. Summit St.*
PATTEN, W. E., C.E. (Cornell) Hydrologist, Soil Conservation Service (Flood Control Surveys), Spartanburg, S.C., *1304 Main St., South Boston, Va.*
PATTERSON, D. O., Ph.D. (Minnesota) Head of Dept., State Teachers Coll., Valley City, N.D. *211 Fifth Ave.*
PATTERSON, Prof. B. C., Ph.D. (Johns Hopkins) Hamilton Coll., Clinton, N.Y.
PATTERSON, G. W., A.M. (Columbia) Research Asso., Moore School of Electrical Engg., Univ. of Penn., Philadelphia, Pa. *623 Yale Ave., Morton, Pa.*
PATTERSON, Asst. Prof. K. B., A.M. (Princeton) Duke Univ., Durham, N.C. *1024 Monmouth Ave.*
PATTON, Asso. Prof. A. C., M.A. (Yale) Clark Univ., Worcester, Mass. *89 Coolidge Rd.*
PAXTON, E. K., A.M. (Columbia) *P.O. Box 754, Lexington, Va.*
PAYDON, Asst. Prof. J. F., Ph.D. (Northwestern) U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
PAYNE, Asso. Prof. C. K., Ph.D. (New York Univ.) Washington Square Coll., New York Univ., New York, N.Y. *24 Valley Road, Butler, N.J.*
PAYNE, MARY H. (Mrs. W. T.), Ph.D. (Brown) Instr., Univ. of Detroit, Detroit, Mich. *17321 Braile Ave., Detroit 19, Mich.*
PEAK, PHILIP, M.S. (Iowa) Instr., Indiana Univ., Bloomington, Ind. *610 N. Faculty St.*
PEASE, D. K., M.S. (Connecticut) Grad. Student, Brown Univ., Providence 12, R.I. *24 Congdon St., Providence 6, R.I.*
PECKHAM, Asst. Prof. C. G., M.S. (Illinois) University of Dayton, Dayton 9, Ohio
PEEBLES, N. A. Engr., Office of President, Atlantic Coast Line RR Co., Wilmington, N.C. *20 N. Eighth St.*
PEGRAM, Prof. ANNIE M., A.M. (Duke) Greensboro Coll., Greensboro, N.C.

PEHRSON, Prof. E. W., A.M. (California) Univ. of Utah, Salt Lake City 1, Utah
PEISER, Asst. Prof. A. M., Ph.D. (Cornell) Hydrocarbon Research, Inc., 115 Broadway, New York, N.Y.
PELLETIER, Prof. ARTHUR. Emerité, Higher Alg., École Poly., Montreal, P.Q., Can. *8456 Drolet St.*
PENCE, Asst. Prof. SALLIE E., Ph.D. (Illinois) Univ. of Kentucky, Lexington 29, Ky. *Dept. of Math.*
PENNELL, W. O., B.S. (Mass. Inst. of Tech.) Retired, Chief Engr., Southwestern Bell Tell. Co., St. Louis, Mo. *69 Court St., Exeter, N.H.*
PENNEY, WALTER, Jr., Cryptanalyst, Navy Dept., Washington, D.C. *1506 17 St. N.W.*
PEPPER, Asso. Prof. P. M., Ph.D. (Cincinnati) Univ. of Notre Dame, Notre Dame, Ind. *Dept. of Math.*
PEPPER, Asst. Prof. R. I., A.M. (Columbia) Winthrop Coll., Rock Hill, S.C. *P.O. Box 1014*
PEREZ, FRANCISCO, Ph.D. (Chicago) University of the Philippines, Coll. of Liberal Arts, Manila, P.I.
PERISHO, C. R., A.M. (Haverford Coll.) Instr., Nebraska Wesleyan Univ., Lincoln 4, Nebraska
PERKINS, Prof. F. W., Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H. *8 Prospect St.*
PERKINS, Prof. L. R., A.M. (Tufts) Emeritus, Middlebury Coll., Middlebury, Vt. *12 Hillcrest Road*
PERLIN, Asso. Prof. I. E., Ph.D. (Chicago) Georgia School of Tech., Atlanta, Ga.
PERRY, D. B., A.M. (Stanford) Internatl. Bus. Machines Corp., Los Angeles 14, Calif. *14321 Valerio St., Van Nuys, Calif.*
PETERHANS, Prof. W. A. Visual Training, Illinois Inst. of Tech., Chicago, Ill. *1346 E. 49th St., Chicago 15, Ill.*
PETERS, Asst. Prof. A. S., Ph.D. (New York Univ.) New York Univ., New York, N.Y. *602 Rockland Ave., Mamaroneck, N.Y.*
PETERS, Asst. Prof. I. D., M.S. (West Virginia) Instr., West Virginia Univ., Morgantown, W. Va.
PETERS, Asst. Prof. J. W., Ph.D. (Johns Hopkins) Univ. of Illinois, Urbana, Ill. *253 Math. Bldg.*
PETERS, MARY A., M.S. (Iowa) Head of Dept., Board of Education, Elgin, Ill. *376 E. Chicago St.*
PETERS, MAX, M.S. (C.C.N.Y.) Chm. of Dept., Long Island City High School, Long Island City, N.Y. *845 West End Ave., New York, N.Y.*
PETERS, Asso. Prof. RUTH M., Ph.D. (Radcliffe) St. Lawrence Univ., Canton, N.Y.
PETERSON, J. C., M.S. in Ed. (North Dakota) Instr., Univ. of North Dakota, Grand Forks, N.D. *Box 486, Univ. Sta.*
PETERSON, Prof. O. J., Ph.D. (Michigan) Head of Dept., State Teachers Coll., Emporia, Kans. *1417 West St.*
PETERSON, R. P., JR., A.B. (U.C.L.A.) Teaching Asst., U.C.L.A. *10729 Ashton Ave., Los Angeles 24, Calif.*
PETERSON, Asso. Prof. T. S., Ph.D. (Ohio State) Univ. of Oregon, Eugene, Ore. *1364 E. 18th Ave.*
PETRIE, Asst. Prof. G. W., III, M.S. (Carnegie Inst. of Tech.) Lehigh Univ., Bethlehem, Pa.; *Special Engr., Bethlehem Steel Co., Bethlehem, Pa. 1439 Lehigh Parkway, Allentown, Pa.*
PETTIS, C. R., Ph.D. (Michigan) Head of Dept., Mississippi State Coll., State College, Miss. *Box 1067*
PETTIT, Prof. H. P., Ph.D. (Illinois) Head of Dept., Marquette Univ., Milwaukee, Wis. *Waterford, Wis.*
PETTUS, Prof. MARY, M.A. (Chicago) Lander Coll., Greenwood, S.C.
PFLAUM, C. W., A.M. (Pennsylvania) Master, Darrow School, New Lebanon, N.Y.
PHALEN, Prof. H. R., Ph.D. (Chicago) Coll. of William and Mary, Williamsburg, Va.
PHELPS, Asst. Prof. C. R., Ph.D. (Harvard) Rutgers Univ., New Brunswick, N.J. *Dept. of Math.*
PHILLIPS, Rev. E. C., Ph.D. (Johns Hopkins) Treas., New York Province of the Society of Jesus, 501 E. Fordham Road, New York 58, N.Y.
PHIPPS, Prof. C. G., Ph.D. (Minnesota) Univ. of Florida, Gainesville, Fla. *Box 2514 Univ. Sta.*
PIERCE, JESSE, Ph.D. (Michigan) *909 Gretna Green Way, Los Angeles 24, Calif.*
PIERSON, A. D., A.M. (Missouri) Chm. of Dept., Jr. Coll., Kansas City, Mo. *7217 Summit Ave., Kansas City 5, Mo.*
PIHLBLAD, MARGARET M., A.B. (Kansas) Grad. Student, Univ. of Kansas, Lawrence, Kans. *1201 W. Campus*

PINKERTON, Prof. R. M., B.S. (Bradley Poly. Inst.) Acting Head of Dept., Texas A. and M. Coll., College Station, Tex. *Aero. Dept.*
PIPES, C. J., B.S. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla. *723 N. Boulevard*
PIRANIAN, Asst. Prof. GEORGE, Ph.D. (Rice) Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*
PIRENIAN, Asso. Prof. Z. M., M.S. (Florida) Univ. of Florida, Gainesville, Fla. *Box 2457,* University Station
PIRRONG, Asst. Prof. C. M., JR., Ed.M. (Okla. City Univ.) Oklahoma City Univ., Oklahoma City, Okla. *912 N.W. 22nd St., Oklahoma City 6, Okla.*
PITCHER, Asso. Prof. A. E., Ph.D. (Harvard) Lehigh Univ., Bethlehem, Pa. *422 W. Broad St.*
PITTS, Prof. R. J., A.M. (Michigan) Fort Valley State Coll., Fort Valley, Ga.
PIXLEY, Asso. Prof. H. H., Ph.D. (Chicago) Wayne Univ., Detroit 1, Mich.
PIZA, P. A., *P.O. Box 627, San Juan, Puerto Rico*
PLANT, Prof. L. C., M.S. (Chicago) Emeritus, Michigan State Coll., East Lansing, Mich. *231 Oakhill Ave.*
PLETHIDES, COSTAS, Degree of Math. (University of Athens, Greece) *2705 Marshall Ave., Newport News, Va.*
PLOENGES, E. W., A.M. (Michigan) Asst. Dean, James Millikin Univ., Decatur, Ill. *317 Linden Place*
PLYMALE, R. B., M.A. (Columbia Univ.) Asso. of Math., Mercer Univ., Macon, Ga. *505 Adams St.*
PODMELE, THERESA L., A.M. (Buffalo) Teacher, East High School, Buffalo, N.Y. *356 Lisbon Ave.*
POHLEY, FLORENCE M., B.S. (Chicago) Mathematician, Country Life Ins. Co., 43 East Ohio St., Chicago 11, Ill. *4715 Farwell Ave., Lincolnwood, Chicago 30, Ill.*
POLAN, Asso. Prof. L. R., M.S. (West Virginia) Alfred Univ., Alfred, N.Y. *Box 824*
POLLARD, Asst. Prof. HARRY, Ph.D. (Harvard) Cornell Univ., Ithaca, N.Y. *112 Auburn St.*
POLLARD, Prof. H. S., Ph.D. (Wisconsin) Acting Head of Dept., Miami Univ., Oxford, Ohio. *350 Patterson Ave.*
POLLARD, Prof. W. G., Ph.D. (Rice) Physics, Univ. of Tennessee, Knoxville 16, Tenn.
POLLEY, Prof. J. C., Ph.D. (Cornell) Wabash Coll., Crawfordsville, Ind.
POLYA, Prof. GEORGE, Ph.D. (Budapest) Stanford Univ., Stanford University, Calif. *660 Dartmouth St., Palo Alto, Calif.*
POOLE, Asst. Prof. A. R., Ph.D. (Calif. Inst. of Tech.) Oregon State Coll., Corvallis, Ore.
POPOW, J. W., B.S. (Washington and Jefferson) Instr., U. S. Naval Acad., Annapolis, Md. *Dept. of Math.*
POPPEN, H. A., A.M. (Northwestern) Consulting Psychologist, Rohrer, Hibler & Replogle, Chicago, Ill. *4240 Linden St., Western Springs, Ill.*
PORCELLI, PASQUALE, B.S. (Ill. Inst. Tech.) Instr., Univ. of Texas, Austin 12, Tex. *Waggener Hall, 17*
PORGES, ARTHUR, M.S. (Ill. Inst. of Tech.) *2743 W. San Marino St., Los Angeles 6, Calif.*
PORITSKY, HILLEL, Ph.D. (Cornell) Consulting Math'n, General Elec. Co., 1 River Road Schenectady, N.Y.
PORTER, Asso. Prof. D. H., A.M. (Indiana) Math. and Physics, Marion Coll., Marion, Ind.
PORTER, Asst. Prof. RUTH E., M.S. (Oregon State Coll.) Berea Coll., Berea, Ky. *Math. Dept.*
POST, Asso. Prof. E. L., Ph.D. (Columbia) Coll. of the City of New York, New York 31, N.Y. *610 West 173 St., New York 32, N.Y.*
POULIOT, ADRIEN, M.A. (Laval) Dean of Faculty of Sciences, Laval Univ., Blvd. de l'Entente, Quebec, P.Q., Can.
POUND, Prof. V. E., Ph.D. (Toronto) Univ. of Buffalo, Buffalo, N.Y. *190 Capen Blvd., Buffalo 14, N.Y.*
POUNDER, D. W., B.A. (Toronto) Grad. Asst., Ill. Inst. of Tech., 3300 S. Federal St., Chicago 16, Ill.
POUNDER, Prof. I. R., Ph.D. (Chicago) Univ. of Toronto, Toronto, Ont., Can.
POWELL, Prof. J. E., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *137 Bogue St.*
PRENOWITZ, Asst. Prof. WALTER, Ph.D. (Columbia) Brooklyn Coll., Brooklyn 10, N.Y.
PRETZ, P. S., A.M. (St. Johns Univ.) Head of Dept., St. Benedict's Coll., Atchison, Kans.
PRICE, Prof. G. B., Ph.D. (Harvard) Univ. of Kansas, Lawrence, Kans. *205 Frank Strong Hall*
PRICE, Prof. H. F., Ph.D. (Pennsylvania) Pacific Univ., Forest Grove, Ore. *303 Second St., S.*
PRICE, Asst. Prof. H. V., Ph.D. (Iowa) Head of Dept., University High School, Iowa City, Iowa

PRICE, IRENE, Ph.D. (Indiana) Chief of Supply Analysis, AAF, War Dept., Wright Field, Dayton, Ohio. *5136 Springfield Pike, Dayton 3, Ohio*
PRICE, R. W., A.M. (Columbia) Head of Dept., Montoursville High School, Montoursville, Pa. *R.D. 1, Hughesville, Pa.*
PRIESTER, G. C., Ph.D. (Michigan) Head of Dept., Math. and Mech., Univ. of Minnesota, Minneapolis 14, Minn.
PROCTOR, S. W., S.M. (Chicago) Instr., Vashon High School, St. Louis, Mo. *2846 Pine Blvd., St. Louis 3, Mo.*
PUCKETT, Asst. Prof. W. T., JR., Ph.D. (Virginia) U.C.L.A., Los Angeles, Calif. *Dept. of Math.*
PUGSLEY, Asso. Prof. D. W., M.S. (Michigan) Berea Coll., Berea, Ky. *Box 114, College Sta.*
PURCELL, Prof. E. J., Ph.D. (Cornell) Univ. of Arizona, Tucson, Ariz. *1702 Lind Rd.*
PURDY, Asso. Prof. J. R., Ph.D. (Illinois) Southern Illinois Normal Univ., Carbondale, Ill. *Box 56*
PUTNAM, Asst. Prof. A. L., Ph.D. (Harvard) Univ. of Chicago, The College, Chicago, Ill. *806 E. 58th St., Chicago 37, Ill.*
PUTNAM, Prof. R. G., Ph.D. (Chicago) New York Univ., New York, N.Y. *115 Riverview Ave., Tarrytown, N.Y.*
PYLE, Prof. H. R., Ph.D. (California) Whittier Coll., Whittier, Calif.

QUAID, Asst. Prof. L. J., B.S. (Illinois) Univ. of Minnesota, Minneapolis, Minn. *236 Melbourne Ave., S.E.*
QUALLEY, A.O., A.M. (Iowa) Instr., Lehigh Univ., Bethlehem, Pa. *411 S. 18th St., Allentown, Pa.*
QUARLES, H. L., A.M. (Alabama) Lt. Col., A.U.S., Inspector General, Div., Port of Embarkation, Fort Mason, Calif.
QUERRY, Prof. J. W., Ph.D. (Iowa) Head of Dept., Sam Houston State Teachers Coll., Huntsville, Tex.
QUILTY, PATRICK, C.E. (Cooper Union) Commissioner, Dept. of Water Supply, Gas and Elec., New York, N.Y. *884 Riverside Drive, New York 32, N.Y.*
QUINN, GRACE S. (Mrs. R. B.), Ph.D. (Ohio State) *3221 Wheeler Road S.E., Washington 20, D.C.*

RADER, Asst. Prof. C. B., SR., M.S. (Houston) Univ. of Houston, Houston, Tex. *5533 McCormick St., Houston 3, Tex.*
RADER, Asst. Prof. M. A., A.M. (Lehigh) Moravian Coll. and Theolog. Sem., Bethlehem, Pa.
RADO, Prof. TIBOR, Ph.D. (Szeged, Hungary) Chm. of Dept., Ohio State Univ., Columbus, Ohio. *92 Walhalla Road, Columbus 2, Ohio*
RAHN, EDRIS P., A.M. (California) Teacher, Union High School, Hayward, Calif. *1456 Glen Drive, San Leandro, Calif.*
RAIFORD, T. E., Ph.D. (Michigan) Instr., Univ. of Michigan, Ann Arbor, Mich. *2511 Newport Road*
RAINE, P. W. A., A.M. (Virginia) Teacher, Newport News Board of Education, Newport News, Virginia. *71 Randolph Road, Hilton Village Branch*
RAINICH, Prof. G. Y., Master in Pure Math. (Kazan) Univ. of Michigan, Ann Arbor, Mich. *602 Oswego St.*
RAINVILLE, Asso. Prof. E. D., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1459 Rosewood St., R.F.D. 7*
RAKER, P. H., A.M. (Michigan) Instr., General Motors Inst., Flint, Mich. *2629 Colby St., Flint 4, Mich.*
RAMBO, Prof. SUSAN M., Ph.D. (Michigan) Smith Coll., Northampton, Mass. *71 Ridgewood Terrace*
RAMLER, Prof. O. J., Ph.D. (Catholic) Catholic Univ. of America, Washington 17, D.C. *12 Girard St., N.E., Washington 2, D.C.*
RAMSDELL, Prof. G. E., A.M. (Harvard) Emeritus, Bates Coll., Lewiston, Me. *40 Mountain Ave.*
RAMSEY, Asst. Prof. L. W., M.S. (Texas A. and M.) Math. and Engg., Texas Christian Univ., Fort Worth 9, Tex. *Box 441, T.C.U.*
RAMSEY, Asso. Prof. MARGARET, A.M. (Oregon) Linfield Coll., McMinnville, Ore. *Box 449*
RANDELS, W. C., Ph.D. (Brown) Physicist, North American Aircraft Corp., Inglewood, Calif. *6009 Mansfield St., Los Angeles 43, Calif.*
RANDOLPH, Prof. J. F., Ph.D. (Cornell) Oberlin Coll., Oberlin, Ohio. *114 Forest St.,*
RANKIN, F. MOZELLE, A.B. (Texas Christian) Instr., Ohio State Univ., Columbus, Ohio. *107–14th Ave., Columbus 1, Ohio*
RANKIN, Prof. J. M., A.M. (California) Coll. of Idaho, Caldwell, Idaho. *1810 Ash St.*

RANKIN, Prof. R. M., M.A. (Chicago) Missouri School of Mines, Univ. of Mo., Rolla, Mo. *1604 N. Pine St.*

RANSOM, Prof. W. R., A.M. (Harvard) Tufts Coll., Medford, Mass. *29 Sawyer Ave., Medford 55, Mass.*

RAO, Prof. B. S. M. Head of Dept., Central Coll., Bangalore, India

RAPOPORT, ANATOL, Ph.D. (Chicago) Instr., Illinois Inst. of Tech., 3300 Federal St., Chicago, Ill.

RAPP, MARY K., A.M. (Illinois) Instr., Illinois Inst. of Tech., Chicago, Ill. *5804 Giddings St., Chicago 30, Ill.*

RASMUSEN, RUTH B., Ph.D. (Chicago) Instr., Chicago City Coll., Wilson Branch, Chicago, Ill. *6105 S. Woodlawn Ave.*

RASMUSSEN, O. M., M.S. (Kansas State T. C., Emporia) Univ. of Kansas, Lawrence, Kans. *814 E. 13th St.*

RASOR, E. A., M.S. (Ohio State) Actuarial Mathematician, Social Security Admin., Washington 25, D.C. *1085 N. Manchester St., Arlington, Va.*

RASOR, Adj. Prof. ELLEN F., A.M. (Duke) Univ. of South Carolina, Columbia, S.C. *Cross Hill, S.C.*

RASOR, Prof. S. E., M.S. (Chicago) Ohio State Univ., Columbus, Ohio. *1594 Neil Ave., Columbus 1, Ohio*

RATHBUN, V. J., B.S. (Univ. of Dayton) Instr., Colegio San Jose, Rio Piedras, Puerto Rico. *Box 515*

RATNER, L. T., M.A. (U.C.L.A.) Asst., U.C.L.A., Los Angeles, Calif. *152 S. Maple Drive, Beverly Hills, Calif.*

RAUCH, L. L., A.B. (Southern California) Research Asst., Princeton Univ., Princeton, N.J. *Fine Hall*

RAUDENBUSH, Asso. Prof. H. W., Ph.D. (Columbia) Queens Coll., Flushing, N.Y.

RAWLINS, Prof. C. H., Jr., Ph.D. (Johns Hopkins) Postgraduate School, U. S. Naval Acad., Annapolis, Md. *13 Franklin St.*

RAYHER, EDWARD, A.M. (Columbia) Head of Math. and Sc. Dept., Bergen Jr. Coll., Teaneck, N.J. *198 Elm Ave.*

RAYL, Asso. Prof. ADRIENNE S., Ph.D. (Chicago) Univ. of Alabama Center, Birmingham, Ala. *1923–8th Ave.*

RAYNOR, Prof. G. E., Ph.D. (Princeton) Lehigh Univ., Bethlehem, Pa. *349 Eighth Ave.*

READ, Prof. C. B., Ph.D. (Colo. State Coll. of Educ.) Head of Dept., Univ. of Wichita, Wichita 6, Kans.

READE, Asst. Prof. M.O., Ph.D. (Rice) Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*

REAGAN, Prof. C. A., A.M. (Kansas) Friends Univ., Wichita 12, Kans.

REAGAN, Asso. Prof. L. M., A.M. (Kansas) Univ. of Wichita, Wichita, Kans. *1056 N. Market St., Wichita 5, Kans.*

REAVES, Prof. CAROLINE M., A.M. (Oklahoma) Emeritus, Coker Coll., Hartsville, S.C. *1100 N. Third St.*

REAVES, Prof. S. W., Ph.D. (Chicago) Emeritus, Dean, Coll. of A & S, Univ. of Oklahoma, Norman, Okla. *527 Chautauqua Ave.*

RECHARD, Prof. O. H., Ph.D. (Wisconsin) Dean, Coll. of Liberal Arts, Univ. of Wyoming, Laramie, Wyo.

RECHARD, O. W., M.A. (Wisconsin) Research Asst., Univ. of Wisconsin, Madison 6, Wis. *North Hall*

RECHT, Prof. A. W., Ph.D. (Chicago) Univ. of Denver, Denver 10, Colo. *2233 S. St. Paul St.*

RECKZEH, Asst. Prof. J. K., A.M. (Kentucky) State Teachers Coll., Jersey City 5, N.J.

RECTOR, R. W., A.M. (Stanford) Instr., U. S. Naval Acad., Annapolis, Md. *13½ Dean St.*

REDDEN, Asso. Prof. J. E., M.S. (Iowa State) John Tarleton Agric. Coll., Stephenville, Tex. *Box 246, Tarleton Sta.*

REDDICK, Adj. Prof. H. W., Ph.D. (Columbia) New York Univ., New York 53, N.Y. *Box 205, N.Y.U.*

REECE, R. H., A.M. (Colorado) President, New Mexico School of Mines, Socorro, N.M.

REED, Prof. F. W., Ph.D. (Virginia) Ohio Univ., Athens, Ohio. *61 Columbia Ave.*

REED, Prof. L. J., Ph.D. (Pennsylvania) Biostatistics, Vice-Pres., Johns Hopkins Univ., Baltimore 5, Md. *615 N. Wolfe St.*

REES, Prof. C. J., Ph.D. (Pennsylvania) Head of Dept., Univ. of Delaware, Newark, Del. *230 E. Main St.*

REES, Asst. Prof. DOROTHY L., A.M. (Texas) Trinity Univ., San Antonio 1, Tex. *220 West Park Ave.*

REES, MINA S., Ph.D. (Chicago) Head, Math. Section, Office of Naval Research, Washington 25, D.C. *1117 N. Pitt St., Alexandria, Va.*

REES, Asso. Prof. P. K., Ph.D. (Rice) Head of Dept., Louisiana State Univ., Baton Rouge, La. *Box 7682 L.S.U.*
REEVE, Prof. W. D., Ph.D. (Minnesota) Teachers Coll., Columbia Univ., 525 West 120 St., New York 27, N.Y.
REGAN, Prof. FRANCIS, Ph.D. (Michigan) St. Louis Univ., St. Louis, Mo.
REHBERG, C. F., A.M. (Columbia) Instr., Elec. Eng., New York Univ., New York, N.Y. *25–28 84th St., Jackson Heights, N.Y.*
REIBER, F. A., M.S. (Florida) Drake Univ., Des Moines, Iowa. *3226 University Ave.*
REICH, EDGAR, B.E.E. (Brooklyn Poly. Inst.) Research Engr., Massachusetts Inst. of Tech., Cambridge 39, Mass. *Graduate House 114A*
REICHELDERFER, Asso. Prof. P. V., Ph.D. (Ohio State) Ohio State Univ., Columbus 10, Ohio. *328 E. Beechwold Blvd., Columbus, Ohio*
REID, Prof. W. T., Ph.D. (Texas) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
REILLY, Sister MARY HENRIETTA, Ph.D. (Catholic Univ.) Head of Dept., Teachers Coll., Norwood, Cincinnati, Ohio. *1409 Freeman Ave., Cincinnati 14, Ohio*
REINGOLD, Asso. Prof. HAIM, Ph.D. (Cincinnati) Illinois Inst. of Tech., Technology Center, Chicago 16, Ill. *Dept. of Math.*
REINSCH, Prof. B. P., Ph.D. (Illinois) Head of Dept. Florida Southern Coll., Lakeland, Fla. *191 Lake Morton Drive*
REISSNER, Asso. Prof. ERIC, Ph.D. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge 39, Mass.
REKLIS, VIRGINIA M. (Mrs. E. P.), Ph.D. (Illinois) Mathematician, Aberdeen Proving Ground, Md. *35 Defense Drive, Aberdeen, Md.*
RENNO, J. G., JR., M.S. (Michigan State) Grad. Student, Univ. of Wis., Madison, Wis. *140 E. Johnson St., Madison 3, Wis.*
REVES, Asst. Prof. G. E., Ph.D. (Cincinnati) Captain, The Citadel, Charleston, S.C.
REX, Asst. Prof. E. C., M.S. (Washington) Geo. Pepperdine Coll., Los Angeles 44, Calif. *911 W. 35th Place, Los Angeles 7, Calif.*
REYNOLDS, Prof. C. N., Ph.D. (Harvard) West Virginia Univ., Morgantown, W. Va. *217 McLane Ave.*
REYNOLDS, Prof. J. B., Ph.D. (Moravian) Head of Dept., Math & Theoretical Mech., Lehigh Univ., Bethlehem, Pa. *721 W. Broad St.*
REYNOLDS, Asso. Prof. J. O., Ph.D. (North Carolina) East Carolina Teachers College, Greenville, N.C. *Dept. of Math.*
REYNOLDS, LENA E., A.M. (California) Instr., Jr. Coll., Fullerton, Calif.
RHODES, A. P., A.M. (Stanford) Coordinator-Apprentice Training, Asso. Civil Engr., U. S. Naval Training Station, San Diego 35, Calif. *1885 Sheridan Ave., San Diego 3, Calif.*
RHODES, Prof. C. E., Ph.D. (Cincinnati) Alfred Univ., Alfred, N.Y.
RHODES, Prof. M. C., Ph.D. (Peabody) Dean, Univ. of Tampa, Tampa 6, Fla.
RICE, Prof. HARRIS, A.M. (Harvard) Worcester Poly. Inst., Worcester 2, Mass.
RICE, Asst. Prof. H. L., M.S. (Iowa) Univ. of Omaha, Omaha, Nebr. *130 Fifth Ave., Council Bluffs, Iowa*
RICE, Asso. Prof. J. N., Ph.D. (Catholic) Catholic Univ. of America, Washington, D.C. *3326–13th St. N.E., Washington 17, D.C.*
RICE, Sister MARY EDMUND, Ed.M. (Cincinnati) Business Mgr., Instr., Our Lady of Cincinnati Coll., Edgecliff, Walnut Hills, Cincinnati 6, Ohio
RICE, R. B., A.B. (Wooster) Mathematician, Research Dept., Phillips Petroleum Co., Bartlesville, Okla.
RICHARDS, W. A., A.M. (Chicago) Head of Dept., Morton Jr. Coll. and High School, Cicero, Ill. *260 Blackhawk Road, Riverside, Ill.*
RICHARDSON, Prof. C. H., Ph.D. (Michigan) Chm. of Dept., Bucknell Univ., Lewisburg, Pa. *401 So. Sixth St.*
RICHARDSON, Asst. Prof. MOSES, Ph.D. (Columbia) Brooklyn Coll., Bedford Ave. and Ave. H, Brooklyn, N.Y.
RICHARDSON, N. L., A.B. (Westminster) Lecturer, St. Louis Univ., St. Louis, Mo. *3821 Botanical Ave.*
RICHARDSON, R. G. D., Ph.D. (Yale) Dean, Grad. School, Brown Univ., Providence, R.I.
RICHERT, Prof. D. H., A.M. (Colorado) Emeritus, Math. and Astr., Bethel Coll., North Newton, Kans.
RICHMOND, C. A., B.S. (Pomona) *Tyngsboro, Mass.*
RICHTMEYER, Prof. C. C., Ph.D. (Colo. State Coll. of Educ.) Head of Dept., Central Michigan Coll. of Educ., Mt. Pleasant, Mich.
RICKARD, Asst. Prof. HORTENSE, A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *79 W. Beaumont Road*

RICKEY, Prof. F. A., Ph.D. (Louisiana State) Louisiana State Univ., University Sta., Baton Rouge, La. *Dept. of Math.*
RIDER, Prof. P. R., Ph.D. (Yale) Washington Univ., St. Louis 5, Mo.
RIESS, Asst. Prof. J. K., Ph.D. (Brown) Tulane Univ., New Orleans 15, La. *17 Audubon Blvd.*
RINE, Asst. Prof. T. E., M.S. (Iowa) Illinois State Normal Univ., Normal, Ill.
RINEHART, Prof. R. F., Ph.D. (Ohio State) Case School of Tech., Cleveland, Ohio. *University Circle*
RIORDAN, JOHN, B.S. (Yale) Tech. Staff, Bell Telephone Labs., Inc., 463 West St., New York 14, N.Y.
RIPANDELLI, J. S., A.B. (Columbia) Actuarial Clerk, Jefferson Standard Life Ins. Co., Greensboro, N.C. *2010 Cornwallis Dr., Greensboro, N.C.*
RITT, Prof. J. F., Ph.D. (Columbia) Columbia Univ., New York 27, N.Y.
ROBB, J. M., A.M. (Michigan) Retired, Univ. of Southern California, Los Angeles, Calif. *1198 W. 29th St., Los Angeles 7, Calif.*
ROBBINS, Asso. Prof. C. K., A.M. (Harvard) Purdue Univ., West Lafayette, Ind. *418 Vine St.*
ROBBINS, E. S., A.B. (Wichita) Teaching Fellow, Univ. of Wichita, Wichita, Kans. *1611 N. Hillside Ave.*
ROBERTS, Prof. B. D., Ph.D. (Iowa) New Mexico Highlands Univ., Las Vegas, N.M.
ROBERTS, Asst. Prof. G. G., A.M. (Kentucky) Berea Coll., Berea, Ky. *6 Estill St.*
ROBERTS, Asso. Prof. J. H., Ph.D. (Texas) Duke Univ., Durham, N.C. *Box 4987, Duke U.*
ROBERTSON, Asst. Prof. FRED, A.M. (Indiana) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
ROBINSON, Asso. Prof. G. DEB., Ph.D. (Cambridge) Univ. of Toronto, Toronto, Ont., Can. *Dept. of Math.*
ROBINSON, G. N., A.B. (Boston Univ.) Oiler, Pumping Station, New Bedford, Mass. *R.F.D. 2, Acushnet Sta.*
ROBINSON, Prof. H. A., Ph.D. (Johns Hopkins) Head of Dept., Agnes Scott Coll., Decatur, Ga.
ROBINSON, Asso. Prof. L. V., Ph.D. (Harvard) Univ. of South Carolina, Columbia, S.C. *Dept. of Math.*
ROBINSON, Asso. Prof. R. M., Ph.D. (California) Univ. of Calif., Berkeley, Calif. *Dept. of Math.*
ROBINSON, Prof. ROBIN, Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H. *16 Allen St.*
ROBINSON, SELBY, Ph.D. (Iowa) Instr., Coll. of the City of New York, New York, N.Y. *92 Hamilton Ave., Yonkers, N.Y.*
ROBINSON, Prof. W. J., Ph.D. (Ohio State) Centre Coll. of Kentucky, Danville, Ky.
ROCHFORD, Sister M. DEPAZZI, Ph.D. (Notre Dame) Dean, Registrar, Briar Cliff Coll., Sioux City 17, Iowa
RODABAUGH, Asso. Prof. L. D., Ph.D. (Ohio State) Southern Ill. Univ., Carbondale, Ill.
RODGERS, T. G., A.M. (Wisconsin) Emeritus, Dean, New Mexico Highlands Univ., Las Vegas, N.M. *1018 Fourth St.*
RODRÍGUEZ, Aux. Prof. MARGARITA, D.P.M.S. (Havana) Mantanzas Inst., Matanzas, Cuba. *Basarrate 56, Havana, Cuba*
ROESSLER, Asso. Prof. E. B., Ph.D. (California) Asso. Statistician, Experiment Sta., Univ. of California, Coll. of Agric., Davis, Calif.
ROEVER, Prof. W. H., Ph.D. (Harvard) Emeritus, Head of Dept., Math. and Astr., Washington Univ., St. Louis, Mo.
ROGERS, Asst. Prof. H. P., A.M. (Illinois) *238 N. Mesilla Ave., Albuquerque, N.M.*
ROGNLIE, Asso. Prof. P. A., M.S. (North Dakota) Univ. of North Dakota, Grand Forks, N.D. *Dept. of Math.*
ROHDE, FLORENCE V., M.A. (Miami). Instr., Univ. of Kentucky, Lexington, Ky. *1613 Versailles Rd.*
ROHR, BEATRICE A., A.M. (Columbia) Instr., Hofstra Coll., Hempstead, N.Y. *Box 93 Halesite, N.Y.*
ROLAND, Brother, B.S. (Laval) Prof., Secretary, Superior School of Commerce, 2 Cook St., Quebec, P.Q., Can.
ROLFE, KATHRYN B. (Mrs. R. W.), M.S. (Washington) Univ. of California, Coll. of Agric., Davis, Calif. *Div. of Math. and Physics*
ROLL, ROSE, A.M. (Columbia) Chm. of Dept., Washington Irving High School, 40 Irving Place, New York, N.Y. *205 East 78 St.*
ROMAN, IRWIN, Ph.D. (Chicago) Senior Geophysicist, Dept. of Interior, Bureau of Mines, Baltimore, Md. *722 Hunting Place, Baltimore 29, Md.*
ROORDA, ETHEL, M.S. (Iowa) Head of Dept., Buena Vista Coll., Storm Lake, Iowa

ROOT, Prof. R. E., Ph.D. (Chicago) Emeritus, Math. and Mech., Postgrad. School, U. S. Naval Acad., Annapolis, Md. *7 Franklin St.*
ROSE, G. F., B.S. (Case School) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *111 N. Randall Ave., Madison 5, Wis.*
ROSE, I. H., A.M. (Brooklyn) N.R.C. Predoctoral Fellow, Harvard Univ., Cambridge, Mass. *1105 Lowell Road, Harvardevens Village, Ft. Devens, Mass.*
ROSEN, Asso. Prof. J. S., Ph.D. (Washington) Chm. of Dept., Univ. of Kansas City, Kansas City 4, Mo.
ROSENBACH, Prof. J. B., M.S. (Illinois) Asst. Head of Dept., Carnegie Inst. of Tech., Pittsburgh, Pa. *2550 Beechwood Blvd., Squirrel Hill, Pittsburgh 17, Pa.*
ROSENBAUM, JOSEPH, Ph.D. (Cornell) Teacher, The Milford School, Milford, Conn. *Box 23, Weatogue, Conn.*
ROSENBAUM, Asst. Prof. LOUISE J. (Mrs. R. A.), Ph.D. (Colorado) Reed Coll., Portland 2, Ore.
ROSENBAUM, Asst. Prof. R. A., A.B. (Yale) Reed Coll., Portland 2, Ore.
ROSENBECK, MARIAN C., M.S. (DePaul) Instr., Chicago City Coll., Wilson Branch, Chicago, Ill. *6200 Kenwood Ave., Chicago 37, Ill.*
ROSENBERG, G. S. Student, Illinois Inst. of Tech., Chicago, Ill. *1501 S. Millard Ave.*
ROSENBLOOM, Asst. Prof. P. C., Ph.D. (Stanford) Brown Univ., Providence, R. I. *Dept. of Math.*
ROSENTHAL, Prof. ARTHUR, Ph.D. (Munich) Purdue Univ., Lafayette, Ind. *Dept. of Math.*
ROSENTHALL, EDWARD, Ph.D. (Calif. Inst. of Tech.) Lecturer, McGill Univ., Montreal, P.Q., Can. *Arts Bldg.*
ROSS, Prof. A. E., Ph.D. (Chicago) Head of Dept., Univ. of Notre Dame, Notre Dame, Ind. *Dept. of Math.*
ROSS, G. J., M.S. in Educ. (C.C.N.Y.) Teacher, Erasmus Hall High School, Brooklyn, N.Y. *935 East 23 St., Brooklyn 10, N.Y.*
ROSS, LOUIS, A.M. in Ed. (Akron) Instr., Univ. of Akron, Akron 2, Ohio. *879 Whittier Ave.*
ROSSER, Prof. J. B., Ph.D. (Princeton) Cornell Univ., Ithaca, N.Y. *Dept. of Math.*
ROSSKOPF, M. F., Ph.D. (Brown) Head of Dept., John Burroughs School, Clayton, Mo. *706 Elm Tree Lane, Kirkwood 22, Mo.*
ROTH, Asst. Prof. S. G., A.M. (North Carolina) Div. of General Educ., New York Univ., New York 3, N.Y.
ROTH, W. E., Ph.D. (Wisconsin) Sampson Coll., Sampson, N.Y.
ROTHE, Asst. Prof. E. H., Ph.D. (Berlin) Univ. of Michigan, Ann Arbor, Mich. *1015 Vaughn St.*
ROTHLISBERGER, Asso. Prof. HAZEL M., B.A. (Iowa S. T. C.) Univ. of Dubuque, Dubuque, Iowa
ROTHROCK, Prof. D. A., Ph.D. (Leipzig) Emeritus, Indiana Univ., Bloomington, Ind. *1000 Atwater Ave.*
ROTHSTEIN, SALLY I., A.B. (Hunter) Asst., Univ. of Wisconsin, Madison, Wis. *Dept. of Math.*
ROUSE, Asst. Prof. L. J., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1137 Michigan Ave.*
ROWLAND, Mrs. ANNIE N., M.S. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex. *2413–22nd St.*
ROWLAND, J. J., A.M. (Oregon) Instr., State Coll. of Washington, Pullman, Wash. *312 Mech. Arts. Bldg.*
ROWLAND, Prof. S. A., A.B. (Ouachita) Ohio Wesleyan Univ., Delaware, Ohio *45 Oak Hill Ave.*
RUBASHKIN, LEILA R., M.A. (Cornell) Teaching Asst., Cornell Univ., Ithaca, N.Y. *123 Highland Place*
RUBIN, HANAN, Student, New York Univ., New York, N.Y. *155 W. 162nd. St., N.Y. 52, N.Y.*
RUDERMAN, H. D., A.M. (Columbia) Teacher, Manhattan H. S. of Aviation Trades, New York 21, N.Y. *1533 Townsend Ave., New York 52, N.Y.*
RULON, Prof. P. J., Ph.D. (Minnesota) Educ., Harvard Univ., Cambridge 38, Mass. *40 Quincy St.*
RUMNEY, ETHEL A., M.S. (Chicago) *309 N. Main St., Sandwich, Ill.*
RUNGE, Asst. Prof. LULU L., A.M. (Wisconsin) Univ. of Nebraska, Lincoln, Nebr.
RUNNING, Prof. T. R., Ph.D. (Wisconsin) Emeritus, Univ. of Michigan, Ann Arbor, Mich. *1019 Michigan Ave.*
RUPP, Prof. C. A., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa.
RUSCH, Prof. C. E., A.M. (Wisconsin) Mission House Coll., R.R. 3, Plymouth, Wis.
RUSH, JACOB, A.B. (Columbia) Teacher, Brooklyn Tech. High School, 29 Ft. Greene Place, Brooklyn 1, N.Y.
RUSK, Prof. EVELYN C. (Mrs. W. S.), Ph.D. (Cornell) Dean, Wells Coll., Aurora, N.Y.
RUSK, Prof. W. J., M.A. (Toronto) Emeritus, Grinnell Coll., Grinnell, Iowa. *1415 Park St.*

RUSS, ALEXANDER, A.M. (Duke) Physicist, Naval Aircraft Factory, Philadelphia, Pa. *2423 Curtin Terrace, Philadelphia 45, Pa.*
RUSSELL, Rev. B. M., M.S. (Michigan) Prof., Gannon Coll., Erie, Pa.
RUSSELL, Asso. Prof. HELEN G., Ph.D. (Radcliffe) Wellesley Coll., Wellesley 81, Mass.
RUTLAND, Asst. Prof. L. W., Jr., M.S. (E. Tex. St. T. C.) Univ. of Miami, Coral Gables, Fla. *Dept. of Math.*
RUTT, Prof. N. E., Ph.D. (Pennsylvania) Louisiana State Univ., University Sta., Baton Rouge, La. *Dept. of Math.*
RYSER, H. J., M.A. (Wisconsin) Univ. of Wisconsin, Madison 6, Wis. *203 North Hall*

SACHS, J. M., Ph.D. (Chicago) Instr., Wilson Jr. Coll., 6800 Stewart Ave., Chicago, Ill.
SADOWSKY, Asso. Prof. M. A., Ph.D. (Berlin) Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill.
SAFFORD, Prof. F. H., Ph.D. (Harvard) Emeritus, Univ. of Pennsylvania, Philadelphia, Pa. *4527 Osage Ave., Philadelphia 43, Pa.*
SAIBEL, Asso. Prof. E. A., Ph.D. (Mass. Inst. of Tech.) Carnegie Inst. of Tech., Pittsburgh 13, Pa.
SALEM, Asso. Prof. RAPHAEL, Sc.D. (Paris) Massachusetts Inst. of Tech., Cambridge, Mass.
SALERNO, JOHN, A.B. (Brooklyn) Mathematician, U. S. Coast and Geodetic Survey, New York, N.Y. *530 Lincoln Ave., Brooklyn 8, N.Y.*
SALKIND, CHARLES, M.S. (C.C.N.Y.) Teacher, S. J. Tilden High School, Brooklyn 3, N.Y. *1304 New York Ave.*
SALTZER, CHARLES, M.S. (Brown) Instr., Brown Univ., Providence, R.I. *351 Thayer St.*
SAMELSON, Asst. Prof. HANS, Dr. sc. math. (Fed. Inst. of Tech.) Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*
SANDERS, Prof. S. T., M.S. (Chicago) Emeritus, Louisiana State Univ., Baton Rouge, La. *1404 E. Linwood Drive, Mobile 8, Ala.*
SANDT, Asst. Prof. J. E., A.M. (Lafayette) Marietta Coll., Marietta, Ohio
SANFORD, Prof. VERA, Ph.D. (Columbia) State Teachers Coll., Oneonta, N.Y.
SANGER, Prof. R. G., Ph.D. (Chicago) Chm. of Dept., Kansas State. Coll., Manhattan, Kans. *1400 Houston St.*
SARD, Asst. Prof. ARTHUR, Ph.D. (Harvard) Queens Coll., Flushing, N.Y. *146-19 Beech Ave.*
SARNO, A. H., M.S. (Coll. of the City of N.Y.) Instr., St. John's Univ., Brooklyn, N.Y. *82-10 60th Road, Elmhurst, Long Island, N.Y.*
SASULY, MAX, M.S. (Chicago) *412 Fifth St., Room 408, Washington 1, D.C.*
SAUNDERS, R. B., Ph.D. (Minnesota) Oregon State Coll., Corvallis, Ore. *2605 Tyler St., Corvallis, Ore.*
SAUTE, Prof. GEORGE, A. M. (Brown) Rollins Coll., Winter Park, Fla.
SCARBOROUGH, Prof. J. B., Ph.D. (Johns Hopkins) U. S. Naval Acad., Annapolis, Md. *Ferry Farms*
SCHAAF, Dr. S. A., Ph.D. (California) Research Mathematician, Univ. of California, Berkeley, Calif. *205 California Hall*
SCHACH, ARTHUR, A.M. (Columbia) Instr., Philosophy, Coll. of the City of New York, New York, N.Y. *4143-43rd St., Long Island City 4, N.Y.*
SCHAEFFER, Sister M. ROSALIN, A.M. (Catholic) Head of Dept., Ursuline Coll., Louisville 6, Ky. *3115 Lexington Road*
SCHATTEN, Asso. Prof. ROBERT, Ph.D. (Columbia) Univ. of Kansas, Lawrence, Kans. *215 Strong Hall*
SCHAWALDER, R. G., B.S. (Univ. of Dayton) Head of Dept., Mt. St. John Normal School, R.D. 2, Dayton 10, Ohio
SCHECTER, SAMUEL, A.B. (Brooklyn) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *North Hall*
SCHEFFE, Asso. Prof. HENRY, Ph.D. (Wisconsin) Engg., U.C.L.A., Los Angeles 24, Calif.
SCHEIER, Rev. M. A., A.M. (St. Bonaventure) Head of Dept., Dean, Div. Natural Sci. and Math., St. Bonaventure Coll., St. Bonaventure, N.Y.
SCHELKUNOFF, S. A., Ph.D. (Columbia) Member, Tech. Staff, Bell Telephone Labs., 463 West St., New York 14, N.Y.
SCHELL, E. D., A.M. (Western Reserve) Statistician, U. S. Bureau of Labor Stat., Washington, D.C. *3440 North 12 Road, Arlington, Va.*
SCHILLING, Asso. Prof. O. F. G., Ph.D. (Marburg) Univ. of Chicago, Chicago 37, Ill. *Dept. of Math.*
SCHLAUCH, Prof. W. S., A.M. (Columbia) Emeritus, New York Univ., New York, N. Y. *14 Lohman Place, Dumont, N.J.*

SCHMID, ERWIN, B.S. (Haverford) Mathematician, U. S. Coast and Geodetic Survey, Washington 25, D.C.
SCHMIDT, O. H., Ph.D. (Brown) Librarian, Skolebakken 37, Virum, pr. Lyngby, Denmark. Danish Tech. Library, Copenhagen, Denmark
SCHNECKENBURGER, Asst. Prof. EDITH R, Ph.D. (Univ. of Mich.) Univ. of Buffalo, Buffalo 14, N.Y. *Dept. of Math.*
SCHNEPP, Prof. R. F., Ph.D. (Fribourg) Head of Dept., St. Mary's Univ. of San Antonio, San Antonio 7, Tex.
SCHOCKEN, KLAUS, Ph.D. (Berlin) Instr., Siena Coll., Loudonville, N.Y.
SCHOENBERG, Asso. Prof. I. J., Ph.D. (Jassy) Univ. of Pennsylvania, Philadelphia, Pa. *333 Dartmouth Ave., Swarthmore, Pa.*
SCHOR, HARRY, M.S. in Ed. (C.C.N.Y.) Chm. of Dept., Manhattan High School of Aviation Trades, New York, N.Y. *2348 Ocean Parkway, Brooklyn 23, N.Y.*
SCHORLING, Prof. RALEIGH, Ph.D. (Columbia) Univ. of Michigan, Ann Arbor, Mich. *403 Lenawee Drive*
SCHRAUT, Asso. Prof. K. C., Ph.D. (Cincinnati) Univ. of Dayton, Dayton 9, Ohio
SCHREIBER, Prof. E. W., A.M. (Chicago) Western Illinois State Teachers Coll., Macomb, Ill. *719 W. Adams St.*
SCHROEDER, Prof. H. F., M.S. (Louisiana) Louiaisna Poly. Inst., Ruston, La. *Tech. Sta.*
SCHULT, VERYL G., A.M. (George Washington) Head of Dept., Washington High Schools, Washington, D.C. *Wardman Park Hotel, Washington 8, D.C.*
SCHULTE, Sister MARY LEONTIUS, Ph.D. (Michigan) Instr., Coll. of St. Teresa, Winona, Minn.
SCHULTZ, O. T., M.S. (Chicago) Project Engr., Sperry Gyroscope Co., Inc., Great Neck, N.Y. *365 Stewart Ave., Garden City, N.Y.*
SCHURRER, AUGUSTA L., A.M. (Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *1731 Regent St., Madison 5, Wis.*
SCHWARTZ, Asst. Prof. ABRAHAM, Ph.D. (Mass. Inst. of Tech.) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
SCHWARTZ, A. A., M.S. in Educ. (C.C.N.Y.) Teacher, Textile Evening High School; Certifying Officer, Railroad Retirement Board, New York, N.Y. *55 Parade Place, Brooklyn 26, N.Y.*
SCHWARTZ, CECIL, B.S. (Mississippi) Instr., Univ. of Illinois, Undergrad. Div., Navy Pier, Chicago, Ill. *Rm. 49*
SCHWARTZ, H. W., Sc.M. (New York) Meteorology Instr., Army Air Forces Tech. Trng. Com., Dept. of Weather. *320 West Pells St., Paxton, Ill.*
SCHWARTZ, H. M., Ph.D. (Pennsylvania) *369 Oak Street, Patchogue, L.I.*
SCHWEITZER, A. R., Ph.D. (Chicago) *452 Oakdale Ave., Chicago, Ill.*
SCHWETYE, R. F., *Vandalia, Mo.*
SCHWID, Asso. Prof. NATHAN, Ph.D. (Wisconsin) Univ. of Wyoming, Laramie, Wyo.
SCIBIORSKI, TRYPHENA HOWARD (Mrs. A. C.), A.M. (Michigan) Instr., Wayne Univ., 5135 Cass Ave., Detroit 1, Mich.
SCOBERT, W. G., M.S. (Oregon) Instr., Univ. of Oregon, Eugene, Ore. *440½ Blair Blvd.*
SCOBLIC, Sister CLAUDETTE, A.M. (Minnesota) Head of Dept., Coll. of St. Benedict, St. Joseph, Minn.
SCOTT, CAROL S. (Mrs. W. C.), M.A. (Western Reserve) Instr., St. Petersburg Jr. Coll., 1190 8th St. N., St. Petersburg 4, Fla.
SCOTT, E. J., Ph.D. (Cornell) Instr., Univ. of Illinois, Urbana, Ill. *117 W. Cedar St., Champaign, Ill.*
SCOTT, P. C., Ph.D. (Peabody) Head of Dept., East Carolina Teachers Coll., Greenville. N.C.
SCOTT, Asst. Prof. W. T., Ph.D. (Rice) Northwestern Univ., Evanston, Ill. *Dept. of Math,*
SEALANDER, Asst. Prof. C. E., Ph.D. (Iowa) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
SEAMONS, R. S., M.S. (Washington) Instr., Grays Harbor Coll., Aberdeen, Wash. *363 N. Division St.*
SEARS, HELEN W., M.A. (Loyola) Instr., Univ. of Ill., Chicago, Ill. *2858 Dickens Ave., Chicago 47, Ill.*
SECADA, C. F., B.S. (Peruvian Naval Acad.) Lt. Commander, Peruvian Navy; Grad. Student, M.I.T., Cambridge, Mass. *107 Appleton St., Arlington, Mass.*
SEEBECK, Asst. Prof. C. L., Jr., Ph.D. (North Carolina) Univ. of Alabama, University, Ala. *Box 814*
SEIDEL, Asso. Prof. WLADMIR, Ph.D. (Munich) Univ. of Rochester, Rochester 3, N.Y.
SEIDENBERG, ABRAHAM, Ph.D. (Johns Hopkins) Instr., Univ. of Calif., Berkeley 4, Calif. *1900 B. Woolsey St., Berkeley, Calif.*
SEIDLIN, Prof. JOSEPH, Ph.D. (Columbia) Dir., Grad. Div., Alfred Univ., Alfred, N.Y.

SELBY, Prof. SAMUEL, Ph.D. (Chicago) Head of Dept., Univ. of Akron, Akron, Ohio
SELITZKY, AUDREY L., Student, Univ. of Chicago, Chicago, Ill. *4915 S. Drexel Blvd., Chicago 15, Ill.*
SEUBERT, G. A., B.S. (Pittsburgh) *Box 107, Latrobe, Pa.*
SEVIER, FRANCIS-ALOYSIUS C., M.S. in Ed. (Pennsylvania). Instr., Penn State College Center, Swarthmore, Pa. *323 S. 46th St., Philadelphia, Pa.*
SEWELL, W. E., Ph.D. (Harvard) Col., GSC, Information and Education Div., War Dept., Washington 25, D.C. *3433 S. Utah St., Arlington, Va.*
SEYBOLD, Asst. Prof. M. ANICE, A.M. (Illinois) North Central Coll., Naperville, Ill.
SHAH, Prof. N. M., M.A. (Trin. Coll., Cambridge, Eng.) Prin. of Math., M.T.B. Coll., Surat, India
SHANKS, Asst. Prof. M. E., Ph.D. (Iowa) Purdue Univ., Lafayette, Ind. *Dept. of Math.*
SHAPIRO, AARON, A.M. (C.C.N.Y.) Teacher, Midwood High School, Brooklyn, N.Y. *120 Kenilworth Place, Brooklyn 10, N.Y.*
SHARMA, Prof. H. G. S. Lingaraj Coll., Belgium, India
SHARP, W. P., Jr., A.M. (Syracuse) Instr., Army Security Agency, Research Analyst, Washington 25, D.C. *5035 25th St., S., Arlington, Va.*
SHAUB, Prof. H. C., Ph.D. (Cornell) Head of Dept., Washington and Jefferson Coll., Washington, Pa.
SHEEHY, M. J., A.B. (U.C.L.A.) Physicist, U. S. Navy Electronics Lab., San Diego 52, Calif.
SHEERAN, Sister M. LAURINE, A.M. (Catholic Univ.) Instr., Mount St. Joseph Jr. Coll., Maple Mount, Ky.
SHEFFER, Prof. I. M., Ph.D. (Harvard) Pennsylvania State Coll., State College, Pa. *212 Sparks Bldg.*
SHELDON, D. C., Ph.D. (California) Head of Dept., Clemson Agric. Coll., Clemson, S.C.
SHELDON, Prof. E. W., Ph.D. (Yale) Univ. of Alberta, Edmonton, Alberta, Can.
SHENTON, Prof. W. F., Ph.D. (Johns Hopkins) American Univ., Washington 16, D.C. *3605 Porter St. N.W.*
SHEPHARD, Asst. Prof. R. W., Ph.D. (California) New York Univ., New York, N.Y. *172 Dorchester Road, River Edge, N.J.*
SHEPHERD, C. M., M.S. in C.E. (Case School) Research Electrochemist, Naval Research Lab., Washington, D.C. *3959 Nichols Ave. S.W.*
SHEPHERD, W. L., M.S. (Okla. A. and M.) Instr., Univ. of Texas, Austin, Tex. *Dept. of Pure Math.*
SHERAK, BERNARD, M.A. (Cornell) Instr., Rutgers Univ., Newark Coll., Newark, N.J. *3314-75th St., Jackson Heights, N.Y.*
SHERER, Prof. C. R., A.M. (Nebraska) Texas Christian Univ., Fort Worth 9, Tex. *Box 245 T.C.U.*
SHERIDAN, Asso. Prof. L. W., Ph.D. (Catholic Univ.) Coll. of St. Thomas, St. Paul 1, Minn.
SHERWOOD, Prof. G. E. F., Ph.D. (Chicago) U.C.L.A., Los Angeles, Calif. *400 N. Barrington Ave., Los Angeles 24, Calif.*
SHEWHART, W. A., Ph.D. (California) Research Engr., Bell Telephone Labs., Murray Hill, N.J. *158 Lake Drive, Mountain Lakes, N.J.*
SHIRK, Prof. J. A. G., M.S. (Kansas) Kansas State Teachers Coll., Pittsburg, Kans. *116 E. Lingburg Ave.*
SHIVELY, Prof. L. S., Ph.D. (Chicago) Ball State Teachers Coll., Muncie, Ind.
SHOBE, Asso. Prof. L. R., M.S. (Kansas State Coll.) Head of Dept., Univ. of Tennessee, Knoxville 16, Tenn. *Dept. of Civil Eng.*
SHOOK, Asso. Prof. C. A., Ph.D. (Johns Hopkins) Lehigh Univ., Bethlehem, Pa. *1122 W. Broad St.*
SHORE, D. P., A.M. (Peabody) Instr., Evening Coll., Texas Christian Univ., Fort Worth, Tex. *2621 Westbrook St.*
SHREVE, Asso. Prof. D. R., Ph.D. (Illinois) Univ. of Tulsa, Tulsa 4, Okla. *2709 E. 10th St.*
SHRINER, Prof. W. O., Ph.D. (Michigan) Indiana State Teachers Coll., Terre Haute, Ind. *2525 E. Ninth St.*
SHULER, Asso. Prof. C. EUCEBIA, Ph.D. (Peabody) Univ. of South Carolina, Columbia, S.C.
SHUMWAY, Prof. R. R., A.B. (Minnesota) Asst. Dean, Univ. of Minnesota, Minneapolis 14, Minn.
SHUSTER, Prof. C. N., Ph.D. (Columbia) Head of Dept. State Teachers Coll., Trenton, N.J.; Visiting Prof., Columbia Univ., N. Y., N.Y., *2393 Pennington Road, Trenton, N.J.*
SICELOFF, Prof. L. P., Ph.D. (Columbia) Columbia Univ., New York 27, N.Y.
SIGLEY, Asso. Prof. D. T., Ph.D. (Illinois) Res. Engr., Inst. of Coop. Res., Johns Hopkins Univ., Silver Springs, Md.

SILBER, Asst. Prof. JACK, B.S. (Chicago) Roosevelt Coll., 430 S. Michigan Ave., Chicago. Ill.
SILLER, HARRY, M.S. (New York Univ.) Univ. of Conn., Extension Center, Waterbury, Conn. *127 Fulton Park Ave., Waterbury 70, Conn.*
SILVERMAN, Prof. L. L., Ph.D. (Missouri) Dartmouth Coll., Hanover, N.H.
SILVERSTEN, I. W., B.A. (Brooklyn) Supervisor, Babcock and Wilcox Co., New York, N.Y. *1123 Sutter Ave., Brooklyn 8, N.Y.*
SIMESTER, Asso. Prof. J. H., M.A. (Toronto) Speed Scientific School, Univ. of Louisville, Louisville, Ky.
SIMMONS, Asso. Prof. H. A., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
SIMON, Prof. W. G., Ph.D. (Chicago) Vice-Pres., Dean of Faculties of Arts & Sci., Western Reserve Univ., Cleveland 6, Ohio
SIMONS, W. H., M.A. (Univ. of B.C.) Lecturer, Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*
SIMPSON, R. C., Jr., A.M. (Syracuse) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *430 N. Lake St., Madison 5, Wis.*
SIMPSON, Prof. T. M., Ph.D. (Wisconsin) Dean, Grad School, Univ. of Florida, Gainesville, Fla.
SIMPSON, Prof. T. M., Jr., Ph.D. (Chicago) Randolph-Macon Coll., Ashland, Va. *Box 345*
SINCLAIR, Prof. MARY EMILY, Ph.D. (Chicago) Emeritus, Oberlin Coll., Oberlin, Ohio. *175 Morgan St.*
SINGER, DAVID, B.C.E. (New York Univ.) Tutor, Drafting Dept., The City Coll., School of Tech., New York 31, N.Y. *1521 Ocean Ave., Brooklyn 30, N.Y.*
SINGER, Asst. Prof. JAMES, Ph.D. (Princeton) Brooklyn Coll., Brooklyn, N.Y. *3054 Bedford Ave., Brooklyn 10, N.Y.*
SINGER, MAURICE, B.S. (Tulane) Lecturer, University Coll., Tulane Univ., New Orleans. *4612 Freret St., New Orleans 15, La.*
SISAM, Prof. C. H., Ph.D. (Cornell) Colorado Coll., Colorado Springs, Colo. *Cutler Hall*
SISK, Prof. AUGUSTUS, Ph.D. (Cornell) Math. and Physics, Maryville Coll., Maryville, Tenn. *117 Miller Ave.*
SKARSTEDT, MARCUS, Ph.D. (California) Librarian, San Francisco Jr. Coll., Phelan Ave., San Francisco, Calif.
SKELDING, A. Z., A.B. (C.C.N.Y.) Asst. Actuary, Natl. Council on Compensation Ins., New York, N.Y. *162 Hamilton Rd., Hempstead, L.I., N.Y.*
SLEIGHT, Prof. E. R., A.M. (Albion) Albion Coll., Albion, Mich. *410 Allen Place*
SLEPIN, BENJAMIN, LL.B. (South Jersey Law Sch.) Mathematical Lab., Dir., 422 S. 57th St., Philadelphia 43, Pa.
SLOAN, Prof. A. R., A.M. (Vanderbilt) Carson-Newman Coll., Jefferson City, Tenn. *150 N. Eastview Ave.*
SLOBIN, Prof. H. L., Ph.D. (Clark) Dean, Grad. School, Univ. of New Hampshire, Durham, N.H.
SLOTNICK, M. M., Ph.D. (Harvard) Supervisor, Geophysical Interpretation, Humble Oil and Refining Co., Houston, Tex. *4421 Roseneath Drive, Houston 4, Tex.*
SMAIL, Prof. L. L., Ph.D. (Columbia) Lehigh Univ., Bethlehem, Pa.
SMALL, W. A., B.S. (U. S. Naval Acad.) Commandant; Instr., DeVeaux School, Lewiston Road, Niagara Falls, N.Y.
SMILEY, Asso. Prof. C. H., Ph.D. (California) Astr., Dir. of Ladd Observatory, Brown Univ., Providence 12, R.I.
SMILEY, Asso. Prof. M. F., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *2616-1 Ridge Ave.*
SMITH, A. A., A.M. (N. Texas State T. C.) Chm. of Dept., San Angelo Coll., San Angelo, Tex. *1311 S. Irving St.*
SMITH, Asst. Prof. A. J., Ph.D. (Pennsylvania) Montana School of Mines, Butte, Montana
SMITH, Asso. Prof. C. D., Ph.D. (Iowa) Univ. of Alabama, University, Ala. *Box 2686*
SMITH, C. V. L., Ph.D. (Harvard) Ratheyon Mfg. Co., Waltham, Mass. *36 Parker St., Newton Centre, Mass.*
SMITH, C. W., A.M. (Wisconsin) Emeritus, State Teachers Coll., Superior, Wis. *401 E. Third St.*
SMITH, Mrs. DOROTHY B., A.B. (Phillips) Special Instr., Univ. of Oklahoma, Norman, Okla. *Box 763*
SMITH, DOROTHY E., B.S. in Ed. (N. Ill. St. T. C.) Teacher, Community High School, Erie, Ill. *Box 231*
SMITH, Prof. D. M., Ph.D. (Chicago) Head of Dept., Georgia School of Tech., Atlanta, Ga.
SMITH, Prof. E. S., Ph.D. (Virginia) Univ. of Cincinnati, Cincinnati 21, Ohio

SMITH, Prof. EDWIN R., Ph.D. (Munich) Iowa State Coll., Ames, Iowa. *1113 Clark Ave.*
SMITH, Prof. ELMER R., A.M. (Vanderbilt) Retired, State Coll. for Women, Tallahassee, Fla. *648 W. Call St.*
SMITH, Asso. Prof. F. C., Ph.D. (Michigan) Coll. of St. Thomas, St. Paul, Minn.
SMITH, Asst. Prof. F. E., Ph.D. (Catholic Univ.) Brooklyn Coll., Brooklyn, N.Y. *Box 49, Wantagh, N.Y.*
SMITH, GENEVA M., A.B. (Maine) Head of Dept., Plymouth Normal School, Plymouth, N.H.
SMITH, Prof. G. W., Ph.D. (Illinois) Univ. of Kansas, Lawrence, Kans. *1730 Illinois St.*
SMITH, Prof. H. L., Ph.D. (Chicago) Louisiana State Univ., Baton Rouge, La. *208A Nicholson Hall*
SMITH, Asso. Prof. H. W., Ph.D. (Texas) Oklahoma A. and M. Coll., Stillwater, Okla. *80 College Circle*
SMITH, Prof. I. W., A.M. (Illinois) Emeritus, North Dakota Agric. Coll.; Actuary, Ancient Order of United Workmen, Fargo, N.D. *203 Tenth St. N.*
SMITH, Asst. Prof. J. C., Ph.D. (Cornell) Lafayette Coll., Easton, Pa. *Dept. of Math.*
SMITH, Rev. J. P., A.M. (Woodstock) Georgetown Univ., Washington 7, D.C.
SMITH, LOIS B. (Mrs. R. B.), A.M. (Kansas) *8357 Allison Ave, La Mesa, Calif.*
SMITH, Prof. L. W., Ph.D. (Washington and Lee) Emeritus, Washington and Lee Univ., Lexington, Va.
SMITH, MALCOLM. Student, Illinois Inst. of Tech., Chicago, Ill. *8705 S. Chicago Ave., Chicago 17, Ill.*
SMITH, M. G., Ph.D. (Illinois) President, Roberts Jr. Coll., North Chili, N.Y.
SMITH, MARION V., A.M. (Minnesota) Head of Dept., State Teachers Coll., Moorhead, Minn. *522 Ninth St. S.*
SMITH, Asso. Prof. R. E., A.M. (North Carolina) Coll. of William and Mary, Williamsburg, Va. *Dept of Math.*
SMITH, Prof. R. G., Ph.D. (Kansas) Chm. of Dept., State Teachers Coll., Pittsburg, Kans.
SMITH, S. J., A.M. (Pittsburgh) Chm. of Dept., State Teachers Coll., Lock Haven, Pa. *75 Susquehanna Ave.*
SMITH, Asst. Prof. S. R., Ph.D. (Indiana) Univ. of Wyoming, Laramie, Wyo. *Dept of Math.*
SMITH, W. A., B.S. (Vermont) Physicist, Johns Hopkins Univ., Appl. Physics Lab., Silver Spring, Md. *1725 17th St., N.W., Washington 9, D.C.*
SMITH, Asst. Prof. W. F., A.M. (Kentucky) Univ. of Detroit, Detroit 21, Mich. *P.O. Box 43, College Park Sta.*
SMITH, Prof. W. M., Ph.D. (Columbia) Head of Dept. Lafayette Coll., Easton, Pa. *2 West College Campus*
SMITH, Asst. Prof. ZENS L., M.S. (Chicago) Asst. Dean, Univ. of Chicago, Chicago 37, Ill. *Box 213, Faculty Exchange*
SMYTH, RUTH B. (Mrs. B. J.), A.M. (Oberlin) Visiting Lecturer, Oberlin Coll., Oberlin, Ohio *273 Morgan St.*
SMYTH, Asso. Prof. S. GRACE, A.M. (Columbia) Dean of Women, Knox Coll., Galesburg, Ill.
SNADER, Prof. D. W., Ed.D. (Columbia) Galesburg Div., Univ. of Illinois, Galesburg, Ill.
SNAPPER, Asst. Prof. ERNEST, Ph.D. (Princeton) Univ. of Southern Calif., Los Angeles 7, Calif. *Dept. of Math.*
SNIVELY, Asst. Prof. L. C., M.S.E.E. (Colorado) Engg. Math., Univ. of Colorado, Boulder, Colo. *814 Pine St.*
SNYDER, Prof. A. D., A.M. (Wisconsin) Union Coll., Schenectady 8, N.Y. *1592 Union St.*
SNYDER, Prof. VIRGIL, Ph.D. (Göttingen) Cornell Univ., Ithaca, N.Y. *214 University Ave.*
SOBCZYK, ANDREW, Ph.D. (Princeton) Chief, Mathematical Research Br., Watson Laboratories, Cambridge, 39, Mass. *53 Pine Ridge Road, Arlington 74, Mass.*
SOGLIN, ALBERT, B.E. (Chicago Teachers Coll.) Instr.; Student, Ill. Inst. of Tech., Chicago, Ill. *4224 W. Grenshaw St., Chicago 24, Ill.*
SOHON, Rev. F. W., Ph.D. (Georgetown) Dean of Grad. School; Dir., Seismolog. Observatory, Georgetown Univ., Washington, D.C.
SOKOLNIKOFF, ELIZABETH S., Ph.D. (Wisconsin) Lecturer, Univ, of Wisconsin, Madison 6, Wis. *North Hall*
SOKOLNIKOFF, Prof. I. S., Ph.D. (Wisconsin) U.C.L.A. Los Angeles 24, Calif. *Dept of Math.*
SOLARI, Mrs. MARY ELIZABETH LADUE Ph.D. (Columbia) *"Dorn" Wolverhampton Rd. Kingswinford, Brierley Hill, Staffs., England*
SOLLINS, A. D., A.B. (George Washington) Mathematician, U.S. Coast and Geodetic Survey, Washington, D.C. *1501–27th St. S.E., Washington 20, D.C.*
SOLLOWAY, C. B., B.S. (Indiana St. T. Coll.) *569 N. Rossmore Ave., Los Angeles 4, Calif.*
SOLOMON, HERBERT, M.A. (Columbia) Special Lecturer, Newark Coll. of Engg., Newark 2, N.J. *Depth. of Math.*

SORGENFREY, Asst. Prof. R. H., Ph.D. (Texas) U.C.L.A. 405 Hilgard Ave., Los Angeles 24, Calif.
SORRELLS, RUTH C. (Mrs. C. C.) A.M. (Columbia) *619 N. Montclair St., Dallas, Tex.*
SOUSLEY, Prof. C. P., Ph.D. (Johns Hopkins) Rose Poly. Inst., Terre Haute, Ind. *92 Potomac Ave.*
SOUTH, Prof. D. E., Ph.D. (Michigan) Univ. of Kentucky, Lexington, Ky.
SOUTHARD, Asst. Prof. T. H., Ph.D. (Ohio State) Wayne Univ., Detroit 1, Mich.
SPARKS, Prof. F. W., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex. *Box 94, Tech. Station*
SPEAR, Prof. JOSEPH, A.M. (Boston Univ.) Chm. of Dept., Northeastern Univ., Boston 15, Mass.
SPEARS, O. S., M.S. (Ala. Poly. Inst.) Instr., Univ. of Oklahoma, Norman, Okla. *Box 2102, Boulevard Station*
SPEARS, W. K., A.B. (Dartmouth) 1st Lt., AUS, Lockbourne AAB, Columbus, 17, Ohio
SPECHT, J. R., A.M. (DePaul) Asst. Prin., Hyde Park High School, Chicago, Ill. *1252 W. 95th Place, Chicago 43, Ill.*
SPECHT, Asst. Prof. R. D., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *North Hall*
SPEEKER, Asso. Prof. G. G., A.M. (Indiana) Mich. State Coll., East Lansing, Mich.
SPEICHER, Asst. Prof. P. I., A.M. (Pennsylvania) Math. and Physics, Albright Coll., Reading, Pa.
SPELTZ, A. H., B.A. (St. Mary's) Instr., College of St. Thomas, St. Paul, Minn. *3625 Pleasant Ave., Minneapolis 8, Minn.*
SPENCELEY, Asso. Prof. G. W., A.M. (Harvard) Miami Univ., Oxford, Ohio *400 E. Church St.*
SPENCER, Asst. Prof. H. E., Ph.D. (Cornell) Virginia Poly. Inst., Blacksburg, Va. *University Club. Box 292*
SPENCER, Prof. MARY C., M.S. (Cornell) Emeritus, Newcomb Coll., New Orleans, La. *1111 Lowerline St., New Orleans 18, La.*
SPENCER, VIVIAN E., Ph.D. (Pennsylvania) Mineral Economist, U.S. Bureau of the Census, Washington, D.C. *9 Albemark St. N.W., Washington, D.C.*
SPERRY, Asso Prof. PAULINE, Ph.D. (Chicago) Univ. of California, Berkeley, Calif. *Box 28, Wheeler Hall*
SPICER, Prof. C. A., Ph.D. (Johns Hopkins) Western Maryland Coll., Westminster, Md. *17 Ridge Road*
SPICKELMIER, J. P., Capt. A.U.S., c/o The Military Attache, American Embassy, London, England
SPINKS, M. J. *Wilmington, Ohio*
SPITZ, HILLEL, M.S. (Brown) Physicist, Naval Research Lab., Washington 20, D.C. *3711 Horner Place S.E.*
SPITZBART, Asst. Prof. ABRAHAM, Ph.D. (Harvard) Univ. of Wisconsin in Milwaukee, 623 W. State St., Milwaukee 3, Wis.
SPOONER, Prof. C. C., A.M. (Amherst) Emeritus, Northern Michigan Coll. of Educ., Marquette, Mich. *117 E. Ridge St.*
SPOONER, GEORGE A., B.S. (Teachers Coll. of Conn.) Instr., Teachers Coll. of Conn., New Britain, Conn.
SPRAGENS, W. H., Jr., M.S. (Kentucky) Taft Teaching Fellow, Georgia State Woman's Coll., Valdosta, Ga. *1413 Livingston St., Covington, Ky.*
SPRINGER, Prof. C. E., Ph.D. (Chicago) Chm. of Dept., Univ. of Oklahoma, Norman, Okla.
SPRUITENBURG, ELSJE, Drs.Math. (Amsterdam) Teacher, High School, The Hague, Netherlands *110 Columbus Street*
SQUIRES, Asst. Prof. P. C., Ph.D. (Princeton) Sampson College, Sampson, N. Y., *Box 301, Geneva, N.Y.*
STABLER, Asst. Prof. E. R., Ed.D. (Harvard) Hofstra Coll., Hempstead, N.Y.
STAIR, R. H., A.B. (Lawrence Coll.) Instr., Clark Jr. Coll., Vancouver, Wash. *805 F East 78th St.*
STALEY, Prof. R. C., Ph.D. (Michigan) Head of Dept., Univ. of North Dakota, Grand Forks, N.D. *321 Cambridge St., Univ. Sta.*
STANLEY, Asst. Prof. E. L., M.S. (Tennessee) Clemson Coll., Clemson, S.C. *Box 106*
STANWICK, C. A., B.S. in E.E. (Washington) Elec. Engr., *131 Rynda Road, South Orange, N.J.*
STARCHER, Prof. G. W., Ph.D. (Illinois) Dean, Univ. Coll., Ohio Univ., Athens, Ohio *Northwood Drive*
STARK, Acting Prof. L. W., A.B. (Kansas City) Atlantic Christian Coll., Wilson, N.C.
STARK, Prof. MARION E., Ph.D. (Chicago) Wellesley Coll., Wellesley 81, Mass.

STARKE, Prof. E. P., Ph.D. (Columbia) Rutgers Univ., New Brunswick, N.J.
STARR, Asso. Prof. D. W., Ph.D. (Illinois) Southern Methodist Univ., Dallas, Tex. *3937 Wentwood, Dallas 5, Texas*
STARRETT, Asso. Prof. A. L., A.M. (Harvard) Georgia School of Tech., Atlanta, Ga. *Dept. of Math.*
STAUFFER, J. R. K., M.S. (Chicago) Instr., Rhode Island State Coll., Kingston, R.I. *Dept. of Math.*
STAYER, Asst. Prof. J. C., A.M. (Pittsburgh) Dean of Men, Juniata Coll., Huntingdon, Pa. *1618 Moore St.*
STECHSCHULTE, Rev. V. C., Ph.D. (California) Prof., Head of Dept., Xavier Univ., Cincinnati 7, Ohio
STEED, Prof. D. V., Ph.D. (California) U.C.L.A. Los Angeles 7, Calif. *1731 W. 80th St., Los Angeles 44, Calif.*
STEEN, Prof. F. H., Ph.D. (Harvard) Head of Dept., Allegheny Coll., Meadville, Pa. *R.D.1*
STEINBERG, H. L., A.B. (U.C.L.A.) Grad. Student, U.C.L.A. Los Angeles, Calif. *10796 Wellworth Ave., Los Angeles 24, Calif.*
STEINHAUS, H. W., Ph.D. (Göttingen) Research Asst., Equitable Life Assur. Soc., 393 7th Ave., New York, N.Y. *Elm Ridge Farm, Scarsdale, N.Y.*
STELFORD, Asst. Prof. NORMA K., A.M. (Northwestern) Northern Illinois State Teachers Coll., De Kalb, Ill.
STELLING, LOIS B., B.S. (Chicago) Instr., Univ. of Illinois, Chicago, Ill. *5909 W. Lake St.*
STELSON, Asso. Prof. H. E., Ph.D. (Iowa) Mich. State Coll., East Lansing. Mich. *300 Beal St.*
STEPHENS, Prof. C. F., Ph.D. (Michigan) Morgan St. Coll., Baltimore 12, Md.
STEPHENS, Prof. R. C., Ph.D. (Iowa) Knox Coll., Galesburg, Ill.
STEPHENS, Prof. R. P., Ph.D. (Johns Hopkins) Univ. of Georgia, Athens, Ga. *230 Woodlawn Ave.*
STETSON, Prof. J. M., Ph.D. (Princeton) Coll. of William and Mary, Williamsburg, Va. *232 Jamestown Rd.*
STEVENS, W. R., A.B. (George Washington) Meteorologist, U.S. Weather Bureau, New Orleans, La. *1124 Second St., New Orleans 13, La.*
STEVENSON, Prof. GUY, Ph.D. (Illinois) Head of Dept., Univ. of Louisville, Louisville, Ky.
STEWART, Asso. Prof. B. M., Ph.D. (Wisconsin) Michigan State Coll., East Lansing, Mich. *318 Albert St.*
STEWART, Asst. Prof. J. C., Ph.D. (Illinois) Lawrence Coll., Appleton, Wis. *Dept. of Math.*
STILWELL, M. F., A.M. (Syracuse) *Weems Creek, R.F.D. #3, Annapolis, Md.*
STIPE, Asso. Prof. C. G., Ph.D. (Michigan) Math. and Surveying, Michigan Coll. of Mining and Tech., Houghton, Mich. *307 Agate St.*
STOKES, ELLEN C., Ph.D. (Chicago) Dean of Women, New York State Coll. for Teachers, Albany 3, N.Y.
STOKES, Asst. Prof. RUTH W., Ph.D. (Duke) Syracuse Univ., Syracuse 10, N.Y. *Dept. of Math.*
STONE, Prof. M. H., Ph.D. (Harvard) Chm. of Dept., University of Chicago, Chicago, Ill. *313 Eckhart Hall*
STONE, Asso. Prof. R. B., A.M. (Harvard) Registrar, Purdue Univ. Lafayette, Ind. *615 Russell St., West LaFayette, Ind.*
STONE, Asst. Prof. W. M., A.M. (Oregon State) Oregon State Coll., Corvallis, Ore. *Dept. of Math.*
STONEHAM, R. G., Sc.M. (Brown) Teaching Asst., Univ. of Calif., Berkeley 4, Calif. *1122 Oxford St., Berkeley 7, Calif.*
STORY, HELEN F., A.M. (Wellesley) Instr., Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
STOUFFER, Prof. E. B., Ph.D. (Illinois) Dean, Univ. of Kansas, Lawrence, Kans. *1019 Maine St.*
STOVALL, W. B., Jr., B.S. (Florida) Grad. Fellow, Univ. of Fla., Gainesville, Fla. *Box 2463, Univ. Sta.*
STOWELL, Prof. C. J., Ph.D. (Illinois) Dean, McKendree Coll., Lebanon, Ill. *810 Belleville St.*
STRAND, H. W., A.M. (Michigan) Instr., Purdue Univ., Lafayette, Ind. *224 S. Grant St., West Lafayette, Ind.*
STRATTON, Prof. W. T., Ph.D. (Washington) Kansas State Coll., Manhattan, Kans.
STRAW, J. A., A.M. (Michigan State) Instr., Math and Mech., General Motors Inst., Flint Mich. *4423 Pengelly Road, Flint 7, Mich.*
STREET, Asso. Prof. A. T., M.A. (Northwestern) Roosevelt College, 430 South Mich. Ave., Chicago 5, Ill.
STREET, R. E., Ph.D. (Harvard) Engr., Dept. of Physics, Univ. of New Mexico, Albuquerque, New Mexico

STRIGHT, Prof. I. L., A.M. (Allegheny) State Teachers College, Indiana, Pa. *301 Blairton Ave.*

STROBEL, Asst. Prof. C. F., Ph.D. (Illinois) Instr., North Carolina State Coll., Raleigh, N.C. *3310 Pollock Place*

STROCK, E. E., A.M. (Yale) Actuarial Supervisor, Prudential Ins. Co., Newark, N.J. *55 Monroe Place, Bloomfield, N.J.*

STRONG, Prof. CORA, A.M. (Michigan) Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *109 Adams St.*

STRUTTON, E. B., M.S. (Toledo) Lecturer, U.C.L.A. Los Angeles, Calif. *P.O. Box 4081, Los Angeles 24, Calif.*

STRUYK, ADRIAN, A.M. (Columbia) Teacher, High School, Clifton, N.J. *232 Jefferson St., Paterson, 2, N.J.*

STUART, H. L., A.B. (Dickinson) *402 Hanover St., Carlisle, Pa.*

STUBBE, J. S., Ph.D. (Cincinnati) Instr., Univ. of New Hampshire, Durham, N.H. *Dept. of Math.*

STUCKEY, C. S., Ph.M. (Wisconsin) Supervisor, Metropolitan Life Ins. Co., One Madison Ave., New York 10, N.Y.

STULKEN, E.J., A.M. (Texas) Seismologist, Geophysical Service, Inc., 6000 Lemmon Ave., Dallas 9, Tex.

STURGES, Mrs. FALKA G., A.M. (California) Teacher, Public Schools, San Francisco, Calif. *790 Hopkins Ave., Redwood City, Calif.*

STURM, Prof. R. G., Ph.D. (Illinois) Engg. Mech., Purdue Univ., Lafayette, Ind. *School of Civil Engg.*

SUDBOROUGH, D. R., M.S. (Michigan) Central Michigan Coll., Mt. Pleasant, Mich.

SUFFA, Prof. MARY C., A.M. (Brown) Math. and Astr., Elmira Coll., Elmira, N.Y.

SULLIVAN, Sister M. HELEN, Ph.D. (Catholic Univ.) Prof., Chm. of Dept., Mount St. Scholastica Coll., Atchison, Kans.

SULLIVAN, Asst. Prof. MILDRED M., Ph.D. (Radcliffe) Queens Coll., 65-30 Kissena Blvd., Flushing, N.Y.

SUMNER, Mrs. RUTH G., Ed.M. (Stanford) Teacher, Oakland High School, Oakland, Calif. *3000 Central Ave., Alameda, Calif.*

SUSSMAN, Prof. IRVING, M.A. (Johns Hopkins) Univ. of California, Berkeley, Calif. *Box 47, Wheeler Hall*

SUTER, M. FRANCES, A.M. (Illinois) Instr., Madison Coll., Harrisonburg, Va. *Ft. Defiance, Va.*

SUTHERLAND, Asso. Prof. ETHEL, Ph.D. (Columbia) State Teachers Coll., Farmville, Va. *509 Beech St.*

SUTTON, Asst. Prof. C. S., M.S. (Mass. Inst. of Tech.) The Citadel, Charleston, S.C.

SVOBODA, A. F., M.S. (DePaul) Instr., DePaul Univ., Chicago, Ill. *1815 S. Troy St., Chicago 23, Ill.*

SWAFFORD, E. G., A.M. (Syracuse) Instr., U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*

SWANK, J. W. Engr., Southern California Edison Co., Ltd., *P.O. Box 351, Los Angeles 53, Calif.*

SWANSON, Asso. Prof. A. G., Ph.D. (Michigan) Head of Dept., Gustavus Adolphus College, St. Peter, Minn.

SWANSON, Asst. Prof. E. L., A.M. (Colorado Coll.) South Dakota State School of Mines, Rapid City, S.D. *P.O. Box 1588*

SWANSON, Asso. Prof. L. W., A.M. (Minnesota) Coe Coll., Cedar Rapids, Iowa *First Avenue East*

SWIFT, Prof. ELIJAH, Ph.D. (Göttingen) Dean, Univ. of Vermont, Burlington, Vt. *415 S. Willard St.*

SWINGLE, Prof. P. M., Ph.D. (Michigan) Univ. of Miami, Coral Gables 34, Fla. *215 Romano Ave.*

SWORDS, R. J., M.A. (Harvard) Instr., Weston Coll., Weston, Mass.

SYNGE, Prof. J. L., Sc.D. (Dublin) Carnegie Inst. of Tech., Pittsburgh 13, Pa. *Dept. of Appl. Math.*

SZASZ, Prof. OTTO, Ph.D. (Budapest) Research lecturer, Univ. of Cincinnati, Cincinnati, Ohio

SZEGÖ, Prof. GABOR, Ph.D. (Vienna) Stanford Univ., Stanford Univ., California

TALIAFERRO, Prof. CARRIE B., A.M. (Columbia) State Teachers Coll., Farmville, Va. *Box 392*

TANZOLA, J. J., A.M. (Columbia) Instr., Cooper Union, New York, N.Y. *2041 Watson Ave., New York, N.Y.*

TAPPAN, Prof. A. HELEN, Ph.D. (Cornell) Western Coll., Oxford, Ohio

TATE, JENNIE L., A.M. (Wisconsin) Head of Dept., McMurry Coll., Abilene, Tex. *1301 Orange St.*
TAUB, Prof. A. H., Ph.D. (Princeton) Univ. of Wash., Seattle, Wash. *Institute for Advanced Study, Princeton, N.J.*
TAUSSIG, R. A., M.B.A. (California) *2421 Durant Ave., Berkeley 4, Calif.*
TAYLOR, Asso. Prof. A. E., Ph.D. (Calif. Inst. of Tech.) U.C.L.A., Los Angeles 24, Calif. *Dept. of Math.*
TAYLOR, B. P., A.M. (U.C.L.A.) Linesman A., Douglas Aircraft Co., Santa Monica, Calif. *11326 Gladwin St., Los Angeles 24, Calif.*
TAYLOR, C. F., B.S. (Northeastern Univ.) Photo. Engr., General Electric Co., Lynn, Mass. *10 Pulaski St., Peabody, Mass.*
TAYLOR, Prof. E. H., Ph.D. (Harvard) Eastern Illinois State Teachers Coll., Charleston, Ill. *885 Seventh St.*
TAYLOR, Asso. Prof. F. J., A.B. (St. Thomas) Coll. of St. Thomas, St. Paul 1, Minn.
TAYLOR, H. E., M.S. (Calif. Inst. of Tech.) Asst., Rice Inst., Houston, Tex. *4235 Emory St., Houston 5, Tex.*
TAYLOR, Prof. J. H., Ph.D. (Chicago) George Washington Univ., Washington 6, D.C.
TAYLOR, Prof. J. S., Ph.D. (California) Univ. of Pittsburgh, Pittsburgh 13, Pa. *Dept. of Math.*
TAYLOR, MARGARET O., M.S. (Pittsburgh) Mathematician, Gulf Research and Development Co., P.O. Drawer 2038, Pittsburgh 30, Pa.
TAYLOR, Prof. MILDRED E., Ph.D. (Illinois) Math. and Astr., Mary Baldwin Coll., Staunton, Va.
TAYLOR, Asst. Prof. W. C., Ph.D. (Wisconsin) Univ. of Cincinnati, Cincinnati 21, Ohio
TAYLOR, Prof. W. E., Ph.D. (Syracuse) Emeritus, Syracuse Univ., Syracuse, N.Y.
TEMMER, H. E., M.S. (Illinois) *address unknown*
TEMPLE, M. M., M.S. (Mississippi) Instr., Tulane Univ., New Orleans 15, La.
TEMPLE, Prof. V. B., A.M. (Texas) Louisiana Coll., Pineville, La.
TEMPLE, Asst. Prof. W. B., A.M. (Louisiana State) Texas A. and M. Coll. College Station, Tex. *Faculty Exchange*
TEMPLETON, MARY M., A.M. (North Carolina) Student, Johns Hopkins Univ., Baltimore, Md.
TERAMI, TAKASHI, Ph.D. (California) Instr., Coll. of St. Thomas, St. Paul, Minn. *1674 Grand Ave., St. Paul 5, Minn.*
THEBAULT, VICTOR, Inspector Honoraire d' Assurances. *"Le Paradis", Tennie, Sarthe, France*
THEILHEIMER, Asst. Prof. FEODOR, Ph.D. (Berlin) Trinity Coll., Hartford 6, Conn.
THIELMAN, Asso. Prof. H. P., Ph.D. (Ohio State) Iowa State Coll., Ames, Iowa *Dept of Math.*
THOMAS, Asst. Prof. BETTY, A.M. (Alabama) Wesleyan Coll., Macon, Ga.
THOMAS, Prof. C. F., M.S. (Case) Case School of Appl. Sci., Cleveland, Ohio
THOMAS, EARL, M.S. (Louisiana State) Geophysical Service, Inc., Dallas 1, Tex. *P.O. Box 72, Maypearl, Tex.*
THOMAS, Prof. J. M., Ph.D. (Pennsylvania) Duke Univ., Durham, N.C. *2215 Cranford Road*
THOMAS, BROTHER L., M.S. (St. Louis) Christian Brothers Coll., St. Louis, Mo.
THOMAS, O. M., A.M. (Colo. St. Coll. of Educ.) Instr., U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*
THOMAS, P. D., A.M. (Oklahoma) Mathematician, U.S. Coast & Geodetic Survey Dept. of Commerce, Wash. 25, D.C. *2881 S. Abingdon St. Arlington, Va.*
THOMAS, R. W., Ph.D. (Pittsburgh) Secretary of the Coll., Washington and Jefferson Coll., Washington, Pa. *College Campus*
THOMAS, T. Y., Ph.D. (Princeton) Chm. of Dept., Indiana Univ., Bloomington, Ind. *Swain Hall*
THOMPSON, Prof. E. L., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex.
THOMPSON, F. B., A.B. (U.C.L.A.) Teaching Asst., Univ. of California, Berkeley, Calif. *Terrace Apt's., Bldg., F. Apt. 28, Richmond, Calif.*
THOMPSON, Prof. J. E., A.M. (Columbia) Pratt Inst., Brooklyn 5, N.Y. *183 Steuben St.*
THOMPSON, L. O., B.S. (New River State) Instr., Univ. of Detroit, 638 Mt. Elliott, Detroit 7, Mich.
THOMPSON, W. I., A.M. (California) Instr., Los Angeles City Coll., Los Angeles, Calif. *122 W. 50th St., Los Angeles 37, Calif.*
THOMSEN, D. L., Jr., Ph.D. (Mass. Inst. of Tech.) Instr., Haverford Coll., Haverford, Pa.
THOMSON, Asst. Prof. J. F., Ph.D. (Michigan) Tulane Univ., New Orleans 15, La.
THORNE, Asso. Prof. C. J., Ph.D. (Iowa State) Univ. of Utah, Salt Lake City, Utah. *Dept. of Math.*

THORNTON, H. B., A.M. (Cincinnati) Patent Examiner, U.S. Patent Office, Washington, D.C. *3765 Jay St., N. E., Apt. 3.*

THORNTON, Mrs. MARIAN W., Ph.D. (Minnesota) *372 N. Cleveland Ave., St. Paul 4, Minn.*

THORP, ELLA, A.B. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *656 Jefferson St. N. E.*

THRALL, Asst. Prof. R. M., Ph.D. (Illinois) Univ. of Michigan, Ann Arbor, Mich. *3004 Angell Hall.*

THROCKMORTON, V. C., A.M. (Southern Calif.) Visiting Instr., Univ. of Southern Calif., Los Angeles, Calif. *821 N. Heliotrope Drive, Los Angeles 27, Calif.*

THULLEN, PETER, Ph.D. (Munster) c/o Cia. Colombiana de Seguros, Bogota, Colombia, S.A.

TIERNEY, J. A., A.M. (Columbia) Instr., U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*

TILLEY, Asso. Prof. ARTHUR, Ph.D. (New York Univ.) New York Univ., New York, N.Y.

TINNER, J. C., M.S. (Chicago) Instr., Florida A. and M. Coll., Tallahassee, Fla.

TODD, JOHN, B.S. (Belfast) Lecturer, King's Coll., Strand, London W. C. 2, England, *Belvedere, 82 Bridge Road, East Molesey, Surrey, England.*

TOLLE, Prof. L. F., A.M. (St. John's Univ.) Chm. of Dept., Teachers Coll., St. John's Univ., Brooklyn, N.Y. *240–23 141 Ave., Rosedale 10, N.Y.*

TOM, D. Y. C., M.S. (Illinois) Structural Engineer, Territorial Highway Dept., Honolulu, Hawaii, *2911 Koali Road, Honolulu 36, Hawaii.*

TOMPKINS, C. B., Ph.D. (Michigan) Mathematician, Eng. Research Associates Inc., Washington 5, D.C. *1221 33rd St., N.W., Washington 7, D.C.*

TONEY, Prof. H. S., A.M. (Ohio State) Mathematician, Engg. Div., Wilberforce Univ., Wilberforce, Ohio *R.F.D., Cedarville, Ohio, RR#2*

TOOPS, Prof. H. A., Ph.D. (Columbia) Psychology, Ohio State Univ., Columbus 10, Ohio *1430 Cambridge, Columbus 12, Ohio*

TOPEL, Prof. B. J., Ph.D. (Notre Dame) Carroll Coll., Helena, Mont.

TOPP, Asst. Prof. C. W., A.M. (Illinois) Fenn Coll., Cleveland, Ohio, *3740 Lowell Rd. Cleveland Hts., Ohio*

TORNHEIM, LEONARD, Ph.D. (Chicago) Instr., Univ. of Michigan, Ann Arbor, Mich. *1454 University Terrace, Ann Arbor, Mich. Apt. 422.*

TORRANCE, Asso. Prof. C. C., Ph.D. (Cornell) Postgrad. School, U.S. Naval Acad., Annapolis, Md. *Eu-De-So, R.F.D.3*

TORREY, Prof. MARIAN M., Ph.D. (Cornell) Goucher Coll., Baltimore 18, Md.

TOWNSEND, R. R., B.S. (Muhlenberg) Instr., Pennsylvania State College, Schuykill Undergraduate Center, Pottsville, Penn.

TRACEY, Asso. Prof. J. I., Ph.D. (Johns Hopkins) Fellow of Timothy Dwight Coll., Yale Univ., New Haven, Conn. *84 McKinley Ave., New Haven 15, Conn.*

TREIBER, H. I., A.B. (Brooklyn) Grad Student, Princeton Univ., Princeton, N.J. *39 N. Hermitage Ave., Trenton 8, N.J.*

TRIGG, C. W., M.S. (Southern Calif.) Instr., Los Angeles City Coll., Los Angeles, Cal., *855 N. Vermont Ave., Los Angeles 27, Calif.*

TRIMBLE, Asso. Prof. H. C., Ph.D. (Wisconsin) Florida State Univ., Tallahassee, Fla. *Box 1013*

TRIPP, Prof. M. O., Ph.D. (Columbia) Emeritus, Wittenberg Coll., Springfield, Ohio *218 W. Cecil St.*

TRUMP, Asso. Prof. P. L., Ph.D. (Wisconsin) Univ. of Wis. Madison, Wis. *2228 Eton Ridge, Madison 5, Wis.*

TUCKER, Prof. A. W., Ph.D. (Princeton) Princeton Univ., Princeton, N. J. *Fine Hall*

TUCKER, Prof. B. A., A.M. (Millsaps) Southeastern Louisiana Coll., Hammond, La.

TUCKER, Asso. Prof. C. B., M.S. (Brown) State Teachers Coll., Emporia, Kans.

TUCKERMAN, BRYANT, Ph.D. (Princeton) Instr., Cornell Univ., Ithaca, N.Y., *White Hall*

TUKEY, Asst. Prof. J. W., Ph.D. (Princeton) Princeton Univ., Princeton, N.J.; Member, Tech. Staff., Bell Telephone Labs., Murray Hill, N.J. *Fine Hall, Princeton, M.J.*

TULLER, ANNITA, Ph.D. (Bryn Mawr) Instr., Hunter Coll., New York, N.Y. *139–62 Pershing Crescent, Jamaica 2, N.Y.*

TULLIER, Asst. Prof. P. M., M.S. (Louisiana State) Loyola Univ., New Orleans 15, La. *2900 Calhoun St.*

TUNELL, Asso. Prof. GEORGE, Ph.D. (Harvard) U.C.L.A., Los Angeles, Calif. *801 Toyopa Drive, Pacific Palisades, Cal.*

TURNER, Prof. A. B., Ph.D. (Pennsylvania) Emeritus, Coll. of the City of New York, New York, N.Y. *245 N. Mountain Ave., Montclair, N.J.*

TURNER, Prof. BIRD M., Ph.D. (Bryn Mawr) Emeritus, West Virginia Univ., Morgantown, W. Va. *61 Wilson Ave.*

TURNER, LOUISE C. (Mrs. P. K.) M.S. (Brown) *255 Shelton Rd., Nichols, Conn.*

TURRITTIN, Asso. Prof. H. L., Ph.D. (Wisconsin) Math. and Mech., Univ. of Minnesota, Minneapolis, Minn. *4046 Beard Ave. S., Minneapolis 10, Minn.*
TYSVER, J. B., A.B. (State Coll. of Washington) Grad. Student, Part-time Instr., State Coll. of Washington, Pullman, Wash. *Box 453 Coll. Sta.*

UHLER, Prof. H. S., Ph.D. (Johns Hopkins) Emeritus, Physics, Yale Univ., New Haven, Conn. *206 Spring St., Meriden, Conn.*
ULLSVIK, Asso. Prof. B. R., Ph.D. (Wisconsin) Illinois State Normal Univ., Normal, Ill.
ULMER, Asst. Prof. GILBERT, Ph.D. (Kansas) Asst. Dean, Coll. of Liberal Arts and Sci., Univ. of Kansas, Lawrence, Kans.
ULRICH, Asso. Prof. F. E., Ph.D. (Harvard) Rice Inst., Houston 1, Tex.
UNDERWOOD, Prof. R. S., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex. *2220 Broadway*
UPTON, Prof. C. B., A.M. (Columbia) Emeritus, Teachers Coll., Columbia Univ., 525 West 120 St., New York 27, N.Y.
URNER, S. E. Ph.D. (Harvard) Chm. of Dept., Los Angeles City Coll., Los Angeles, 27, Calif. *5141 Fulton Ave., Sherman Oaks, Calif.*
UTZ, W. R., A.M. (Missouri) Instr., Univ. of Virginia, Charlottesville, Va. *1115 Wertland St.*

VALENTINE, Asst. Prof. F. A., Ph.D. (Chicago) U.C.L.A., 405 Hilgard Ave., Los Angeles, Calif.
VAN ARKEL, G. H., M.S. (Washington) Instr., Math. and Physics, Everett Jr. Coll., Everett Wash.
VAN ARNAM, Asst. Prof. R. N., M.S. (Cornell) Math. and Astr., Lehigh Univ., Bethlehem, Pa. *705 First Ave.*
VAN BUSKIRK, Prof. H. C., Ph.D. (Cornell) Emeritus, California Inst. of Tech., Pasedana, Calif. *390 S. Holliston Ave., Pasadena 5, Calif.*
VANCE, Asst. Prof. E. P., Ph.D. (Michigan) Oberlin Coll., Oberlin, Ohio *174 Forest. St.*
VANDIVER, Prof. H. S., Univ. of Texas, Austin, Texas
VANDORT, H. J., A.B. (West. Mich. Coll. of Educ.) Theological Student, Bexley Theological Sem., Kenyon Coll., Gambier, Ohio, *130 Langdon Ave., N.E. Grand Rapids 3, Mich.*
VAN ENGEN, HENRY, Ph.D. (Michigan) Head of Dept., Iowa State Teachers Coll., Cedar Falls, Iowa
VAN HORN, Prof. C. E., Ph.D. (Chicago) Fisk Univ., Nashville, Tenn. *916–17th Ave. N., Nashville 8, Tenn.*
VAN ORSTRAND, C. E., M.S. (Michigan) Retired, Geophysicist, U.S. Geol. Survey, Washington, D.C. *Route 2, Manito, Ill.*
VAN SCHAACK, Asst. Prof. G. B., Ph.D. (Harvard) Washington Univ., St. Louis 5, Mo. *Dept. of Math.*
VAN VOORHIS, Asst. Prof. W. R., Ph.D. (Pennsylvania State) Fenn Coll., Cleveland, Ohio *Dept. of Math.*
VARINEAU, Asst. Prof. V. J., Ph.D. (Wisconsin) Univ. of Wyoming, Laramie, Wyo. *Dept. of Math.*
VARNHORN, Asso. Prof. MARY C., Ph.D. (Catholic Univ.) Trinity Coll., Washington D.C. *1120 Poplar Grove Ave., Baltimore 16, Md.*
VARNUM, E. C., M.S. (Michigan) Mathematician, Barber-Colman Co., Rockford, Ill. *1214 Van Wie Ave.*
VASS, J. I., Ph.D. (Wisconsin) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *5141 N. Santa Monica Blvd.*
VATNSDAL, Asso. Prof. J. R., Ph.D. (Michigan) State Coll. of Washington, Pullman, Wash. *1916 B Street*
VAUGHAN, Asst. Prof. H. E., Ph.D. (Michigan) Univ. of Illinois, Urbana, Ill. *907 S. Vine St.*
VAUSE, R. Z., Jr., B.S. (South Carolina) Clemson Coll., Clemson, S.C. *Box 1249*
VEATCH, Asso. Prof. R. W., A.M. (Northwestern) Univ. of Tulsa, Tulsa 4, Okla. *332 N. Santa Fe St.*
VEBLEN, Prof. OSWALD, Ph.D. (Chicago) Inst. for Advanced Study, Princeton, N.J. *58 Battle Road*
VEDOVA, Prof. G. C., Ph.D. (Maryland) Chm. of Dept., Newark Coll. of Engg. 365 High St., Newark 2, N.J.
VEHSE, Asso. Prof. C. H., Ph.D. (Brown) West Virginia Univ., Morgantown, W. Va. *505 Cambridge Ave.*
VEST, Asst. Prof. M. L., A.M. (Michigan) West Virginia Univ., Morgantown, W. Va. *Dept. of Math.*
VEZEAU, Asst. Prof. W. A., M.S. (St. Louis) St. Louis Univ., St. Louis 3, Mo. *6464 Lloyd, St. Louis 10, Mo.*

VICK, Asst. Prof. G. R., A.M. (Sam Houston St. T.C.) Sam Houston State Teachers Coll., Huntsville, Tex. *Box 97*

VITALE, R. L., M.E. (Poly. Inst. of Brooklyn) Asst Electrical Engr., Board of Transportation, City of New York, N.Y. *366 Avenue T., Brooklyn, N.Y.*

VON NEUMANN, Prof. JOHN, Ph.D. (Budapest) Inst. for Advanced Study, Princeton, N.J. *26 Wescott Rd.*

WADE, T. L., JR., Ph.D. (Virginia) Head of Dept., Florida State Univ., Tallahassee, Fla. *1003 Washington St.*

WAGNER, Asst. Prof. R. D., Ph.D. (Wisconsin) Marinette Extension Div., Marinette, Wis. *1006 Elizabeth St.*

WAGNER, Asst. Prof. R. W., Ph.D. (Michigan) Oberlin Coll., Oberlin, Ohio *61 S. Professor St.*

WAHLERT, Asst. Prof. H. E., A.M. (Princeton) Washington Square Coll., New York Univ., New York, N.Y. *50 Strickland Pl., Manhasset, N.Y.*

WAHLIN, Prof. G. E., Ph.D. (Yale) Univ. of Missouri, Columbia, Mo. *1401 Anthony St.*

WAIDER, K. J., A.M. (California) Instr., Physics, Univ. of San Francisco, 2130 Fulton St., San Francisco, Calif.

WAITE, ETTA A., A.M. (Columbia) Teacher, Prospect Heights H. S., Brooklyn 25, N.Y. *491 New Milford Ave., Oradell, N.J.*

WAKERLING, Asst. Prof. R. K., Ph.D. (California) Physicist, Head Tech., Information Div., Univ. of California, Berkeley, Calif. *2834 Derby St., Berkeley 5, Calif.*

WALBERT, Brother LADILAUS, B.S. (St. Mary's) Instr., Christian Brothers Coll., Memphis, Tenn. *656 E. Parkway S.*

WALDEN, Asso. Prof. EARL, Ph.D. (Illinois) Head of Dept., New Mexico Coll. of A. and M.A., State College, N.M. *Box 65*

WALDER, Prof. O.E., A.M. (Nebraska) South Dakota State Coll., Brookings, S.D. *College Station*

WALKER, Asst. Prof. G. L., Ph.D. (Cornell) Purdue Univ., West Lafayette, Ind. *Dept. of Math.*

WALKER, Rev. G. W., M.A. (Hamilton) Pastor, Walden Presbyterian Church, 2065 Bailey Ave., Buffalo 11, N.Y.

WALKER, Prof. HELEN M., Ph.D. (Columbia) Educ., Teachers Coll., Columbia Univ., New York, N.Y. *106 Morningside Dr., New York 27, N.Y.*

WALKER, L. A., A.M. (Stanford) Teacher, Math. and Physics, High School, San Mateo, Calif. *336 San Antonio St.*

WALKER, Asso. Prof. R. J., Ph.D. (Princeton) Cornell Univ., Ithaca, N.Y. *White Hall*

WALKLEY, S. E., B.S. in E.E. (Tufts Coll.) Asst., Univ. of Ill., Urbana, Ill. *160 Math. Bldg.*

WALL, Asso. Prof. B. M., A.M. (Sam Houston St. T. C.) Sam Houston State Teachers Coll., Huntsville, Tex.

WALL, Prof. H. S., Ph.D. (Wisconsin) Univ. of Texas, Austin, Tex., *11 Waggener Hall*

Wallace, Asst. Prof. A. D., Ph.D. (Virginia) Tulane Univ., New Orleans, La. *Gibson Hall*

WALLACE, Asso. Prof. R. T., A.B. (British Columbia) Victoria Coll., Victoria, B.C., Can. *830 Richmond Rd.*

WALLACE, WILLIAM, B.S. (Northeastern) Instr., Northeastern Univ., 360 Huntington Ave., Boston, Mass.

WALLACH, ISRAEL, M.S. (New York) Teacher, Thomas Jefferson High School, Brooklyn, N.Y. *816 St. John's Place, Brooklyn 16, N.Y.*

WALLICK, E. E., Ed.M. (Temple) Head of Dept., Senior High School, Lakewood, N.J. *426 Third St.*

WALSH, FRANCES E., B.S. (Mt. St. Scholastica) Grad. Student., St. Louis Univ., St. Louis, Mo. *321 North 12 St., Atchison, Kans.*

WALSH, Prof. J. L., Ph.D. (Harvard) Harvard Univ., Cambridge 38, Mass. *474 Widener Library*

WALTER, Asso. Prof. R. M., A.M. (Columbia) New Jersey Coll. for Women, New Brunswick, N.J.

WALTERS, Mrs. LILLIE C., A.M. (Colorado) Instr. Engg. Math., Univ. of Colorado, Boulder, Colo.

WALTON, JEAN B., A.M. (Brown) Instr., Univ. of Pennsylvania, Philadelphia 4, Pa. *Linton Lodge, R.F.D. 3, Media, Pa.*

WALTON, Prof. T. O., Ph.D. (Michigan) Kalamazoo Coll., Kalamazoo 49, Mich.

WALTZ, Asso. Prof. A. K., Ph.D. (Cornell) Clarkson Coll., Potsdam, N.Y. *7 Pleasant St.*

WANG, CHIH-YI, B.S. (Yenching) Grad. Student, Univ. of Minnesota, Minneapolis, Minn. *Dept. of Math.*

WANG, E. H., Dip. Engg., Techn. (Vienna) Instr., Univ. of Cincinnati, Engg. Coll., 3630 Wabash Ave., Cincinnati 7, Ohio

WAPPLE, A. R., A.M. (California) Instr., Texas A. and M. Coll., College Station, Tex. *1300 23rd St., Bryan, Tex.*

WARD, Asso. Prof. J. A., Ph.D. (Wisconsin) Asst. Head of Dept., Univ. of Georgia, Athens, Ga.

WARD, L. E., Ph.D. (Harvard) Mathematician, Fire Control Section, Res. and Dev. Dept., N.O.T.S., Inyokern, Calif.

WARD, Prof. MORGAN, Ph.D. (Calif. Inst. of Tech.) California Inst. of Tech., Pasadena 4, Calif.

WARDWELL, Asso. Prof. J. F., Ph.D. (Johns Hopkins) Colgate Univ., Hamilton, N.Y. *62 Broad St.*

WARNER, F. C., A.B. (Wooster) Instr., Univ. of Buffalo, Buffalo, N.Y. *Hayes Hall*

WARNOCK, Asso. Prof. W. G., Ph.D. (Illinois) Rensselaer Poly. Inst., Troy, N.Y. *18 Avenue A., Melrose, N.Y.*

WARREN, Prof. K. L., Ph.D. (Michigan State) Millsaps Coll., Jackson, Miss. *432 Marshall St., Apt. 1*

WARREN, Prof. L. A. H., Ph.D. (Chicago) Actuarial Sci., Univ. of Manitoba, Winnipeg, Man., Can. *64 Niagara St.*

WASHBURNE, A. C., Grad. (U.S. Milit. Acad.) Emeritus, Actuary, Berkshire Life Ins. Co., Pittsfield, Mass. *134 E. Housatonic St.*

WASOW, Asst. Prof. W. R., Ph.D. (New York Univ.) Swarthmore Coll., Swarthmore, Pa.

WASSON, Asst. Prof. H. P., A.M. (Columbia) Newark Coll. of Engg. Newark, N.J. *282 William St., East Orange, N.J.*

WATKEYS, Prof. C. W., A.M. (Harvard) Chm. of Dept., Univ. of Rochester, Rochester, N.Y. *287 Dartmouth St.*

WATSON, Asst. Prof. G. C., A.M. (Virginia) North Carolina State Coll., Raleigh, N.C. *Dept. of Math.*

WATT, Asso. Prof. MARTHA W., A.M. (Columbia) Emeritus, Wheaton Coll., Norton, Mass. *35 Humboldt Ave., Providence 6, R.I.*

WATTS, C. B., A.B. (Indiana) Astronomer, U.S. Naval Observatory, Washington 25, D.C.

WAYNE, ALAN, M.S. in Educ. (C.C.N.Y.) Teacher, Automotive H.S., Brooklyn 6, N.Y. *141-21 78th Road, Flushing, N.Y.*

WEAR, Asso. Prof. L. E., Ph.D. (Johns Hopkins) California Inst. of Tech., Pasadena, Calif. *2247 Lambert Drive*

WEAVER, C. L., B.S. (Kent State) Asst. Actuary, New England Mut. Life Ins. Co., 501 Boylston St., Boston, Mass.

WEAVER, WARREN, Ph.D. (Wisconsin) Dir. for the Natural Sciences, Rockefeller Foundation, New York, N.Y. Chief, Applied Math. Panel, N.D.R.C. *160 Brite Ave., Scarsdale, N.Y.*

WEBB, Asst. Prof. D. L., Ph.D. (Calif. Inst. of Tech.) Univ. of Arizona, Tucson, Ariz. *Dept. of Math.*

WEBBER, Asso. Prof. G. C., Ph.D. (Chicago) Univ. of Delaware, Newark, Del.

WEBER, BETTY R., A.B. (Columbia Coll.) Instr., Univ. of South Carolina, Columbia, S.C., *1516 Columbia College Drive*

WEBER, C. J. Trust Officer, Chase National Bank, New York, N.Y. *P.O. Box 63, Chappaqua, N.Y.*

WEBSTER, Asst. Prof. M. S., Ph.D. (Pennsylvania) Purdue Univ., Lafayette, Ind.

WEDDERBURN, Prof. J. H. M., D.Sc. (Edinburgh) Emeritus, Princeton Univ., Princeton, N.J. *Fine Hall*

WEDEL, Asst. Prof. E. B., A.M. (Oklahoma) Univ. of Wichita, Wichita, Kans. *3600 E. 21st, Route 3*

WEEBER, Asso. Prof. MARGARET C., A.M. (Chicago) Teachers Coll. of Connecticut, New Britain, Conn.

WEGNER, Asso. Prof. K. W., Ph.D. (Wisconsin) Carleton Coll., Northfield, Minn.

WEHAUSEN, J. V., Ph.D. (Michigan) Mathematician, David Taylor Model Basin, Cardarock, Md. *126 Winchester Way, Falls Church, Va.*

WEIDA, Prof. F. M., Ph.D. (Iowa) Head of Dept. of Statistics, George Washington Univ., Washington 6, D.C. *7130 Hampden Lane, Bethesda 14, Md.*

WEINGARTEN, HARRY, A.M. (Columbia) Tutor, Coll. of the City of New York; Instr., Academic Dept., New York Univ., New York, N.Y. *347 West 55th St., New York 19, N.Y.*

WEISNER, Asso. Prof. LOUIS, Ph.D. (Columbia) Hunter Coll., Bedford Park Blvd. and Navy Ave., New York, N.Y.

WEISS, Prof. MARIE J., Ph.D. (Stanford) Sophie Newcomb Coll., New Orleans, 18, La.

WELCHONS, A. M., A.M. (Indiana) Head of Dept., Arsenal Tech. Schools, Indianapolis, Ind. *509 N. Drexel Ave., Indianapolis 1, Ind.*

WELLS, Asso. Prof. C. P., Ph.D. (Iowa State) Michigan State Coll., East Lansing, Mich. *Dept. of Math.*

WELLS, Asso. Prof. E. D., A.M. (Minnesota) Univ. of Pittsburgh, Pittsburgh, Pa. *735 S. Negley Ave., Pittsburgh 6, Pa.*

WELLS, Prof. MARY EVELYN, Ph.D. (Chicago) Chm. of Dept., Vassar Coll., Poughkeepsie, N.Y.

WELLS, N. W., A.M. (Rice) Instr., Texas A & M Coll., College Sta., Tex.

WELLS, Prof. V. H., Ph.D. (Michigan) Williams Coll., Williamstown, Mass. *3 Chapin Court*

WELMERS, Asst. Prof. E. T., Ph.D. (Michigan) Engr., Bell Aircraft Corp., Niagara Falls, N.Y. *165 Fayette Ave., Kenmore, N.Y.*

WELMERS, INA W. (Mrs. E. T.) A.M. (Michigan) Instr., Univ. of Buffalo, Buffalo, N.Y. *165 Fayette Ave., Kenmore, 17, N.Y.*

WENTE, IRENE L., M.S. (Chicago) Instr., South Dakota State Coll., Brookings, S.D.

WERKMAN, MARY K., M.S. (Chicago) Teacher, Parker High School, Chicago, Ill. *7139 S. Normal Blvd., Chicago 21, Ill.*

WESCOTT, Asst. Prof. M. E., Ph.D. (Northwestern) Northwestern Univ., Evanston, Ill. *2011 Beechwood Drive, Wilmette, Ill.*

WEST, ADA H., A.M. (Kansas) *615 N. Hennepin St., Dixon, Ill.*

WEST, MARION L., Student, Univ. of Tulsa, Tulsa, Okla. *1161 N. Elwood, Tulsa 6, Okla.*

WESTBROOK, H. S., M. Sgt., Sq. B., 131st AAF, BU, Offutt Field, Ft. Crook, Nebr.

WESTERN, D. W., Ph.D. (Brown) Instr., Brown Univ., Providence 12, R.I.

WESTGATE, CHRISTINE, M.S. (Chicago) Instr., Univ. of North Dakota, Grand Forks, N.D. *1025 Hill Ave., Grafton, N.D.*

WESTHAFER, Asso. Prof. R. L., Ph.D. (Ohio State) New Mexico Coll. of A. and M.A., State College, N.M. *510½ W. Amador Ave., Las Cruces, N.M.*

WETZIG, C. U., A.M. (Texas) Instr., Arkansas A. and M. Coll., Magnolia, Ark.

WEXLER, Prof. CHARLES, Ph.D. (Harvard) Head of Dept., State Teachers Coll., Tempe, Ariz. *1215 Maple Ave.*

WEYL, F. J., Ph.D. (Princeton) Mathematician, Bureau of Ordnance, Navy Dept., Washington, D.C. *R.F.D. 3, Box 833, Fairfax, Va.*

WHEELER, A. H., A.M. (Clark) Retired, Prof. Affiliate, Clark Univ., Worcester, Mass. *44 Beverly Road, Worcester 5, Mass.*

WHEELER, Prof. ANNA PELL, Ph.D. (Chicago) Bryn Mawr Coll., Bryn Mawr, Pa.

WHEELER, Prof. C. H. III, Ph.D. (Johns Hopkins) Treas., Univ. of Richmond, Richmond, Va.

WHEELER, Asso. Prof. J. J., A.M. (Kansas) Univ. of Kansas, Lawrence, Kans. *1024 Alabama St.*

WHELAN, Asst. Prof. A. MARIE, Ph.D. (Johns Hopkins) Hunter Coll., New York, N.Y.

WHETSTONE, Asst. Prof. G. A., Ph.D. (Washington) Texas Tech. Coll., Lubbock, Tex. *Dept. of Civil Engg.*

WHITE, Prof. A. E., M.S. (Purdue) Kansas State Coll., Manhattan, Kansas

WHITE, E. V., A.M. (Baylor) Dean, Texas State Coll. for Women, Denton, Tex. *Box 3895, T.S.C.W. Station*

WHITE, HELEN R., A.B. (North Carolina) Statistician, Bureau of Census, Dept. of Commerce; Grad. Student, George Washington Univ., Washington, D.C. *4415 S. 36th St., Apt. A-2, Arlington, Va.*

WHITE, J. H., M.A. (Columbia) Instr., U. S. Naval Acad., Annapolis, Md. *Gen. Del., Riva, Md.*

WHITE, M. E., B.A. (Wesleyan) Instr., Hunter Coll., New York, N.Y. *240 W. 102nd St., New York 25, N.Y.*

WHITE, MARELENA, M.S. (Louisiana) Asst., Louisiana State Univ., University Sta., Baton Rouge, La. *Dept. of Math.*

WHITE, MARION B., Ph.D. (Chicago) *1949 Woodlyn Road, Pasadena 7, Calif.*

WHITE, R. L., A.B. (U.C.L.A.) Student, U.C.L.A., Los Angeles, Calif. *2122 W. Washington Blvd., Los Angeles 7, Calif.*

WHITE, T. J., A.B. (Rice) Asst., Rice Inst., Houston, Tex. *2509 McClendon St., Houston 5, Tex.*

WHITEHEAD, G. S., A.M. (Oxford) *Bay Pines, Florida*

WHITEMAN, A. L., Ph.D. (Pennsylvania) Mathematician, Office of Chief of Naval Operations, Washington, D.C. *2514–14th St. N.W., Apt. 606, Washington 9, D.C.*

WHITFORD, Prof. A. E., A.M. (Wisconsin) Dean., Coll. of Lib. Arts, Alfred Univ., Alfred, N.Y.

WHITFORD, Asst. Prof. D. E., Ed.M. (Harvard) Poly. Inst. of Brooklyn, Brooklyn, N.Y. *85–99 Livingston St.*

WHITING, MABEL G., A.M. (Oberlin) Head of Dept., Jr. Coll., Santa Ana, Calif. *506 E. Chestnut Ave.*
WHITMAN, Asso. Prof. E. A., A.M. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *521 Locust St., Pittsburgh 18, Pa.*
WHITMAN, Asst. Prof. P. M., Ph.D. (Harvard) Tufts Coll., Medford 55, Mass. *Robinson Hall*
WHITMORE, Asst. Prof. R. M., M.A. (Texas) Southwestern Univ., Georgetown, Tex. *Box 15, S.U. Sta.*
WHITNEY, D. R., A.M. (Princeton) Grad. Student, Ohio State Univ., Columbus, Ohio, *39 Tibet Rd., Columbus 2, Ohio*
WHITTED, Prof. J. A., A.M. (Southwestern) Emeritus, Ohio Northern Univ., Ada, Ohio *75 Forest Ave., Delaware, Ohio*
WHITTEMORE, Asso. Prof. J. K., A.M. (Harvard) Emeritus, Yale Univ., New Haven, Conn.
WHYBURN, Prof. G. T., Ph.D. (Texas) Chm. of Dept., Univ. of Virginia, Charlottesville, Va. *Colonnade Club*
WHYBURN, W. M., Ph.D. (Texas) President, Texas Tech. Coll., Lubbock, Tex.
WICKER, Prof. B. R., A.M. (Nebraska) Head of Dept., Loyola Univ. of Los Angeles, Los Angeles, Calif.
WIDDER, Prof. D. V., Ph.D. (Harvard) Chm. of Dept., Harvard Univ., Cambridge, Mass. *68 Snake Hill Road, Belmont 78, Mass.*
WIGGIN, Prof. EVELYN P., Ph.D. (Chicago) Randolph-Macon Woman's Coll., Lynchburg, Va.
WILCOX, Asso. Prof. L. R., Ph.D. (Chicago) Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill.
WILCZEWSKI, Rev. JOSEPH, Ph.D. (St. Louis) Prof., Marquette Univ., Milwaukee, Wis. *1131 Wisconsin Ave.*
WILDER, Prof. C. E., Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H.
WILDER, Prof. R. L., Ph.D. (Texas) Univ. of Michigan, Ann Arbor, Mich. *1617 Cambridge Road*
WILDERMUTH, Prof. R. B., A.M. (Ohio State) Capital Univ., Columbus, Ohio *2285 E. Mound St.*
WILEY, Prof. F. B., Ph.D. (Chicago) Head of Dept., Denison Univ., Granville, Ohio *Box 476*
WILKINS, J. E., JR., Ph.D. (Chicago) Mathematician, American Optical Co., Box A, Buffalo 15, N.Y.
WILKINS, Prof. P. D., M.S. (Case) Bates Coll., Lewiston, Maine *420 College St.*
WILKS, Prof. S. S., Ph.D. (Iowa) Member of Advisory Comm., N.D.R.C.; Princeton Univ., Princeton, N.J. *Fine Hall*
WILLEY, MAUD, A.M. (Mills) *Box 3058, T.S.C.W., Denton, Tex.*
WILLIAMS, Asst. Prof. A. R., Ph.D. (California) Univ. of California, Berkeley, Calif. *455 Wheeler Hall*
WILLIAMS, Mrs. BERYL W., A.M. (Maine) Instr., Agric. and Tech. Coll., Greensboro, N.C. *Box 356 A. and T. Coll.*
WILLIAMS, Asst. Prof. ERNEST, Ph.D. (Michigan) Alabama Poly. Inst., Auburn, Ala. *452½ Magnolia Ave.*
WILLIAMS, F. G., Ph.D. (Cornell) Headmaster, Atlantic Air Acad., Rye Beach, N.H.
WILLIAMS, Prof. G. A., A.M. (California) Oregon State Coll., Corvallis, Ore. *306 N. 32nd St.*
WILLIAMS, G. T., Brookhaven National Lab., Upton, N.Y.
WILLIAMS, Prof. K. P., Ph.D. (Princeton) Indiana Univ., Bloomington, Ind.
WILLIAMS, Asst. Prof. MARY E., A.M. (Kentucky) Skidmore Coll., Saratoga Springs, N.Y. *Dept. of Math.*
WILLIAMS, Prof. W. L., Ph.D. (Chicago) Head of Dept., Univ. of South Carolina, Columbia, S.C.
WILLIAMSON, Prof. C. O., Ph.D. (Chicago) Acting Head of Dept., Coll. of Wooster, Wooster, Ohio *1141 Beall Ave.*
WILLIAMSON, Asso. Prof. JOHN, Ph.D. (Chicago) Queens Coll., Flushing, N.Y.
WILSON, Prof. A. H., Ph.D. (Chicago) Emeritus, Haverford Coll., Haverford, Pa.
WILSON, C. C., B.Ed. (Chicago T. C.) Instr., Univ. of Ill., Chicago 11, Ill. *758 Gardner Rd., Westchester, Ill.*
WILSON, Prof. E. B., Ph.D. (Yale) Emeritus, Vital Statistics, Harvard Univ., Cambridge, Mass., *55 Shattuck St., Boston 15, Mass.*
WILSON, G. H., Ph.D. (Pennsylvania) Instr., Physics, Univ. of Pennsylvania, Philadelphia, Pa. *N. Rockland Road, Merion, Pa.*
WILSON, HAZEL S. (Mrs. L. T.), Ph.D. (Cornell) Chm. of Dept., High School, Annapolis, Md. *20 Thompson St.*
WILSON, MARY S., B.S. (Georgia) Grad. Student, Univ. of Georgia, Athens, Ga. *112 W. Adair St., Valdosta, Ga.*

WILSON, Asst. Prof. R. H., JR. Ph.D. (Pennsylvania) Temple Univ., Philadelphia, Pa., *437 Wellesley Rd., Philadelphia 19, Pa.*

WILSON, R. L., A.M. (Wisconsin) Instr., Univ. of Tennessee, Knoxville, Tenn., *Dept. of Math.*

WILSON, Prof. W. A., Ph.D. (Yale) Yale Univ., New Haven, Conn. *1960 Chapel St.*

WILSON, W. E., Ph.D. (Iowa) Director of Research, Sundstrand Machine Tool Co., Rockford, Ill. *2112 Harlem Blvd.*

WILSON, W. H., Ph.D. (Illinois) Asso. Dean, Coll. of Arts and Sci., Univ. of Florida, Gainesville, Fla. *Box 2227 University Sta.*

WINCHESTER, Mrs. REBECCA W., A.B. (Greensboro Coll.) *204 S. Main St., Kannapolis, N.C.*

WING, G. M., A.B. (Rochester) *4819 Regent St., Philadelphia 43 Pa.*

WINGER, Prof. R. M., Ph.D. (Johns Hopkins) Univ. of Washington, Seattle 6, Wash.

WINN, W. F. *305 W. 15th St., Ada, Okla.*

WINSTON, CLEMENT, Ph.D. (Pennsylvania) Principal Industrial Economist, War Production Board, Washington, D.C. *1420 Tuckerman St., Washington 11, D.C.*

WIRSCHING, FLORENCE A., A.B. (N. J. St. T. C.) Instr., Purdue Univ., West Lafayette, Ind. *Dept. of Math.*

WIRTH, Asst. Prof. H. P., Ph.D. (New York Univ.) Coll. of the City of New York, Convent Ave. and 139 St., New York, N.Y. *Dept of Math.*

WOLF, Asst. Prof. FRANTISEK, Ph.D. (Masaryk) Univ. of California, Berkeley 4, Calif. *Box 49W*

WOLF, Asst. Prof. LOUISE A., Ph.D. (Wisconsin) Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *3700 S. 116th St., Milwaukee 14, Wis.*

WOLFE, ALBERTA, M.S. (Iowa State) Instr., Miami Univ., Oxford, Ohio *P.O. Box 93*

WOLFE, C. H., A.M. (Wesleyan, Ohio) Supt., Danbury High School, Lakeside, Ohio *P.O. Box 373*

WOLFE, Asso. Prof. H. E., Ph.D. (Indiana) Indiana Univ., Bloomington Ind. *812 S. Fess Ave.*

WOLFE, Asst. Prof. J. M., Ph.D. (New York Univ.) Brooklyn Coll., Bedford Ave. and Ave. H, Brooklyn 10, N.Y.

WOLFE, R. S., A.M. (Washington) Instr., Northwestern Univ., Evanston, Ill. *Dept. of Math.*

WOLLAN, Asst. Prof. G. N., M.S. (Iowa) Sampson Coll., Sampson, N.Y.

WONG, Asso. Prof. Y. K., Ph.D. (Chicago) Lecturer, Univ. of North Carolina, Chapel Hill, N.C.

WOOD, FREDERICK, Ph.D. (Wisconsin) Dean, Coll. of Arts and Sci., Univ. of Nevada, Reno, Nev.

WOOD, Asso. Prof. F. E., Ph.D. (Chicago) Univ. of Oregon, Eugene, Ore. *Dept. of Math.*

WOOD, H. A., Ph.D. (Mass. Inst. of Tech.) Project Analytical Engr., Chance Vought Aircraft, Stratford, Conn. *1843 Elm St.*

WOODBRIDGE, MARGARET Y., Ph.D. (New York Univ.) Instr., Brooklyn Coll., Brooklyn, N.Y. *169 Columbia Heights, Brooklyn 2, N.Y.*

WOODS, C. L., M.S. (Cincinnati) Dean, Pacific Union Coll., Angwin, Calif. *Box 235*

WOODS, Prof. F. S., Ph.D. (Göttingen) Emeritus, Massachusetts Inst. of Tech., Cambridge, Mass. *123 Sumner St., Newton Center, Mass.*

WOODS, Asso. Prof. Roscoe, Ph.D. (Illinois) Univ. of Iowa, Iowa City, Iowa. *517 S. Lucas St.*

WOODSON, Prof. G. F., Jr., A.M. (Ohio State) Head of Dept., Johnson C. Smith Univ., Charlotte, N.C.

WOODWARD, Asst. Prof. H. E., A.M. (Texas Tech.) Texas Tech. Coll., Lubbock, Tex. *Tech. Station, Box 82.*

WOOLARD, E. W., Ph.D. (George Washington) Asst. Director Nautical Almanac, U.S. Naval Observatory, Washington, D.C. *1232-30th St. N.W., Washington 7, D.C.*

WORTHINGTON, Asst. Prof. Euphemia R., Ph.D. (Yale) U.C.L.A., Los Angeles 24, Calif.

WRAY, W. D., Ph.D. (Cornell) U.S. Navy Dept., Washington, D.C. *R.F.D. 3, Box 370, Alexandria, Va.*

WREN, Prof. F. L., Ph.D. (Chicago) George Peabody Coll., Nashville 4, Tenn.

WRENCH, J. W., Ph.D. (Yale) Mathematician, Catholic Univ., Washington, D.C. *4711 Davenport St., N.W., Washington 16, D.C.*

WRESTLER, FERNA E., A.M. (Kansas) Instr., Univ. of Wichita, Wichita, Kans. *1704 N. Holyoke Ave.*

WRIGHT, Asst. Prof. Alice Kelsey, A.M. (Illinois) Southern Illinois Normal Univ., Carbondale, Ill. *804 W. Main St.*

WRIGHT, Prof. C. B., Ph.D. (Pittsburgh) Chm. of Dept., East Texas State Teachers Coll., Commerce, Tex. *1603 Bonham St.*

WRIGHT, Frances M., A.M. (Brown) Instr., Syracuse Univ., Triple Cities Coll., Endicott, N.Y. *9 Minden Ave., Binghamton, N.Y.*

WRIGHT, Prof. H. A., Ph.D. (New York Univ.) Head of Dept., Transylvania Coll., Lexington, Ky.
WRIGHT, Prof. H. N., Ph.D. (California) Pres., Coll. of City of New York, N.Y. *3900 Greystone Ave., New York 63, N.Y.*
WURSTER, MARIE, Ph.D. (Chicago) Instr., Temple Univ., Philadelphia, Pa. *6635 McCallum St., Philadelphia 19, Pa.*
WYCKOFF, J. F., A.M. (Yale) Conn. General Life Ins. Co., Research Div., Actuarial Dept., Hartford, Conn. *78 W. Cedar St., Newington 11, Conn.*
WYLIE, Prof. C. C., Ph.D. (Illinois) Astr., Univ. of Iowa, Iowa City, Iowa *Univ. Observatory*
WYLIE, Prof. C. R., JR., Ph.D. (Cornell) Chm. of Dept., Army Air Forces, Inst. of Tech., Wright Field, Dayton, Ohio. *119 James Drive, Osborn, Ohio*

YANNEY, Prof. B. F., Ph.D. (Chicago) Emeritus, Coll. of Wooster, Wooster, Ohio. *354 E. Bowman St.*
YANOSIK, Asso. Prof. G. A., C.E. (New York Univ.) New York Univ., New York, N.Y. *52 Greenvale Ave., Yonkers, N.Y.*
YARBROUGH, H. M., Ph.D. (Indiana) Head of Dept., Western Kentucky State Teachers Coll., Bowling Green, Ky. *Route No. 4.*
YATES, R. C., Ph.D. (Johns Hopkins) Lt. Col., A.U.S. U.S. Military Acad., West Point, N.Y.
YEAGER, E. N., M.S. (Notre Dame) Exec. Vice-Pres., Napoleon Products Co., Napoleon, Ohio. *323 W. Clinton St.*
YEATON, Prof. C. H., Ph.D. (Chicago) Oberlin Coll., Oberlin, Ohio *189 Forest St.*
YEATON, MARIE M. (Mrs. C. H.) Ph.D. (Chicago) *189 Forest St., Oberlin, Ohio*
YOOD, BERTRAM, M.S. (Calif. Inst. of Tech.) Instr., Cornell Univ., Ithaca, N.Y. *White Hall*
YOUNG, F. H., B.A. (Ore. S.T.C.) Instr., Oregon State Coll., Corvallis, Ore. *Route 1, Box 226*
YOUNG, G. J., M.S. (Chicago) Principal Physicist, Clinton Laboratories, Oak Ridge, Tenn.
YOUNG, Asso. Prof. J.W.A., Ph.D. (Clark) Emeritus, Univ. of Chicago, Chicago, Ill. *5422 Blackstone Ave., Chicago 15, Ill.*
YOUNG, Prof. Mabel M., Ph.D. (Johns Hopkins) Emeritus, Wellesley Coll., Wellesley, Mass. *6 Norfolk Terrace*
YOUNG, Asso. Prof. P. M., Ph.D. (Ohio State) Kansas State College, Manhattan, Kans.
YOUNGS, Asso. Prof. J.W.T., Ph.D. (Ohio State) Indiana Univ., Bloomington, Ind., *Swain Hall*
YOWELL, Prof. E. I., Ph.D. (Cincinnati) Emeritus, Observatory, Univ. of Cincinnati, Cincinnati, Ohio. *3127 Griest Ave. Cincinnati 8, Ohio*

ZAHLEN, JEAN-PIERRE, Student, Univ. of Nancy, Nancy, France; Dir., Matico-pedagogical periodical, Pythagore. *4 rue de l'acier, Differdange, Grand—Duche de Luxembourg, France*
ZANOLAR, Asso. Prof. A. J., M.S. (Catholic Univ.) St. Joseph's Coll., Collegeville, Ind.
ZANT, J. H., Ph.D. (Columbia) Asst. Head of Dept., Okmulgee Branch, Oklahoma A. and M. Coll., Okmulgee, Okla.
ZARISKI, Prof. OSCAR, Ph.D. (Rome) Harvard Univ., Cambridge 38, Mass., *12 Hunt Hall*
ZEIGLER, R. K., M.S. (Nevada) Grad. Asst., State Univ. of Iowa, Iowa City, Iowa. *117 Quonset Park*
ZELDIN, Asso. Prof. S. D., Ph.D. (Clark) Mass. Inst. of Tech., Cambridge, Mass. *Dept. of Math.*
ZELLER, SISTER M. CLAUDIA, Ph.D. (Michigan) Asst. Prof. Coll. of St. Francis, Joliet, Ill.
ZEMMER, J. L., JR., M.S. (Tulane) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *927 Lake Ct.*
ZERBE, H. M., M.S. (Pennsylvania State) Instr., Pennsylvania State Coll., Hazleton Undergrad. Center, Hazleton, Pa. *586 N. Vine St.*
ZIEROFF, SISTER GERTRUDE MARIE, M.S. (St. Louis) Prof. Marian Coll., Indianapolis, Ind. *R.R. 17, Box 56, Indianapolis 44, Ind.*
ZIMMERBERG, Asst. Prof. H. J., Ph.D. (Chicago) Rutgers Univ., New Brunswick, N.J. *Dept. of Math.*
ZIMMERMAN, B. C., A.M. (St. Louis) *Cayo, British Honduras, C.A.*
ZIMMERMAN, L. J., B.A. (Goshen) Instr., Goshen Coll., Goshen, Ind.
ZORN, Prof. M. A., Dr. res. mat. (Hamburg) Indiana Univ., Bloomington, Ind. *Swain Hall*
ZUBAY, E. A., A.M. (Minnesota) Instr., Drake Univ., Des Moines, Iowa. *Dept. of Math.*
ZUCKERMAN, Asst. Prof. H. S., Ph.D. (California) Univ. of Washington, Seattle 5, Wash.

Total membership, January 1, 1948..........................................2,913

# GEOGRAPHICAL DISTRIBUTION OF MEMBERS

(The name of a member was omitted from this list when his location was unknown)

## UNITED STATES

### Alabama

Auburn.
*Alabama Poly. Inst.* Bancroft, Doner, Fuller, Williams.
Birmingham. Hess, Locke, Moore, Rayl.
Florence. Culmer.
Montevallo. Jackson.
Montgomery, Green.
St. Bernard. Capesius.
Talladega. Brothers.
Troy. Collins.
Tuscaloosa. Grau.
University
*Univ. of Alabama.* Burton, Cathey, Hadnot, Hummel, Lewis, Mancill, Seebeck, Smith.

### Alaska

Nome. Beberman.

### Arizona

Flagstaff. Butchart, Lampland.
Phoenix. Leithold.
Tempe. Wexler.
Thatcher. Olpin.
Tucson.
*Univ. of Arizona.* Denton, Graesser, Leonard, Purcell, Webb.
Yuma. Johnson.

### Arkansas

Arkadelphia. Foster.
Clarksville. Levy.
Fayetteville. Adkisson, Eagle.
Magnolia. Wetzig.
Monticello. Garrett.

### California

Albany. de Regt.
Angwin. Woods.
Bell. Niersbach.
Berkeley. Danskin, Evans, Taussig.
*Univ. of California.* Bernstein, Chin, Cunningham, Harary, Hughes, Kelley, Lehmer, McDonald, Neustadter, Noble, Robinson, Seidenberg, Schaaf, Sperry, Stoneham, Sussman, Thompson, Wakerling, Williams, Wolf.
Claremont. Hamilton, Jaeger.
Culver City. Butter.
Davis.
*Univ. of California, Coll. of Agriculture.* Baker, Burdette, Fulton, Roessler, Rolfe.
Fort Mason. Quarles.
Fresno. Morris.
Fullerton. Ernsberger, Reynolds.
Glendale. Lausman.
Hayward. Rahn.
Hollywood. Bachmann, Campbell, Hand.
Inglewood. Randels.
Inyokern U.S.N.O.T.S. Crow, Lawrence, Ward.
La Jolla. McEwen.
La Mesa. Smith.
Long Beach. Black, McClellan.
Los Angeles. Alexander, Beenken, Campbell, Collier, Cothran, Duncan, Floris, Isaacs, Lawrence, McConnell, Olson, Perry, Pierce, Porges, Rex, Solloway, Steinberg, Swank, Wicker.
*Los Angeles City Coll.* Herrera, Hills, Horton, Kaelin, Thompson, Trigg, Urner.
*Univ. of California at Los Angeles.* Beckenbach, Bell, Daus, Edmundson, Glazier, Gourrich, Green, Hestenes, Hoel, Horton, Hunt, James, Justin, Mason, Peterson, Puckett, Ratner, Scheffe, Sherwood, Sokolnikoff, Sorgenfrey, Strutton, Taylor, Tunell, Valentine, White, Worthington.
*Univ. of Southern California.* Busemann, Clarke, Hurt, Hyers, Kelly, Robb, Snapper, Steed, Throckmorton.
Madera. Fuller.
Marysville. Miller.
Modesto. Osner.
Moffett Field. Heaslet.
Oakland. Eggett, Hesse, McClelland, Sumner.
Pasadena. Damsgard, Glenn, Lay, White.
*California Inst. of Tech.* Bell, Birchby, Bohnenblust, Dilworth, Karlin, Kimme, Lagerstrom, Michal, Van Buskirk, Ward, Wear.
Redlands. Albert.
Richmond. Fay, Good.
St. Mary's College. Dominic.
San Diego. Donaldson, Hickman, Klauber, Livingston, Mador, Rhodes, Sheehy.
San Francisco. Dernham, Frank, Graeber, Ivanoff, Sturges, Waider.
*San Francisco Junior Coll.* Bass, Hanson, McCarty, McKenzie, Skarstedt.
San Jose. Clarke.
*San Jose State Coll.* Bird, Jamison, Myers, Olds.
San Mateo. Francis, Walker.
San Rafael. Beckwith.
Santa Ana. Whiting.
Santa Monica, Adams Hastings, Lehman, Norden, Taylor.
Stanford University.
*Stanford University.* Bacon, Baez, Grad, Polya, Szegö.

STOCKTON. Corbin.
WHITTIER. Pyle.

CANAL ZONE

BALBOA. McNair.

COLORADO

BOULDER.
*Univ. of Colorado.* Barrick, Bock, Britton, Farnell, Holubar, Hutchinson, Kempner Kendall, Nelson, Snively, Walters.
COLORADO SPRINGS. Hansman, Leavens, Sisam.
DENVER. Duffner, Gorsline, Hoffman, Hurry.
*Univ. of Denver.* Carmichael, Gorrell, Gysland, Lewis, Noble, Recht.
FORT COLLINS.
*Colorado State Coll.* Clark, Guard, Hayward, MacDonald, Madison.
GOLDEN.
*Colorado School of Mines.* Carpenter, Cook, Everett Hebel.
GREELY. Mallory.
LORETTO. Cook.

CONNECTICUT

GLENBROOK. Fraser.
HARTFORD. Bronstein, Donchian, Elston, Keffer, Kelly.
*Trinity Coll.* Dadourian, Ogilvy, Theilheimer.
MIDDLETOWN.
*Wesleyan Univ.* Arnold, Camp, Howland.
MILFORD. Rosenbaum.
NEW BRITAIN.
*Teachers Coll. of Connecticut.* Fuller, Spooner, Weeber.
NEW HAVEN
*Yale University.* Bateman, Begle, Bornmann, Davis, Dunford, Hille, Kovarik, Longley, Miles, Montgomery, Norwood, Ore, Tracey, Uhler, Whittemore, Wilson.
NEW LONDON. Bower, Dimick.
NICHOLS. Turner.
NORWALK. MacLane.
SIMSBURY. Howard.
STORRS. Cheney.
SRATFORD.
*Chance Vought Aircraft.* Jonah Loring, Wood.
WATERBURY. Siller.
WATERTOWN. Gillette.
WEST HAVEN. Anderton.

DELAWARE

NEWARK.
*Univ. of Delaware.* Botts, Jones, McDougle, Rees, Webber.

DISTRICT OF COLUMBIA

WASHINGTON. Anderson, Berry, Blake, Blanche, Branson, Bumer, Burington, Campaigne, Clark, Cromwell, Curtiss, Dribin, Engstrom, Federico, Fisher, Fox, Gabrielle, Getchell, Goldberg, Greville, Harman, Hyman, E. Kaplan, S. Kaplan, Kinsman, Kopp, Lennahan, Lloyd, Manson, McCamman, Mertie, Moulton, Nowlan, O'Brien, Penney, Quinn, Rasor, Sasuly, Schell, Schult, Sewell, Sharp, Shenton, Smith, Sohon, Thornton, Tompkins, Van Orstrand, Varnhorn, Watts, Weyl, Whiteman, Winston, Woolard, Wray.
*Catholic Univ. of America.* Finan, Landry, Ramler, Rice, Wrench.
*George Washington Univ.* Johnson, Johnston, Kullback, Mears, Taylor, Weida.
*U.S. Bureau of Census.* Daly, Lucas, Spencer, White.
*U.S. Coast & Geodetic Survey.* Adams, Darling, Duerksen, Lambert, Orlin, Schmid, Sollins, Thomas.
*U.S. Naval Research Lab.* Burns, Rees, Shepherd, Spitz.

FLORIDA

BAY PINES. Whitehead.
CORAL GABLES.
*Univ. of Miami.* Clement, MacNeish, Rutland, Swingle.
DELRAY BEACH. Hoffman, LeStourgeon.
EAST PALATKA. Hundertmark.
GAINESVILLE.
*Univ. of Florida.* Blake, Cowan, Davis, Dostal, Gager, Gormsen, Kokomoor, Lang, Mason, McInnis, Meyer, Owens, Phipps, Pirenian, Simpson, Stovall, Wilson.
JACKSONVILLE. Armstrong.
JACKSONVILLE BEACH. Harrje.
LAKELAND. Reinsch.
MIAMI. Ablow, Boyle.
ST. PETERSBURG. Baird, Burt, Scott.
TALLAHASSEE. Clarke, Tinner, Trimble.
*State Coll. for Women.* Larson, Smith, Wade.
TAMPA. Rhodes.
WINTER PARK. Jones, Sauté.

GEORGIA

ATHENS.
*Univ. of Georgia.* Barrow, Beckwith, Bryan, Callaway, Cohen, Fort, Hill, Huff, Levit, Park, Parker, Stephens, Ward, Wilson.
ATLANTA. Neisius.
*Georgia School of Tech.* Bailey, Field, Fulmer, Hefner, Holton, Hook, Martin, Perlin, Smith, Starrett.
COLLEGEBORO. Moye.
DAHLONEGA. Barnes.
DECATUR. Gaylord, Robinson.
EMORY UNIVERSITY. Latimer, Messick.
FORT VALLEY. Pitts.
LA GRANGE. Bailey.
MACON. Bruce, Plymale, Thomas.
MARIETTA. Howe.
MILLEDGEVILLE. Nelson.
ROME. Hightower.
VALDOSTA. Moore, Spragens.

HAWAII

HONOLULU. Tom.

IDAHO

BOISE. Buck.
CALDWELL. Rankin.

ILLINOIS

AURORA Miksa.
BLOOMINGTON. Hunt.
BLUFFS. Carter.
CARBONDALE.
*Southern Illinois Univ.* McDaniel, Purdy, Rodabaugh, Wright.
CARTHAGE. Boatman.
CHAMPAIGN. Fry.
CHARLESTON. Hendrix, Taylor.
CHICAGO, Andris, Anthony, Badger, Ballinger, Bay, Boardman, Brady, Campbell, Christman, Esposito, Ettinger, Finch, Georges, Gerst, Gibney, Greer, Hadley, Herlihy, Holland, Kaufman, Koch, Kurzin, Liolios, London, Mansfield, Mary Canisia, Mary Esther, Monell, Moran, Nagle, Nicolet, Pohley, Poppen, Schweitzer, Specht, Werkman.
*Chicago City Coll.* Buelow, Georges, Kurzin, Moran.
*DePaul Univ.* D'Arco, Fischer, Svoboda.
*Illinois Inst. of Tech.* Alberti, Arenson, Ballard, Barten, Bibb, Bradfield, Comfort, DeCicco, Eulenberg, Ford, Hazleton, Higgins, Krathwohl, Levy, McDowell, Menger, Miller, Oldenburger, Pall, Peterhans, Pounder, Rapoport, Rapp, Reingold, Rosenberg, Sadowsky, Smith, Soglin, Wilcox.
*Roosevelt Coll.* Gore, Johnson, Kingsley, Silber, Street.
*Univ. of Chicago.* Albert, Barnard, Bartky, Bliss, Dickson, Everett, Flanders, Gaffney, Graves, Hartung, Kaplansky, Lane, Leech, Loch, Logsdon, Lunn, MacLane, Meyer, Northrop, Ogawa, Putnam, Schilling, Selitzky, Smith, Stone, Young.
*Univ. of Illinois.* Bailey, Berglund, Corliss, Croft, Davis, Feinstein, Grenard, Hornacek, Lariviere, Latham, Levin, Levitt, Moureau, Nolan, Olsen, Ondrak, Schwartz, Sears, Stelling, Wilson.
*Wilson Jr. Coll.* Feltges, Kinney, Lange, Rasmussen, Rosenbeck, Sachs.
CICERO. Richards.
DECATUR. Coulter.
*J. Millikin Univ.* B. K. Brown, F. R. Brown, Kiefer, Ploenges.
DE KALB. Hellmich, Stelford.
DIXON. West.
ELGIN. Peters.
ELMHURST. Baumgart.
ERIE. Smith.
EUREKA. Newson.
EVANSTON. Christian.
*Northwestern Univ.* Bloom, Buell, Curtiss, Hellinger, Hildebrandt, Kliphardt, Moulton, Lindstrum, Lonseth, Reid, Scott, Simmons, Smiley, Wescott, Wolfe.
FREEPORT. Baumgartner.
GALESBURG. Lafferty, Snader.
*Knox Coll.* Clare, Heren, Smyth, Stephens.
JACKSONVILLE. Hallerberg, Miller.
JOLIET. Zeller.
LAKE FOREST. Curtis.
LEBANON. Stowell.
LINCOLN. Balof.
MACOMB.
*Western Illinois State Teachers Coll.* Ayre, Ginnings, Schreiber.
MONMOUTH. Beveridge, Cramer.
NAPERVILLE. Seybold.
NORMAL.
*Illinois State Normal Univ.* Atkin, Bey, Flagg, McCormick, Mills, Rhine, Ullsvik.
PAXTON. Schwartz.
PEORIA.
*Bradley Univ.* Comstock, Gault, Moore.
RIVER FOREST. Dobbin.
ROCKFORD. Varnum, Wilson.
ROCK ISLAND. Cederberg, Olmsted.
SANDWICH. Rumney.
TAYLORVILLE. Dappert.
URBANA.
*Univ. of Illinois.* Armstrong, Carmichael, Chanler, Coble, Frank, Hartley, Hattan, Hoersch, Ketchum, Landin, Levy, Meserve, Miles, Miller, Moore, Peters, Scott, Vaughan, Walkley.
WHEATON. Boyce, Martin.
WILMETTE. Pancoe.
WINNETKA. Humphrey.

INDIANA

BLOOMINGTON.
*Univ. of Indiana.* Graves, Gustin, Peak, Rothrock, Thomas, Williams, Wolfe, Youngs, Zorn.
COLLEGEVILLE. Zanolar.
CRAWFORDSVILLE. Carscallen, Polley.
EARLHAM. Long.
EAST CHICAGO. Burns.
EVANSVILLE. Kronsbein.
FORT WAYNE. Olson.
GARY. Copp, Oursler.
GOSHEN. Hartzler, Zimmerman.
GREENCASTLE.
*DePauw Univ.* Arnold, Edington, Greenleaf.
HAMMOND. Groves.
HOLY CROSS. Edward.
INDIANAPOLIS, Beal, Crull, Heyda, McColgin, Welchons, Zieroff.
LAFAYETTE.
*Purdue Univ.* Ayres, Black, Bolks, Burr, Byrne, Crain, Derflinger, Erickson, Golomb, Gould, Graves, Hazard, Hodge, Holtom, Hughes, Jonah, Keller, Klinger, Leone, Miller, Robbins, Rosenthal, Shanks, Stone, Strand, Sturm, Walker, Webster, Wirsching.
MARION. Porter.
MUNCIE. Edwards, Shively.
NORTH MANCHESTER. Dotterer.

NOTRE DAME.
*Univ. of Notre Dame.* Caparó, LaSalle, Nastucoff, Pepper, Ross.
REYNOLDS. Erwin.
TERRE HAUTE. Martin, Shriner, Sousley.
UPLAND. Draper.
WEST BADEN SPRINGS. Hausmann. Muehlmann

IOWA

AMES. M. M. McKelvey.
*Iowa State Coll.* Anderson, Beach, Brandner, Daniells, Davis, Gaskell, Gouwens, Herr, Hinrischsen, Holl, Horvitz, Kreider, Lambert, McKelvey, Robertson, Sealander, Smith, Thielman.
CEDAR FALLS. Van Engen.
CEDAR RAPIDS. Coffin, Swanson.
DAVENPORT. Hratz.
DES MOINES.
*Drake Univ.* Canfield, Gillam, Harper, Neff, Reiber, Zubay.
DUBUQUE. Burns, Ernsdorff, Rothlisberger.
EPWORTH. Earhart.
ESTHERVILLE. Hammer.
FAYETTE. Deming.
GRINNELL. McLenon, Rusk.
IOWA CITY. Price.
*Univ. of Iowa.* Chittenden, Conkwright, Cosby, Craig, Forman, Knowler, Oberg, Woods, Wylie, Zeigler.
LAMONI. Jacobsen.
MOUNT VERNON.
*Cornell Coll.* Davis, McGaw, Moots.
SIOUX CITY. Bushyager, Rochford.
STORM LAKE. Roorda.
WAVERLY. Lyche.

KANSAS

ATCHISON. Pretz.
*Mt. St. Scholastica Coll.* Obrist, Sullivan, Walsh.
BALDWIN. Garrett.
EMPORIA.
*State Teachers Coll.* Laird, Peterson, Tucker.
HAYS. Grabbe.
HESSTON. Driver.
LAWRENCE.
*University of Kansas.* Babcock, Black, Dougherty, Jordan, Kneale, Pihlblad, Price, Rasmussen, Schatten, Smith, Stouffer, Ulmer, Wheeler.
LINDSBORG. Marm.
MANHATTAN.
*Kansas State Coll.* Babcock, Chatelain, Furman, Greer, Hyde, Janes, Lewis, Mossman, Parker, Sanger, Sigley, Stratton, White, Young.
NORTH NEWTON. Ewy, Richert.
OTTAWA. Bemmels.
PITTSBURG.
*Kansas State Teachers Coll.* Curfman, Shirk, Smith.
SALINA. Arnoldy.
STERLING. Bell.
TOPEKA. Messick.
*Washburn Univ.* Breneman, Eberhart, Greene, Martinson.
WICHITA. Longenecker, Reagan.
*Univ. of Wichita.* Hanna, Hoare, Read, Reagan, Robbins, Wedel, Wrestler.
WINFIELD. Kruger.
XAVIER. Ann Elizabeth.

KENTUCKY

BEREA
*Berea Coll.* Hutcherson, Porter, Pugsley, Roberts.
BOWLING GREEN. Yarbrough.
DANVILLE. Robinson.
DAYTON. Blakeley.
GEORGETOWN. Hatfield.
LEXINGTON. Wright.
*Univ. of Kentucky.* Boyd, Brown, Downing, Jasper, Latimer, Pence, Rohde, South.
LOUISVILLE. Bullitt, Fields, Ford, Morrison, Schaeffer.
*Univ. of Louisville.* Moore, Simester, Stevenson.
MAPLE MOUNT. Sheeran.
MURRAY. Carman.
RICHMOND. Park.
WINCHESTER. Howard.

LOUISIANA

BATON ROUGE.
*Louisiana State Univ.* Currie, Freas, Karnes, Nichols, O'Quinn, Rees, Rickey, Rutt, Sanders, Smith, White.
HAMMOND.
*Southeastern La. Coll.* Davis, McClimans, Tucker.
LAFAYETTE.
*Southwestern La. Inst.* Buchanan, Guthrie, Loflin, Nolan.
LAKE CHARLES. Bradford.
NATCHITOCHES.
*Northwestern State Coll.* Gandy, Killen, Maddox.
NEW ORLEANS. Frankenbush, Singer, Stevens, Tullier.
*Sophie Newcomb Coll.* Beard, Humphreys, Many, Spencer, Weiss.
*Tulane.* Buchanan, Duren, Gilmore, Riess, Singer, Temple, Thomson, Wallace.
PINEVILLE. Temple.
RUSTON.
*La. Poly. Inst.* Garrison, Gentry, Schroeder
SHREVEPORT.
*Centenary Coll.* Carlton, Griffith, Hardin.

MAINE

BATH. Brown.
BRUNSWICK
*Bowdoin Coll.* Christie, Hammond, Holmes, Korgen.

LEWISTON. Ramsdell, Wilkins.
ORONO.
*Univ. of Maine.* Hohn, Kimball, Lapidus.
WATERVILLE. Ashcraft.

MARYLAND

ABERDEEN PROVING GROUND. Coleman, Dederick, Dimsdale, Giese, Golub, Hart, Lotkin, Madrill, Norden, Reklis.
ANNAPOLIS. Bingley, Stilwell, Wilson.
*U.S. Naval Acad.* Ayres, Bailey, Ball, Bleick, Buikstra, Clements, Currier, Dillingham, Gorman, Gras, Hammond, Kells, Lamb, Lyle, Mayer, McLaughlin, Milkman, Milos, Moore, Paydon, Popow, Rector, Scarborough, Swafford, Thomas, Tierney, White.
*Naval Postgrad. School.* Church, Denbow, Jennings, Mewborn, Rawlins, Root, Torrance.
BALTIMORE. Cook, Karl, Roman, Smith, Stephens.
*Goucher Coll.* Lewis, Torrey.
*Johns Hopkins Univ.* Bourne, Clifford, Cohen, Edmonson, Haviland, Kennedy, Light, Mehr, Morrill, Murnaghan, Reed, Sigley, Smith, Templeton.
BRENTWOOD. Meade.
CARDAROCK. Wehausen.
CHEVERLY. Huck.
CHEVY CHASE. Cramer.
COLLEGE PARK.
*Univ. of Maryland.* Good, Hall, Lancaster, Lewis, Martin, Mitchell.
EMMITSBURG. Burke, Klos.
FREDERICK. Brown, Maloney.
FROSTBURG. Hallett.
STEVENSON. Morrel.
WESTMINSTER. Spicer.
WOODSTOCK. Hennessey.

MASSACHUSETTS

AMHERST. Boutelle, Graff.
ARLINGTON. Sobczyk.
BOSTON. Carnahan, Gould, Hemenway, Hoskins, Marie Laurentine, Miller, Weaver.
*Boston Univ.* Bruce, Johanson, Mode.
*Northeastern Univ.* Brown, Combellack, Spear, Wallace.
BROOKLINE. McCarthy.
CAMBRIDGE. Boas.
*Harvard Univ.* Ahlfors, Beatley, Birkhoff, Brown, Coolidge, Emmons, Gleason, Huntington, Kravetz, LeVeque, Mosteller, Newman, Rose, Rulon, Walsh, Widder, Wilson, Zariski.
*Massachusetts Inst. of Tech.* Douglass, Franklin, Harvey, Haskins, Martin, Moon, Reich, Reissner, Salem, Secada, Woods, Zeldin.
CHESTNUT HILL. Marcou, O'Donnell.
CHICOPEE. Madden.
FALL RIVER. Connors.
FITCHBURG. Bissinger.
FORT DEVANS. Hubbard.
GROTON. Nash.
LYNN. Taylor.
MEDFORD.
*Tufts Coll.* Mergendahl, Ransom, Whitman.
MILFORD. Dennison.
NEW BEDFORD. Robinson.
NORTHAMPTON. Johnson, McCoy, Munroe, Rambo.
NORTON.
*Wheaton Coll.* Garabedian, Nickerson, Watt.
PITTSFIELD. Washburne.
RICHMOND. Buchanan.
SOUTHBORO. Harrison.
SOUTHBRIDGE. Boeder.
SOUTH HADLEY. Bates, Litzinger.
SOUTH LANCASTER. Durham.
TYNGSBORO. Richmond.
WALTHAM. Smith.
WELLESLEY.
*Wellesley Coll.* Copeland, Merrill, Russell, Stark, Young.
WELLESLEY HILLS. Kehl.
WESTON. Burke, Lewis, Swords,
WILLIAMSTOWN.
*Williams Coll.* Agard, Miser, Wells.
WORCESTER. Burns, Johnson, McBrien.
*Clark Univ.* Melville, Patton, Wheeler.
*Worcester Poly Inst.* Brown, Cobb, Gay, Morley, Rice.

MICHIGAN

ALBION.
*Albion Coll.* Ingalls, Larsen, Sleight.
ANN ARBOR. Hamilton.
*Univ. of Michigan.* Anning, Bradshaw, Chacalos, Churchill, Coburn, Coe, Copeland, Cox, Craig, Crispin, Curtis, Dickinson, Dwyer, Faulkner, Field, Fischer, Ford, Hay, Hildebrandt, Hopkins, Jehn, Jones, Kaplan, Karpinski, Love, Mela, Myers, Nemerever, Nyswander, Opatowski, Piranian, Raiford, Rainich, Rainville, Reade, Rothe, Rouse, Running, Samelson, Schorling, Thrall, Tornheim, Wilder.
BERRIEN SPRINGS. Lashier.
DEARBORN. Maguire.
DETROIT. Bagby, Johnson, Paula.
*Univ. of Detroit.* Johnston, Markle, McCarthy, McGrail, Mehlenbacher, Payne, Smith, Thompson.
*Wayne Univ.* Baldwin, Borgman, Folley, Loweke, Morrow, Nelson, Pixley, Scibiorski, Southard.
EAST GRAND RAPIDS. Bellardo.
EAST LANSING.
*Michigan State Coll.* Barbour, Bell, Black, Carr, Frame, Grove, Herzog, Hill, Nordhaus, Plant, Powell, Speeker, Stelson, Stewart, Wells.
FLINT.
*General Motors Inst.* DeMoss, Grotts, Raker, Straw,
HART, Burdick.
HOLLAND, Lampen.

HOUGHTON. Park, Stipe.
IRONWOOD. Field.
KALAMAZOO. Myers, Walton.
*Western Michigan Coll.* Bartoo, Beeler, Blair, Butler, Cain, Everett.
KALKASKA. Dunlap.
MARQUETTE. Spooner.
MILFORD. McNeal.
MT. PLEASANT.
*Central Michigan Coll.* Brown, Foust, Richtmeyer, Sudborough.
YPSILANTI. Frankel.
*Michigan State Normal Coll.* Erikson, Lindquist, Pate.

MINNESOTA

COLERAINE. Kearney.
DULUTH. Cothran, Lockwood, McEwen, Mercedes, Morin.
MINNEAPOLIS.
*Univ. of Minnesota.* Amundson, Bearman, Brink, Brooke, Bussey, Cameron, Carlson, Colson, Eggers, Fischer, Gibbens, Hall, Hart, Hartig, Hatfield, Jensen, Johnson, Johnston, Kalisch, Kirchner, Koehler, Loud, McCutcheon, McEwen, Munro, Ness, Ohnsorg, Olmsted, Priester, Quaid, Shumway, Thorp, Turritin, Wang.
MOORHEAD. Andersen, Mundhjeld, Smith.
NORTHFIELD. Carlson.
*Carleton Coll.* Beasley, Gingrich, May, Wegner.
ST. JOSEPH. Muggli, Scoblic.
ST. PAUL. Boehm, Bracewell, Brown, Camp, Gibbons, Hill, Morgan, Thornton.
*Coll. of St. Thomas.* Bush, Godderz, Montgomery, Norris, Sheridan, Smith, Speltz, Taylor, Terami.
ST. PETER. Nelson, Swanson.
VIRGINIA. Hancock.
WINONA. Lasalle, Gregory, Kloyde, Schulte

MISSISSIPPI

BLUE MOUNTAIN. Gillis.
COLUMBUS. Erickson.
HATTIESBURG. Foote, Johnson.
JACKSON. Babbitt, McCoy, Mitchell, Warren.
PASS CHRISTIAN. Keyes.
STATE COLLEGE
*Mississippi State Coll.* Goen, Grimes, Hopkins, Murray, Ollivier, Pettis.
UNIVERSITY. Bickerstaff, Hume.

MISSOURI

CAPE GIRARDEAU. Michel.
CLAYTON. Rosskopf.
COLUMBIA. Cosby.
*Univ. of Missouri.* Blumenthal, Ewing, Haynes, Kelly, Koken, Wahlin.
FAYETTE. Denny, Helton.
FULTON. Lacy.
HANNIBAL. Moore.
JEFFERSON CITY. Jason.
KANSAS CITY. Cutting, Doyle, Lackay, Pierson, Rosen.
LIBERTY. Jones.
ROLLA.
*Missouri School of Mines.* Carpenter, Erkiletian, Goodhue, Johnson, Rankin.
ST. CHARLES. Karr.
ST. LOUIS. Bickley, Gove, Lewis, Marth, Proctor, Thomas.
*St. Louis Univ.* Andrews, Case, Regan, Richardson, Vezeau.
*Washington Univ.* Dunkel, Gottlieb, Leighton, Mathews, Middlemiss, Rider, Roever, Van Schaack.
SPRINGFIELD. Graves, H'Doubler.
VANDALIA. Schwetye.
WARRENSBURG. Brown, Hadley.
WEBSTER GROVES. Clarke.

MONTANA

BOZEMAN. Hurst.
BUTTE. Smith.
GARRISON. Canning.
HELENA. Topel.
MISSOULA. Carey, Merrill.

NEBRASKA

CHADRON. Berry.
CRETE. Johnson.
FT. CROOK. Westbrook.
HASTINGS. Hadlock, McDill.
LINCOLN. Howie, Ogden, Perisho.
*Univ. of Nebraska.* Basoco, Brenke, Camp, Cox, Runge.
OMAHA. Becker, Bettinger, Clarkson, Earl, Rice.
PERU. Huck.
WAYNE. Boyce.
YORK. Feemster.

NEVADA

RENO. Beesley, Wood.

NEW HAMPSHIRE

CONCORD. Conwell.
DURHAM. Slobin, Stubbe.
EXETER. Adkins, Funkhouser, Lynch, Pennell.
HANOVER.
*Dartmouth Coll.* Brown, Doyle, Durfee, Forsyth, Mathewson, Morgan, Nordstrom, Perkins, Robinson, Silverman, Wilder.
KEENE. Goodrich.
MANCHESTER. O'Leary.
PLYMOUTH. Smith.
RYE BEACH. Williams.

NEW JERSEY

BLOOMFIELD. Oergel.
CALDWELL. Anita.
CLIFTON. Struyk.
CONVENT STATION. Kenna.
EAST ORANGE. LePori, Nordgaard.
HIGHLAND PARK. Hamilton.

HIGHTSTOWN. Litterick.
HOBOKEN. Murray.
JERSEY CITY. Kruse, Reckzeh.
LAKEWOOD. Wallick.
LAWRENCEVILLE. Kimball, Mikesh.
MADISON. Battin.
MAPLEWOOD. Hazeltine.
MURRAY HILL. Hamming, Shewart.
NEWARK. Mosesson, Strock.
*Newark Coll., Rutgers Univ.* Ammerman, Henry, McCarthy, Sherak.
*Newark Coll. of Engg.* Celaura, Jaffe, Molina, Solomon, Vedova, Wasson.
NEW BRUNSWICK.
*Rutgers Univ.* Barlaz, Biser, Bunyan, Firestone, Galbraith, Goodman, Grant, Hazard, LeLeiko, Meder, Morris, Nelson, Ott, Phelps, Starke, Walter, Zimmerberg.
PATERSON. Daugherty.
PRINCETON. Houghton.
*Inst. for Advanced Study.* Alexander, Dolciani, Eberlein, Gelbart, Goldstine, Gottschalk, Halmos, Morse, Veblen, von Neumann.
*Princeton Univ.* Adams, Artin, Lefschetz, Michael, Mills, Rauch, Treiber, Tucker, Tukey, Wedderburn, Wilks.
RIDGEWOOD. Muller.
RIVER EDGE. Coleman.
SOUTH ORANGE. Davis, Stanwick.
TEANECK. Rayher.
TRENTON. Levine, Shuster.
UPPER MONTCLAIR. Campbell
*State Teachers Coll.* Clifford, Davis, Fehr, Mallory.
WEST ORANGE. Edison.

NEW MEXICO

ALBUQUERQUE. Bauer, Mathany, Rogers.
*Univ. of New Mexico.* Boldyreff, Hildner, LaPaz, Street.
LAS VEGAS. Roberts, Rodgers.
ROSWELL. Harp.
SANTA FE. Cyprian Luke.
SOCORRO. Reece.
STATE COLLEGE.
*New Mexico Coll. of A. and M. A.* Branson Heinzman, Walden, Wells, Westhafer.

NEW YORK

ALBANY. Noel Marie.
*New York State Coll. for Teachers.* Beaver, Birchenough, Lester, Stokes.
ALFRED.
*Alfred Univ.* Nevins, Polan, Rhodes, Seidlin, Whitford.
AURORA. Hollcroft. Rusk.
BROOKLYN. Appuhn, Berkofsky, Byrne, Charosh, Chernofsky, Epsell, Finkel, First, Francis Xavier, Gerst, Kalish, Karnow, Klevan, Kramer-Lassar, Lavoie, Lazar, Leeds, Lieber, Lonner, Maddaus, Miller, Mitchell, Palladino, Ross, Rush, Salkind, Sarno, Shapi Tolle, Waite, Wallach.
*Brooklyn Coll.* Baten, Borofsky, Boy Fleisher, Forman, Griffin, Johnsc Karlin, Kennison, Kieval, Lande Maria, Moore, Prenowitz, Richardsc Singer, Smith, Wolfe, Woodbridge.
*Poly. Inst. of Brooklyn.* Berry, Forra Foster, Hutchinson, Lowe, Whitford.
*Pratt Inst.* Cowles, Helme, Moore, Nc man, Thompson.
BUFFALO. Browne, Maloney, Podmel Walker, Wilkins.
*Univ. of Buffalo.* Gehman, Montagu Montgomery, Noller, Pound, Schneckeı burger, Warner, Welmers.
CANTON. Peters.
CLINTON.
*Hamilton Coll.* Brown, Ferry, Gere, Pat terson.
ELMIRA. Suffa.
ENDICOTT. Dubisch, Wright.
FAR ROCKAWAY. Moore.
FLUSHING. Bakst.
*Queens Coll.* Archibald, Brown, Cope Dean, Eaton, Feld, Raudenbush, Sard Sullivan, Williamson.
FOREST HILLS. Hertzig.
GARDEN CITY. Bowden, Clark.
GARRISON. Davis.
GENEVA. Durfee. Hubbs.
GREAT NECK. Schultz.
HAMILTON.
*Colgate Univ.* Aude, Downie, Munshower, Wardwell.
HEMPSTEAD.
*Hofstra Coll.* Charlesworth, Hawthorne, Hove, Ollmann, Rohr, Stabler.
HOUGHTON. Davison, Luckey.
ITHACA.
*Cornell Univ.* Agnew, Beinert, Carver, Feller, Flexner, Gunder, Holzinger, Hurwitz, Jones, Kac, Karapetoff, Laush, Lee, Luippold, Pollard, Rosser, Rubashkin, Snyder, Tuckerman, Walker, Yood.
JAMAICA. Jordan.
KINGS POINT. Nickl.
LONG ISLAND CITY. Peters.
LOUDONVILLE.
*Siena Coll.* Hanhauser, Kuhn, Schocken.
NEW LEBANON. Pflaum.
NEW ROCHELLE. Kiely.
NEW YORK. Alfieri, Arnold, Berger, Bernard Alfred, Boehm, Braverman, Burgess, Conlan, Crane, Croci, Darraugh, D'Atri, Ginsburg, Gray, Grossman, Heath, Hlavaty, Hobbs, Jablonower, Joffe, Katz, Keeler, Kruskal, Lehner, Levine, Mandel, Mayerson, McGrath, McKenna, McMahon, Mirick, Nehrbas, Peiser, Phillips, Quilty, Roll, Ruderman, Salerno, Schor, Schwartz, Silversten, Skelding, Steinhaus, Stuckey, Vitale, Wayne, Weaver, Weber.
*Bell Telephone Labs.* Clos, Fry, Gray,

Jones, MacColl, Mead, Riordan, Schelkunoff.
*Coll. of the City of New York.* Fagerstrom, Gill, Grove, Hubert, Hurwitz, Linehan, MacEwen, Mortola, Nathan, Post, Robinson, Schach, Singer, Turner, Weingarten, Wirth, Wright.
*Columbia Univ.* Aurora, Bolton, Dillon, Eilenberg, Fite, Gentzler, Goldman, Kasner, Lewis, Littauer, Mino, Mullins, Reeve, Ritt, Siceloff, Upton, Walker.
*Cooper Union.* Anderson, Eastham, Lehman, Levenson, Miller, Tanzola.
*Fordham Univ.* Kirby, Kubis, Oehler.
*Hunter Coll.* Anderson, Aroian, Bradley, Brock, J. Hobart, Bushey, J. H. Bushey, Cooper, Darkow, Eisele, Hill, Kutman, Landers, Rees, Simons, Tuller, Weisner, Whelan, White.
*New York Univ.* Adler, Bernardi, Cooley, Courant, Graham, F. W. John, Fritz John, Kline, Payne, Peters, Putnam, Reddick, Rehberg, Roth, Rubin, Schlauch, Shephard, Tilley, Wahlert, Yanosik.
NIAGARA FALLS. Small, Welmers.
NIAGARA UNIVERSITY. Banks.
NORTH CHILI. Smith.
ONEONTA. Callahan, Sanford.
PATCHOGUE. Schwartz.
POTSDAM.
*Clarkson Coll.* Buxton, Greene, Myatt, Waltz.
POUGHKEEPSIE.
*Vassar Coll.* Asprey, Baker, McDonald, Newton, Wells.
ROCHESTER. Chesna, Foard, Harding, Merrill.
*Univ. of Rochester.* Atkins, Bernstein, Betz, Gale, Gunderson, Klimczak, Long, Marchand, Seidel, Watkeys.
ST. ALBANS. Deutsch.
ST. BONAVENTURE. Scheier.
SAMPSON.
*Sampson Coll.* Albert, Casey, Caulum, Roth, Squires, Wollan.
SARATOGA SPRINGS. Williams.
SCHENECTADY. Poritsky.
*Union Coll.* Burkett, Fox, Morse, Snyder.
SPRINGVILLE. Harrington.
SYRACUSE.
*Syracuse Univ.* Cairns, Carroll, Cole, Decker, Harwood, Heilman, Loewner, Stokes, Taylor.
TROY.
*Rensselaer Poly. Inst.* Allen, Bartholomay, Biggerstaff, Campbell, Fraser, Jones, Nash, Nickol, Warniock.
UPTON. Williams.
UTICA. Bush, Kotler.
WEST POINT.
*U. S. Military Acad.* Bessell, Jones, Yates.
WHITE PLAINS. Mary Benedicta.
WYOMING. Hartnell.

## NORTH CAROLINA

CHAPEL HILL.
*Univ. of North Carolina.* Blyth, Bradley, Brauer, Browne, Cameron, Eaves, Garner, Henderson, Hickerson, Hill, Hsu, Lasley, Mackie, Morrow, Wong.
CHARLOTTE. Hoyle, Woodson.
DAVIDSON. McGavock, Mebane.
DURHAM.
*Duke Univ.* Clark, Dressel, Elliott, Gergen, Hickson, LaRoe, Miles, Patterson, Roberts, Thomas.
GREENSBORO. Carpenter, Jeffries, Ripandelli, Williams.
*Woman's Coll. of the Univ. of North Carolina.* Barton, Lewis, Pegram, Strong.
GREENVILLE.
*East Carolina Teachers Coll.* Graham, Reynolds, Scott.
HICKORY. Dodson.
HIGH POINT. Adams.
KANNAPOLIS. Winchester.
MARS HILL. Howell.
MOORESVILLE. Templeton.
RALEIGH. Downing.
*North Carolina State Coll.* Baker, Bullock, Cell, Harris, Nahikian, Strobel, Watson.
SALISBURY. Dearborn.
WILMINGTON. Peebles.
WILSON. Stark.

## NORTH DAKOTA

FARGO. Arena, Smith.
GRAND FORKS.
*Univ. of North Dakota.* Mason, McBride, Peterson, Rognlie, Staley, Westgate.
JAMESTOWN. Jackson.
VALLEY CITY. Patterson.

## OHIO

ADA. Whitted.
AKRON.
*Univ. of Akron.* Mauch, Ross, Selby.
ALLIANCE. Freese.
ATHENS.
*Univ. of Ohio.* Marquis, Reed, Starcher.
BEREA. Annear.
BOWLING GREEN.
*Bowling Green State Univ.* Krabill, Mathias, Overman.
CHILLICOTHE. Clinton.
CINCINNATI. Lukacs, Reilly, Rice, Stechschulte.
*Univ. of Cincinnati.* Barnett, Brand, Justice, Lipsich, Lubin, Merriman, Moore, Smith, Szász, Taylor, Wang, Yowell.
CLEVELAND. Burwell, Garvin, Joliat, Johnson, Musselman, Simon.
*Case School.* Brown, Focke, Green, McCuskey, Morris, Nassau, Rinehart, Thomas.
*Fenn Coll.* Brown, Kelly, Topp, Van Voorhis.

COLUMBUS. Byrd, Heinke, Spears, Wildermuth.
*Ohio State Univ.* Alden, Bareis, Beatty, Caris, Dressler, Fawcett, Helsel, Jones, Kuhn, Mickle, Millsaps, Morris, Parrish, Radó, Rankin, Rasor, Reichelderfer, Rickard, Toops, Whitney.
DAYTON. Fettis, Price, Schawalder.
*A. A. F. Inst. of Tech.* Carson, Downing, Gatewood, Wylie.
*Univ. of Dayton.* Cassel, Gephart, Hafner, Peckham, Schraut.
DEFIANCE. Godfrey, MacCullough.
DELAWARE. Crane, Rowland.
GAMBIER. Berg, MacNeille, Vandort.
GRANVILLE. Ladner, Wiley.
HIRAM. Clarke.
KENT. Lowenstein.
*Kent State Univ.* Brooks, Harshbarger, Jenkins, Johnson, Manchester, Oslon, Rogers.
LAKESIDE. Wolfe.
MARIETTA. Bennett, Sandt.
MT. ST. JOSEPH. Corona, Dimond.
NAPOLEON. Yeager.
NEW CONCORD. Knight.
NEW LEXINGTON. Hoops.
NORTH BALTIMORE. Blackall.
NORTH CANTON. Mummery.
OBERLIN. Yeaton.
*Oberlin Coll.* Cairns, Carr, Newsom, Randolph, Sinclair, Smyth, Vance, Wagner, Yeaton.
OXFORD. Tappan.
*Miama Univ.* Anderson, Pollard, Spenceley, Wolfe.
SPRINGFIELD. Krueger, Tripp.
TIFFIN. Menke.
TOLEDO. Brandeberry, Dancer, Koley, Mercedes.
WESTERVILLE. Glover.
WILBERFORCE. Toney.
WILMINGTON. Spinks.
WOOSTER.
*Coll. of Wooster.* Fobes, Knight, Williamson, Yanney.
YELLOW SPRINGS. Astrachan.

OKLAHOMA

ADA. Canada, Heimann, Winn.
ALVA. Huneke.
BARTLESVILLE. Rice.
CLAREMORE. Nemecek.
DURANT. Dwight.
NORMAN.
*University of Oklahoma.* Bernhart, Brixey, Court, Dragoo, Hassler, Huff, Huskey, Krattiger, LaFon, McFarland, McKnelly, Palmer, Pipes, Reaves, Smith, Spears, Springer.
OKLAHOMA CITY. Meador, Pirrong.
OKMULGEE. Zant.
SHAWNEE. Doerfler.
STILLWATER.
*Oklahoma A. & M. Coll.* Allen, Barnett, Caskey, Diamond, Flanders, Hamilton, Lewis, McDole, Mitchell, Smith.
TULSA. Argue, Doll, Duncan, Eisen, Ellis, Lewis.
*Univ. of Tulsa.* Carter, Shreve, Veatch. West.
WEATHERFORD. Linscheid.

OREGON

ADRIAN. Bunch.
CORVALLIS.
*Oregon State Coll.* Beaty, Clark, Eves, Hammer, Kirkham, Li, Milne, Poole, Saunders, Stone, Williams, Young.
EUGENE.
*Univ. of Oregon.* Civin, DeCou, Ghent, Gilbert, Moursund, Niven, Peterson, Scobert, Wood.
FOREST GROVE. Noble, Price.
McMINNVILLE. Ramsey.
PORTLAND. Frederickson, Merriss.
*Reed Coll.* Griffin, L. J. Rosenbaum, R. A. Rosenbaum.
SALEM. Luther.

PENNSYLVANIA

ALLENTOWN. Billig, Kunkel.
*Muhlenberg Coll.* Deck, Holt, Koehler, Nelson.
ANNVILLE. Black.
BEAVER FALLS. Cleland.
BETHLEHEM. Ashbaugh, Radar.
*Lehigh Univ.* Beer, Chellevold, Cowling, Cutler, Hailperin, Latshaw, Petrie, Pitcher, Qualley, Raynor, Reynolds, Shook, Smail, Van Arnam.
BRADFORD. Cummings.
BRYN MAWR. Atkinson, Innis.
*Bryn Mawr Coll.* Lehr, Oxtoby, Wheeler.
CARLISLE. Ayres, Kuebler, Stuart.
CHAMBERSBURG. Johnson.
CHESTER. Helms.
COLLEGEVILLE.
*Ursinus Coll.* Clawson, Dennis, Manning.
DENVER. Marburger.
EASTON.
*Lafayette Coll.* Benner, Cawley, Hatch, J. C. Smith, W. M. Smith.
ERIE. Kraus, Russell.
GEORGE SCHOOL. Fraser.
GREENSBURG. McNeil.
GROVE CITY. Carpenter.
HAVERFORD.
*Haverford Coll.* Allendoerfer, Oakley, Thomsen, Wilson.
HAZLETON. Liechty, Zerbe.
HERSHEY. Haag.
HUNTINGDON. Stayer.
IMMACULATA. Amator.
INDIANA. Stright.
KUTZTOWN. Knedler.
LANCASTER. Murray.
LATROBE. Seubert.
LEWISBURG.
*Bucknell Univ.* Gold, MacCreadie, Miller, Richardson.
LOCK HAVEN. Smith.

McKees Rock. Arnold.
Meadville. Steen.
Millersville. Boyer.
Montoursville. Price.
New Wilmington. McGaughey.
Philadelphia. Brown, Campbell, Cavalli, Constable, Davis, Durand, Eggert, Eisenhart, Fudge, Hearn, Keralla, Koch, Latshaw, Levy, McDonough, McNeary, Moliver, Neale, Russ, Slepin, Wing.
*Univ. of Pennsylvania.* Caris, Clarkson, Evans, Fine, Kline, Makarov, Patterson, Safford, Schoenberg, Walton, Wilson.
*Temple Univ.* Moses, Wilson, Wurster.
Picture Rocks. Price.
Pittsburgh. Briant, Buker, Calkins, Frankel, Harmon, Leifer, Miller, Mullan, Taylor.
*Carnegie Inst. of Tech.* Dines, Epstein, Hoover, Johnson, Lahti, Maple, Moskovitz, Neelley, Olds, Rosenback, Saibel, Synge, Whitman.
*Univ. of Pittsburgh.* Blumberg, Bryson, Foraker, Hovey, Taylor, Wells.
Pleasantville. Kerr.
Reading. Speicher.
Ridgeway. Bauser.
Scranton. Bertrand, Daniel.
Spring Grove. Martin.
Slippery Rock. Lady.
State College.
*Pennsylvania State Coll.* Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, Harrington, Herpel, Johnson, Krall, F. W. Owens, H. B. Owens, Rupp, Schwartz, Sheffer, Story, Townsend.
Swarthmore.
*Swarthmore Coll.* Brinkmann, Dresden, Marriott, Wasow.
Upper Darby. Houghton.
Washington.
*Washington & Jefferson Coll.* Bert, Dorwwart, Shaub, Thomas.
Waynesburg. Moston.

Rhode Island

Kingston.
*Rhode Island State Coll.* Bender, Brown, Stauffer.
Newport. Cavanagh, Chase.
Providence. McKenney, McMurtrie.
*Brown Univ.* Adams, Archibald, Bennett, Buck, Carlen, Gilman, Heins, Kaufman, Keck, Lin, Manning, Moore, Pease, Richardson, Rosenbloom, Saltzer, Smiley, Western.

South Carolina

Charleston.
*The Citadel.* Dye, Folsom, Hair, Hutchison, Reves, Sutton.
Clemson.
*Clemson Agric. Coll.* Brewster, Brown, Lagrone, Sheldon, Stanley, Vause.
Clinton. Coleman.
Columbia.
*Univ. of South Carolina.* Dinkines, E. C. Douglas, N. C. Douglas, Durst, Hedberg, Jackson, Lytle, Novak, Rasor, Robinson, Shuler, Weber, Williams.
Greenville. Blackwell, Mays.
Greenwood. Pettus.
Hartsville. Frierson, Reaves.
Newberry. Gaver.
Rock Hill. Pepper.
Spartanburg. Patten.

South Dakota

Brookings.
*South Dakota State Coll.* MacDougal, Walder, Wente.
Mitchell. Knox.
Rapid City. Swanson.
Springfield. Hoopes.
Vermillion. Meyer.
*Univ. of South Dakota.* Bedwell, Ekman Miller, H. W. Morrow, K. W. Morrow.
Yankton. Hoffman, Howell.

Tennessee

Alcoa. Harris.
Chattanooga. Massey.
Clarksville. Layton.
Cookeville. Hutchinson, Moorman.
Fountain City. Keller.
Harrogate. Bowling.
Jefferson City. Sloan.
Johnson City. Carson.
Knoxville. Parker.
*Univ. of Tennessee.* Albert, Cooley, Eaves, Ficken, Givens, Lee, Miller, Pollard, Shobe, Wilson.
Maryville. Sisk.
Memphis. Coker, Kaltenborn, McBride, Walbert.
Nashville. Boswell, Gasaway, Van Horn, Wren.
*Vanderbilt Univ.* Blair, Boyce, Clark, Graham, Hyden, Lundberg, Martin, Mrs. W. L. Miser, Prof. W. L. Miser.
Oak Ridge. Householder, Hurd, Jones, Young.

Texas

Abilene. Burnam, Mullings, Tate.
Amarillo. Adams, Davis.
Arlington. Howard.
Austin.
*Univ. of Texas.* Anderson, Batchelder, Craig, Decherd, Ettlinger, Greenwood, Guy, Lubben, Osborn, Porcelli, Shepherd, Vandiver, Wall.
Belton. Mason.
Brownsville. de la Garza.
Brownwood. Johnson.
Canyon. Murray.
College Station.
*A. & M. College of Texas.* Basye, Beeman, Daum, Klipple, Luther, McCulley, Moore, Pinkerton, Temple, Wapple.
Commerce. Wright.
Daingerfield. Kennedy.

DALLAS. McNabb, Mouzon, Sorrells, Starr, Stulken, Thomas.
DENTON.
*N. Texas Teachers Coll.* Brown, Cooke, Hanson.
*Texas State Coll. for Women.* Ashburn, Miller, White, Willey.
EL PASO. Lane.
FORT WORTH.
*Texas Christian Univ.* Bramblett, Morgan, Ramsey, Sherer, Shore,
GEORGETOWN. Whitmore.
HOUSTON. Blau, Howe, Newhouse, Rader, Slotnick.
*Rice Inst.* Bray, Brunk, Calkin, Dean, Lovett, Taylor, Ulrich, White.
HUNTSVILLE.
*Sam Houston State T. C.* Hardy, Lane, Querry, Vick, Wall.
KINGSVILLE. Dorroh.
LONGVIEW. Falvey.
LUBBOCK. Parker.
*Texas Tech. Coll.* Cross, Hazlewood, Heineman, May, Michie, Rowland, Sparks, Thompson, Underwood, Whetstone, Whyburn, Woodward.
ODESSA. Felder.
SAN ANGELO. Bright, Smith.
SAN ANTONIO. Dobbins, Jenke, Mary of Mercy, McNelly, Morgan, Schnepp.
*Trinity Univ.* Newton, Oesch, Rees.
SAN MARCOS. Cude.
STEPHENVILLE. McSweeny, Redden.
TEAGUE. Notley.

UTAH

LOGAN. Calvert, Hunsaker.
ST. GEORGE. Everett.
SALT LAKE CITY.
*Univ. of Utah.* Biesele, Bridger, Hayes, Heriques, Horsfall, Pehrson, Thorne.

VERMONT

BURLINGTON.
*Univ. of Vermont.* Bullard, Butterfield, Larrivee, Millington, Swift.
MIDDLEBURY.
*Middlebury Coll.* Ballou, Bowker, Hazeltine, Perkins.
NORTHFIELD. Dix.
SWANTON. Alliot.

VIRGINIA

BLACKSBURG.
*Virginia Poly. Inst.* Hatcher, Horne, McFadden, O'Shaughnessy, Spencer.
BUENA VISTA. Durham.
CHARLOTTESVILLE.
*Univ. of Virginia.* Aylor, DeFrancesco, Floyd, Fort, Hedlund, Klee, Linfield, McShane, Oglesby, Utz, Whyburn.
DAHLGREN. Bramble.
FARMVILLE. Sutherland, Taliaferro.
FREDRICKSBURG. Frick.
HAMPTON. Claytor.
HARRISONBURG. Ikenberry, Suter.
LEXINGTON. Byrne, Knox, Meadows, Paxton, Smith.
LYNCHBURG.
*Randolph-Macon Coll.* Baker, Larew, Simpson, Wiggin.
MIDDLEBURG. Keppler.
NEWPORT NEWS. Plethides, Raine.
NORFOLK. Norris.
PETERSBURG. Hunter.
RICHMOND. Drew.
*Univ. of Richmond.* Gaines, Grable, Harris, Wheeler.
SALEM. Carpenter.
STAUNTON. Taylor.
SWEET BRIAR. Morenus.
WILLIAMSBURG.
*Coll. of William and Mary.* Calkins, Phalen, A. Lee Smith, R. E. Smith, Stetson.

WASHINGTON

ABERDEEN. Seamons.
CHENEY. Bell.
EVERETT. Van Arkel.
LACEY. Cebula.
PORT TOWNSEND. Fitzgerald.
PULLMAN.
*St. Coll. of Washington.* Butler, Caton, Hacker, Irwin, Klotz, Knebelman, Rowland, Tysver, Vatnsdal.
SEATTLE. Beegle, Hull, Lanczos.
*Univ. of Washington.* Ballantine, Beaumont, Cramlet, Haller, Jerbert, Kingston, McFarlan, Taub, Winger, Zuckerman.
SPOKANE. Carlson.
TACOMA. Goman.
VANCOUVER. Stair.

WEST VIRGINIA

COWEN. Haller.
HUNTINGTON. Goins.
MONTGOMERY. Babcock.
MORGANTOWN.
*West Virginia Univ.* Bauserman. Cunningham, Davis. Eiesland, Peters, Reynolds, Turner, Vehse, Vest.

WISCONSIN

APPLETON. Berry, Stewart.
BELOIT. Conwell, Huffer.
EAU CLAIRE. Otteson.
GREEN BAY. DeCleene.
LA CROSSE. Adkins, Malin.
MADISON. Ericksen, Otis.
*Univ. of Wisconsin.* Allen, Andree, Arnold, Bing, Bruck, Chessin, Cohen, Colvin, Davies, Evans, Fuller, Hart, Hood, Ingraham, Jarmain, Kleene, Langer, MacDuffee, March, Mark, Mayor, Otis,

Rechard, Remo, Rose, Rothstein, Ryser, Schecter, Schurrer, Simpson, Sokolnikoff, Specht, Trump, Zemmer.
MARINETTE. Wagner.
MILWAUKEE. Arashiro, Baumann, Bigelow, Boehmer, Clark, Felice, Fetterer, Gertrude, Norris.
*Marquette Univ.* Fitzpatrick, Glander, Moeller, Pettit, Wilczewski.
*Univ. of Wisconsin in Milwaukee.* Bardell, Battig, Kenney, Marden, Nemmers, Parkinson, Spitzbart, Vass, Wolf.
PLATTEVILLE. Harrell.
PLYMOUTH. Rusch.
RACINE. Jautz.
RIVER FALLS. Eide, McLaughlin.
SUPERIOR. Flogstad, Smith.
WAUKESHA. Dancey, Hopkins.
WHITEFISH BAY. Anderson.

WYOMING

LARAMIE. Bellamy.
*Univ. of Wyoming.* Barr, Brady, Neubauer, Rechard, Schwid, Smith, Varineau.

## CANADA

AURORA. Lane.
EDMONTON.
*Univ. of Alberta.* Campbell, Cook, Sheldon.
FREDERICTON. Edwards.
HAMILTON. Bankier, Findlay.
KINGSTON.
*Queens Univ.* Halperin, Jeffery, Miller.
LONDON. Cole, Kingston.
MONTREAL. Ayoub, Frechette, Gauthier, Gough, Lalonde, Pelletier, Rosenthall.
OTTAWA. Dubé, Duffie, Keyfitz.
QUEBEC. Pouliot, Roland.
SACKVILLE. Crawford.
SASKATOON. Ferns, Miller.
TORONTO. Dobson, Grant.
*Univ. of Toronto.* Beatty, Brauer, Burk, Coxeter, Pounder, Robinson.
VANCOUVER.
*Univ. of British Columbia.* Buchanan, Chapman, Free, Gage, James, Jennings Kent, Murdoch, Nowlan, Simons.
VICTORIA. Wallace.
WINNIPEG.
*Univ. of Manitoba.* Fleming, Mendelsohn, McEwen, Moser, Warren.
WOLFVILLE. MacPhail.

## FOREIGN MEMBERS

ARGENTINA

BUENOS AIRES. Baidaff, Barral-Souto.

BELGIUM

BRUGES. Goormaghtigh.
BRUXELLES. Errera.
MONS. Deaux.

BRITISH HONDURAS

BELIZE. Zimmerman.

CEYLON

VADDUKODDAI. Lockwood.

CHILE

SANTIAGO. Moreno.

COLOMBIA

BOGOTA. Thullen.

CUBA

HAVANA. Aleman, Gonzáles, Novoa, Rodrígues.
SANTIAGO. Muguercia.

DENMARK

COPENHAGEN. Schmidt.

ENGLAND

BRIERLEY HILL. Solari.
CAMBRIDGE. Hardy.
CHESSINGTON. O'Beirne
CHIPPING NORTON. O'Hara.
ENGLEFIELD GREEN. McCrea.
LONDON. Dalal, Spickelmier, Todd.

FRANCE

BOURG-LA-REINE. Minois.
CLERMONT-FERRAND. Droussent.
LUXEMBOURG. Zahlen.
PARIS. Belgodère.
TENNIE. Thébault.

GREECE

ATHENS. Lanckton.

GUATEMALA

GUATEMALA. Engel.

INDIA

BANGALORE. Madhava Rao.
BELGAUM. Sharma.
HYDERABAD. Ghani.
SURAT. Shah.

IRELAND

DUBLIN. Broderick.

LEBANON

BEIRUT. Jurdak, Kennedy.

MEXICO

MEXICO. Nápoles.

NETHERLANDS

THE HAGUE. Spruitenburg.

NEW ZEALAND

DUNEDIN. Martyn.

PANAMA

PANAMA CITY. Linares.

PERU

LIMA. de Losada y Puga.

PHILIPPINE ISLANDS

MANILA. Perez.

PORTUGAL

LISBON. Caraca.

PUERTO RICO

MAYAGUEZ. Garcia.
RIO PIEDRAS. Rathburn.
SAN JUAN. Piza.

RUSSIA

MOSCOW. Kryloff.

SOUTH AFRICA

BLOEMFONTEIN. Arndt.

SPAIN

MADRID. Bachiller.

SWITZERLAND

BASLE. Ostrowski.
FRIBOURG. Bays.
NEUCHÂTEL. DuPasquier.
ZURICH. Burckhardt.

TURKEY

ISTANBUL. Miller.

URUGUAY

MONTEVIDEO. Calcagno.

WALES

BRIDGEND TRADING ESTATE. Nicholls.

VENEZUELA

CARACAS. Michalup.

## BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC).

(As amended to January 1, 1948)

### ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

### ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission, endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

### ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members in the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

## Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

### Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

### Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

### Article VII—Dues

1. Members of the Association shall pay an initiation fee of Two Dollars ($2) at the tim of election.

2. The annual dues of each member shall be Four Dollars ($4), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a resonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of one dollar.

### Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

## PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION

(The periods were for the calendar years except that after 1942 the terms of Regional Governors began and ended July 1.)

### Honorary President for Life

H. E. Slaught, December 1933–May 1937

### Presidents

| | |
|---|---|
| E. R. Hedrick | 1916 |
| Florian Cajori | 1917 |
| E. V. Huntington | 1918 |
| H. E. Slaught | 1919 |
| D. E. Smith | 1920 |
| G. A. Miller | 1921 |
| R. C. Archibald | 1922 |
| R. D. Carmichael | 1923 |
| H. L. Rietz | 1924 |
| J. L. Coolidge | 1925 |
| Dunham Jackson | 1926 |
| W. B. Ford | 1927–1928 |
| J. W. Young | 1929–1930 |
| E. T. Bell | 1931–1932 |
| Arnold Dresden | 1933–1934 |
| D. R. Curtiss | 1935–1936 |
| A. J. Kempner | 1937–1938 |
| W. B. Carver | 1939–1940 |
| R. W. Brink | 1941–1942 |
| W. D. Cairns | 1943–1944 |
| C. C. MacDuffee | 1945–1946 |

### Vice-Presidents

| | |
|---|---|
| E. V. Huntington | 1916 |
| G. A. Miller | 1916 |
| D. N. Lehmer | 1917, 1918 |
| Oswald Veblen | 1917 |
| J. W. Young | 1918, 1926 |
| R. G. D. Richardson | 1919 |
| H. L. Rietz | 1919 |
| Helen A. Merrill | 1920 |
| E. J. Wilczynski | 1920 |
| R. C. Archibald | 1921 |
| R. D. Carmichael | 1921, 1922 |
| B. F. Finkel | 1922 |
| A. B. Chace | 1923 |
| L. P. Eisenhart | 1923 |
| J. L. Coolidge | 1924 |
| Dunham Jackson | 1924, 1925 |
| A. A. Bennett | 1925, 1933, 1934 |
| W. B. Ford | 1926 |
| A. J. Kempner | 1927, 1928, 1935 |
| Clara E. Smith | 1927 |
| F. D. Murnaghan | 1928, 1939 |
| E. T. Bell | 1929, 1930 |
| W. C. Graustein | 1929, 1930, 1940 |
| Arnold Dresden | 1931 |
| C. N. Moore | 1931 |
| W. H. Bussey | 1932 |
| G. C. Evans | 1932 |
| E. B. Stouffer | 1933 |
| E. P. Lane | 1934 |
| L. L. Dines | 1935 |
| N. A. Court | 1936 |
| T. C. Fry | 1936 |
| T. H. Hildebrandt | 1937 |
| E. J. Moulton | 1937, 1938 |
| H. E. Buchanan | 1938 |
| W. L. Hart | 1939 |
| R. W. Brink | 1940 |
| B. H. Brown | 1941–1942 |
| R. E. Langer | 1941 |
| Tomlinson Fort | 1942–1943 |
| C. C. MacDuffee | 1943–1944 |
| W. M. Whyburn | 1944–1945 |
| W. F. Cheney, Jr. | 1945–1946 |

### Secretary-Treasurer

(Appointed by the Board after 1918)

| | |
|---|---|
| W. D. Cairns | 1916–1942 |
| W. B. Carver | 1943–1947 |

### Committee on Official Journal

Appointed by the Board. Discontinued after 1939)

| | |
|---|---|
| H. E. Slaught | 1916–1937 |
| R. D. Carmichael | 1916–1918 |
| W. H. Bussey | 1916–1918, 1926–1931 |
| R. C. Archibald | 1919–1921 |
| W. A. Hurwitz | 1919–1921 |
| A. A. Bennett | 1922 |
| H. P. Manning | 1921–1922 |
| W. B. Ford | 1923–1925 |
| J. L. Coolidge | 1923 |
| A. J. Kempner | 1924–1939 |
| W. B. Carver | 1932–1936, 1937–1939 |
| E. J. Moulton | 1937–1939 |

### Editors-in-Chief after 1939

| | |
|---|---|
| E. J. Moulton | 1940–1941 |
| L. R. Ford | 1942–1946 |

ADDITIONAL MEMBERS OF THE BOARD

D. N. LEHMER.......1916–1918, 1922–1924, 1930–1932
R. E. MORITZ..................1916–1918
F. D. MURNAGHAN..............1935–1937
G. C. EVANS...................1936–1941
MARY EMILY SINCLAIR..........1936–1938
K. D. SWARTZEL...............1916
OSWALD VEBLEN..........1916, 1920–1922, 1926–1931
R. C. ARCHIBALD.....1916–1917, 1923–1930
FLORIAN CAJORI...........1916, 1918–1923, 1926–1930
M. B. PORTER..................1916–1917
J. W. YOUNG.........1916–1917, 1920–1922
B. F. FINKEL.........1916–1921, 1930–1935
E. H. MOORE........1916–1921, 1923–1928
ALEXANDER ZIWET.............1916–1918
E. R. HEDRICK.......1917–1922, 1924–1929, 1932–1937
J. N. VAN DER VRIES...........1916–1918
HELEN A. MERRILL.............1917–1919
D. E. SMITH.........1917–1919, 1921–1926, 1937–1939
ELIZABETH B. COWLEY..........1918–1920
G. A. MILLER........1918–1920, 1922–1924
E. J. WILCZYNSKI.....1918–1919, 1922–1926
L. P. EISENHART.....1919–1922, 1925–1930
E. V. HUNTINGTON.........1917, 1919–1927, 1933–1935
E. L. DODD....................1920
R. D. CARMICHAEL.........1920, 1924–1929, 1939–1941
A. A. BENNETT..1921, 1930–1932, 1939–1941
H. L. RIETZ..........1921–1923, 1925–1930, 1934–1936
C. F. GUMMER................1921–1925
DUNHAM JACKSON.............1923–1929
CLARA E. SMITH...............1923–1925
A. B. CHACE..................1924–1925
J. L. COOLIDGE.................1926–1931
E. T. BELL....................1927–1928
E. P. LANE....................1928–1933
W. B. FORD...................1929–1934
E. R. SMITH...................1929
W. L. HART...................1930–1935
LAO G. SIMONS................1930–1931
L. L. DINES...................1931–1933
T. C. FRY....................1931–1933
J. W. GLOVER.................1931–1933
H. E. BUCHANAN..............1932–1937
W. R. LONGLEY......1932–1934, 1936–1938
E. J. MOULTON.......1933–1936, 1943–1945
R. W. BRINK..................1934–1939
D. R. CURTIS...1934, 1937–1939, 1940–1942
J. L. WALSH...................1934–1936
ARNOLD DRESDEN....1935–1940, 1943–1945
J. O. HASSLER..................1935–1936
J. M. THOMAS........1937–1939, 1940–1941
MARIE J. WEISS................1937–1938
WILLIAM BETZ.................1938–1940
A. B. COBLE...................1938–1940
J. H. WEAVER..................1938–1940
PHILIP FRANKLIN...............1940–1942
F. L. GRIFFIN..................1940–1942
MAYME I. LOGSDON.............1940–1942
G. T. WHYBURN................1940–1942
C. V. NEWSOM.................1940–1941
O. J. PETERSON.................1940–1941
F. B. WILEY...................1940–1941
H. M. BACON..................1941–1943
H. J. ETTLINGER................1941–1943
CORNELIUS GOUWENS...........1941–1943
W. C. KRATHWOHL.............1941–1943
E. J. MCSHANE.................1941–1943
F. W. OWENS..................1941–1943
S. T. SANDERS.................1941–1943
W. L. AYRES...................1942–1944
R. L. WILDER..................1942–1944
SAUNDERS MAC LANE...........1943–1945
R. P. AGNEW...................1942–1944
L. M. BLUMENTHAL............1942–1944
W. F. CHENEY, JR..............1942–1944
C. G. LATIMER.................1942–1944
W. E. MILNE...................1942–1944
O. H. RECHARD.................1942–1944
H. A. ROBINSON................1942–1944
H. E. BRAY....................1943–1945
D. W. HALL....................1943–1945
C. G. JAEGER..................1943–1945
A. L. NELSON..................1943–1945
W. V. PARKER.................1943–1945
K. W. WEGNER.................1943–1945
R. G. SANGER..................1944–1946
MORGAN WARD.................1944–1946
W. R. RANSOM.................1944–1946
H. F. MACNEISH...............1944–1946
J. W. LASLEY, JR...............1944–1946
W. L. MISER...................1944–1946
G. W. SMITH...................1944–1946
E. J. PURCELL..................1944–1946
F. S. NOWLAN..................1944–1946
L. L. DINES....................1945–1947
GILLIE A. LAREW...............1945–1947
H. L. SMITH....................1945–1947
K. P. WILLIAMS................1945–1947
R. C. HUFFER..................1945–1947
N. A. COURT...................1945–1947
SOPHIE L. MCDONALD...........1945–1947

# The Rhind Mathematical Papyrus

The late CHANCELLOR ARNOLD BUFFUM CHACE of Brown University rendered signal honor to the ASSOCIATION by publishing under its auspices his RHIND MATHEMATICAL PAPYRUS. The entire receipts from the sale of this great work go to form an endowment fund of the ASSOCIATION, known as the ARNOLD BUFFUM CHACE FUND.

Volume I, over 200 pages (11¼″ × 8″), contains the free Translation, Commentary, and Bibliography of Egyptian Mathematics.

Volume II, 140 plates (11¼″ × 14″) in two colors with Text and Introductions, contains the Photographic Facsimile, Hieroglyphic Transcription, Transliteration, and Literal Translation.

This exposition of one of the very oldest mathematical documents in the world is of great value, not only to students of mathematics, but to any one interested in the work of a civilization existing nearly 4,000 years ago.

The special price of $20.00 per set (postage prepaid) has been made for members of the ASSOCIATION when purchased through the SECRETARY; to all others the price is $25.00 per set (postage prepaid), obtainable only through the OPEN COURT PUBLISHING COMPANY, LaSalle, Illinois.

# The Carus Mathematical Monographs

The CARUS MATHEMATICAL MONOGRAPHS are an expression of the desire of Mrs. Mary Hegeler Carus, and of her son, Dr. Edward H. Carus, to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics.

The expositions of mathematical subjects which the monographs contain are set forth in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises.

## MONOGRAPHS THUS FAR PUBLISHED

No. 1. *Calculus of Variations,* by Professor G. A. Bliss. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935.)

No. 2. *Analytic Functions of a Complex Variable,* by Professor D. R. Curtiss. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by Professor H. L. Rietz. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936.)

No. 4. *Projective Geometry,* by Professor J. W. Young. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by Professors David Eugene Smith and Jekuthiel Ginsburg. (First Impression, 1934.)

No. 6. *Fourier Series and Orthogonal Polynomials,* by Professor Dunham Jackson. (First Impression, 1941.)

No. 7. *Vectors and Matrices,* by Professor C. C. MacDuffee. (First Impression, 1943.)

Price $1.25 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary, H. M. Gehman, University of Buffalo, Buffalo 14, N.Y.

Additional copies for members, and copies for non-members, must be purchased from the Open Court Publishing Co., La Salle, Illinois, at the regular price of $2.00 per copy.